Normally Regular Digraphs

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Abstract

A normally regular digraph with parameters \((v, k, \lambda, \mu)\) is a directed graph on \(v\) vertices whose adjacency matrix \(A\) satisfies the equation \(AA^t = kI + \lambda(A + A^t) + \mu(J - I - A - A^t)\). This means that every vertex has out-degree \(k\), a pair of non-adjacent vertices have \(\mu\) common out-neighbours, a pair of vertices connected by an edge in one direction have \(\lambda\) common out-neighbours and a pair of vertices connected by edges in both directions have \(2\lambda - \mu\) common out-neighbours. We often assume that two vertices can not be connected in both directions.

We prove that the adjacency matrix of a normally regular digraph is normal. A connected \(k\)-regular digraph with normal adjacency matrix is a normally regular digraph if and only if all eigenvalues other than \(k\) are on one circle in the complex plane. We prove a Bruck-Ryser type condition for existence and give a combinatorial proof for a restriction excluding existence in some cases with small values of \(\lambda\). There is a structural characterization of normally regular digraphs with \(\mu = 0\) or \(\mu = k\). For other values of \(\mu\) we give several constructions of normally regular digraphs. In many cases these graphs are Cayley graphs of abelian groups and the construction is then based on a generalization of difference sets. In particular, if \(4t + 1\), \(4s + 3\) and \(q\) are prime powers and \(r\) is not divisible by \(3\) we get normally
regular Cayley digraphs with the following parameters

\[((4t + 1)(4s + 3), (4t + 2)(2s + 1), 4st + 3s + t + 1),
((4s + 3)(2s + 1), 4s + 1, s, 1),
(q^{2r} + q^r + 1, q^r - q, q^2, q^2 + q + 1)\]

and, if \(q \equiv 1 \mod 3\)

\(\left(\frac{1}{3}(q^2 + q + 1), q - 1, 1, 3 \right)\).

We also show connections to other combinatorial objects: strongly regular graphs, symmetric 2-designs and association schemes.

Mathematics Subject Classifications: 05E30, 05B05, 05C20, 05C50

1 Introduction

In this section we introduce normally regular digraphs and other basic concepts. In Section 2 we prove that the adjacency matrices of normally regular digraphs are normal and we give a Bruck-Ryser type condition for existence. In Section 3 we show that complements of normally regular digraphs are normally regular and we prove bounds on the parameters. In Section 4 we characterize normally regular digraphs with \(\mu = 0\) or \(\mu = k\). We consider eigenvalues of normally regular digraphs in Section 5 and show that a regular digraph with normal adjacency matrix is a normally regular digraph if and only if the non-trivial eigenvalues are on a circle in the complex plane. In Section 6 we consider relations to association schemes. The subject of Section 7 is partitions of the vertex set and in particular group divisible digraphs, i.e., orientations of complete multipartite graphs. In Section 8 we exclude existence for some parameter sets with small \(\lambda\). Section 9 describes applications of normally regular digraphs to partitions of designs in smaller designs. In Section 10 we give several constructions of normally regular digraphs, primarily constructions as Cayley graphs.

The adjacency matrix of a digraph with vertex set \(\{x_1, \ldots, x_v\}\) is a \(v \times v\) matrix \(A\) in which the \((i, j)\)-entry is

\[A_{ij} = \begin{cases} 1 & \text{if there is an edge directed from } x_i \text{ to } x_j \\ 0 & \text{otherwise.} \end{cases}\]
Thus any square \(\{0,1\}\)-matrix is the adjacency matrix of a digraph if and only if all its diagonal entries are 0. In this paper we consider such matrices that satisfy an equation involving \(AA^t\). The \((i,j)\) entry of \(AA^t\) (respectively \(A^tA\)) is the number of common out-neighbours (respectively in-neighbours) of \(x_i\) and \(x_j\).

We say that a digraph is normal if its adjacency matrix \(A\) is normal, i.e., if \(AA^t = A^tA\). It follows that a digraph is normal if and only if for any two (not necessarily distinct) vertices \(x\) and \(y\) the number of common out-neighbours of \(x\) and \(y\) is equal to the number of common in-neighbours of \(x\) and \(y\).

We will use the notation \(x \rightarrow y\) if there is an edge directed from \(x\) to \(y\) (and possibly also an edge from \(y\) to \(x\)). If \(x \rightarrow y\) then we say that \(x\) dominates \(y\). We write \(x \leftrightarrow y\) if \(x \rightarrow y\) and \(x \leftarrow y\), and identify these two directed edges with an undirected edge.

The set \(\{y \mid x \rightarrow y\}\) of out-neighbours of a vertex \(x\) is denoted by \(x^+\). Similarly \(x^-\) denotes the set of in-neighbours. \(d^+(x) = |x^+|\) and \(d^-(x) = |x^-|\) denotes the out-degree and in-degree of \(x\), respectively.

We will now introduce normally regular digraphs. We first give a matrix free definition.

**Definition 1.** A normally regular digraph with parameters \((v, k, \lambda, \mu)\), also denoted by \(NRD(v, k, \lambda, \mu)\), is a directed graph on \(v\) vertices so that

- every vertex has out-degree \(k\)
- any pair of non-adjacent vertices have exactly \(\mu\) common out-neighbours,
- any pair of vertices \(x, y\) such that exactly one of the edges \(x \rightarrow y\) or \(x \leftarrow y\) is present have exactly \(\lambda\) common out-neighbours,
- any pair of vertices \(x, y\) such that \(x \leftrightarrow y\) have exactly \(2\lambda - \mu\) common out-neighbours.

A normally regular digraph is said to be asymmetric if there is no pair \(x, y\) so that \(x \leftrightarrow y\).
Proposition 1. A $v \times v \{0,1\}$-matrix $A$ is the adjacency matrix of a normally regular digraph if and only if every diagonal entry is 0 and

$$AA^t = kI + \lambda(A + A^t) + \mu(J - I - A - A^t),$$

where $I$ is the identity matrix and $J$ is the matrix in which all entries are 1.

This normally regular digraph is asymmetric if and only if $A + A^t$ is a \{0,1\} matrix.

The author first intended to study only asymmetric normally regular digraphs. However, most of the results hold in the general case, so we will usually not assume that graphs are asymmetric, but for connections to association schemes and similar results we need to assume that the graph is asymmetric.

Asymmetric normally regular digraphs with $\mu = \lambda$ have been studied by Ito [13], [14], [15], [16], and also by Ionin and Kharaghani [12].

Fossorier, Ježek, Nation and Pogel [6] introduced what they call ordinary graphs. Their definition is similar to our Definition 1, but the number of common out-neighbours (and common in-neighbours) of $x$ and $y$ in the three cases is $a$, $b$ and $c$, respectively. They do not assume that $c = 2b - a$ (although this is satisfied in some of their results). Note that the equation $c = 2b - a$ is essential for the formulation of the definition of a normally regular digraph as a matrix equation, and thus it is essential for the theory.

U. Ott [28] considered Cayley graph construction from “generalized difference set” that leads to normally regular digraphs.

Another variation of normally regular digraphs is Deza digraphs. A regular digraph is said to be a Deza digraph if the number common out-neighbours of two vertices is either $b$ or $c$, for some constants $b$ and $c$, but it need not depend on whether the vertices are adjacent or not. Deza digraphs have been studied by Wang and Feng [34].

Many constructions of normally regular digraphs uses Cayley graphs of a group. Let $G$ be a group and let $S$ be a subset of $G$ not containing the group identity. Then the Cayley graph Cay($G$, $S$) is the graph whose vertices are the elements of $G$ and with edge set

$$\{x \rightarrow y \mid x^{-1}y \in S\}.$$
every \( g \in G, g \neq 1 \), the number of pairs \((x,y)\in S \times S\) satisfying \(yx^{-1} = g\) is \(\mu\) if \( g \notin S \cup S^{(1)} \), \(\lambda\) if \( g \) is in exactly one the sets \(S, S^{(1)}\) and \(2\lambda - \mu\) if \( g \in S \cap S^{(1)}\).

In [22] we prove the following multiplier theorem.

**Theorem 2.** Suppose that \( G \) is an abelian group and that \( \text{Cay}(G,S) \) is an \( \text{NRD}(v,k,\lambda,\mu) \). Let \( w \) be the smallest positive number so that for every \( g \in G \) the order of \( g \) divides \( w \). Let \( m \) be a natural number relatively prime to \( v \), so that \( m \) divides \( \eta = k - \mu + (\lambda - \mu)^2 \) and let \( t \) be relatively prime to \( v \). Suppose that for every prime \( p \) dividing \( m \) there exist an integer \( f \) so that \( t \equiv pf \mod w \). If either \( m > \mu = \lambda + 2 \) or \( m > 2\lambda - \mu \), \( \lambda \geq \mu \) then (in additive notation) \( tS = S \).

Furthermore, in [23] we enumerate small normally regular digraphs and prove some results related these graphs. In [24], group divisible normally regular digraphs, i.e, the digraphs considered in Section 4.4 and Section 7 are investigated.

It is well-known (see Godsil and Royle [7]) that a strongly regular graph with parameters \((v,k,\lambda,\mu)\) is an undirected graph with \( v \) vertices in which

- every vertex has degree \( k \)
- any pair of adjacent vertices have exactly \( \lambda \) common neighbours
- any pair of non-adjacent vertices have exactly \( \mu \) common neighbours.

Equivalently, a strongly regular graph is a graph whose adjacency matrix \( A \) satisfies

\[
A^2 = kI + \lambda A + \mu(J - I - A) \quad \text{and} \quad AJ = JA = kJ.
\]

Thus any normally regular digraph where all edges are undirected (i.e., \( x \rightarrow y \) if and only if \( x \leftarrow y \)) is a strongly regular graph. Note however that we use \( \lambda \) in a different meaning. For a normally regular digraph we will use \( \lambda_2 = 2\lambda - \mu \) to denote the number of common out-neighbours of a pair of vertices joined by two edges.

In the theory of normally regular digraphs we will require that \( \lambda \) is an integer and thus \( \mu \) and \( \lambda_2 \) are congruent modulo 2. Thus not every strongly regular graph is a normally regular digraph.

A strongly regular with \((v,k,\lambda,\mu) = (4\mu + 1,2\mu,\mu - 1,\mu)\) is called a conference graph. Since \( \lambda \) and \( \mu \) have different parity a conference is not a
normally regular digraph, but it will be used in some constructions. The most important construction of conference graphs are the Paley graphs which are constructed as follows. Let $F$ be a field of $q$ elements, $q \equiv 1 \pmod{4}$ and let $Q$ be the non-zero squares in $F$. Then the Cayley graph of the additive group $\text{Cay}(F, Q)$ is a conference graph, see [7].

There are some directed analogues of strongly regular graphs other than normally regular digraphs. Duval [5] introduced directed strongly regular graphs which have adjacency matrix $A$ satisfying

$$A^2 = tI + \lambda A + \mu(J - I - A) \quad \text{and} \quad AJ = JA = kJ.$$  

Many proof techniques from strongly regular graphs, especially the use of eigenvalues, are more easily applied to directed strongly regular graphs than to normally regular graphs, see [5] or [19].

Another well-known combinatorial structure to which normally regular digraphs are related are 2-designs (or Balanced Incomplete Block Designs). A 2-$(v, k, \lambda)$ design is an incidence structure with $\{0, 1\}$ incidence matrix $N$ of size $v \times b$, $b = \frac{\lambda(v^2 - 1)}{k(k-1)}$ satisfying

$$NN^t = (k - \lambda)I + \lambda J.$$  

A 2-design is said to be symmetric if $b = v$. The parameter $k - \lambda$ is called the order of the symmetric design. For information on design theory, see Beth, Jungnickel and Lenz [2].

Let $A$ be the adjacency matrix of a normally regular digraph. If $\mu = \lambda$ then $A$ is incidence matrix of symmetric 2-design. If $\mu = \lambda + 1$ then $A + I$ is incidence matrix of a symmetric 2-design. In this paper we will often assume that $\mu \notin \{\lambda, \lambda + 1\}$.

A tournament is a digraph with the property that for any two distinct vertices $x$ and $y$ exactly one of the edges $x \rightarrow y$ or $y \rightarrow x$ is present.

We will need the following property of regular tournaments in Section 4.

**Lemma 3** (Rowlinson [31]). A tournament is normal if and only if it is regular.

**Proof.** If $A$ is an adjacency matrix of a regular tournament, i.e., $AJ = JA = kJ$, for some number $k$ then, since $A^t = J - I - A$, $AA^t = AJ - A - A^2 = JA - A - A^2 = A^tA$.

Conversely, in a normal digraph every vertex has the same in-degree and out-degree. \qed
If a tournament is a normally regular digraph then it is called a doubly regular tournament. It satisfies \( k = 2\lambda + 1 \). \( \mu \) is arbitrary, so we may take \( \mu = \lambda \). Such tournaments are also called homogenous tournaments by Kotzig [25], and Ito [17] used the term Hadamard tournaments, as these tournaments are equivalent to skew Hadamard matrices of order \( v + 1 \) (see Reid and Brown [29])

Thus it is possible that doubly regular tournaments of order \( v \) exists for all \( v \equiv 3 \mod 4 \).

The most important construction of a doubly regular tournament is the Paley tournament which is constructed as follows. Let \( F \) be a field of \( q \) elements \( q \equiv 3 \mod 4 \) and let \( Q \) be the non-zero squares in \( F \). Then the Cayley graph \( \text{Cay}(F,Q) \) is a doubly regular tournament.

We conclude this section with two small asymmetric normally regular digraphs.

**Example 1.** Let \( Q = \{1, i, j, k, -1, -i, -j, -k\} \) be the quaternion group. Then \( \text{Cay}(Q,\{i,j,k\}) \) is an NRD(8,3,1,0) with the following adjacency matrix

\[
\begin{pmatrix}
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 
\end{pmatrix}
\]

This is the smallest non-trivial normally regular digraph with \( \mu = 0 \). Normally regular digraphs with \( \mu = 0 \) or \( \mu = k \) are characterized in Section 4.

**Example 2.** \( \text{Cay}(\mathbb{Z}_{19},\{1,4,6,7,9,11\}) \) is an NRD(19,6,1,3). This is the smallest normally regular digraph with \( \mu \notin \{ k, 0, \lambda, \lambda + 1 \} \). It belongs to an infinite family constructed in Theorem 31. This digraph is asymmetric and in fact \( \lambda_2 = 2\lambda - \mu \) is negative.
2 Matrix equations

It is convenient to introduce two further parameters of a normally regular digraph:

\[ \eta = k - \mu + (\mu - \lambda)^2 \]

and

\[ \rho = k + \mu - \lambda. \]

The parameter \( \eta \) will play a role similar to that of the order of a symmetric design. Ma [27] uses the parameter \( \Delta = 4\eta \) in the study of strongly regular graphs. The factor 4 is necessary in order get an integer for a general strongly regular graph.

The matrix equation in Proposition 1 is equivalent to the following equation.

\[
(A + (\mu - \lambda)I)(A + (\mu - \lambda)I)^t = \eta I + \mu J. \tag{2}
\]

Thus for \( B = (A + (\mu - \lambda)I) \) we have

\[ BB^t = \eta I + \mu J, \]

and since \( AJ = kJ \) (every vertex has out-degree \( k \)),

\[ BJ = \rho J. \]

We will now prove that a normally regular digraph is normal. The following lemma is a generalization of a proof of the fact that the dual of a symmetric 2-design is also a 2-design, see [2].

**Lemma 4.** Suppose that \( B \) is a non-singular \( v \times v \) matrix so that \( BB^t = \eta I + \mu J \) and \( BJ = \rho J \) for some constants \( \rho, \eta, \mu \). Then \( B \) is normal and \( \mu v = \rho^2 - \eta \).

**Proof.** From \( BJ = \rho J \) we get \( \rho^{-1} J = B^{-1} J \) and

\[
B^t = B^{-1}(BB^t) = B^{-1}(\eta I + \mu J) = \eta B^{-1} + \mu \rho^{-1} J. \tag{3}
\]

Using that \( J \) is symmetric, we get from this

\[ \rho J = (BJ)^t = JB^t = \eta JB^{-1} + \mu \rho^{-1} J^2 = \eta JB^{-1} + \mu \rho^{-1} v J. \]

This implies that

\[ JB^{-1} = \frac{\rho - \mu \rho^{-1} v}{\eta} J, \]
and so
\[ vJ = J^2 = (JB^{-1})(BJ) = \frac{\rho - \mu \rho^{-1}v}{\eta} \rho vJ. \]

Thus
\[ \frac{\rho - \mu \rho^{-1}v}{\eta} = \rho^{-1}, \]

and $JB^{-1} = \rho^{-1}J$ or $\rho J = JB$. Now equation 3 implies
\[ B^tB = \eta I + \mu J = B B^t. \]

Rewriting equation 4 we get
\[ \mu v = \rho^2 - \eta. \]

**Corollary 5.** Every normally regular digraph is normal.

*Proof.* Let $A$ be the adjacency of a normally regular digraph and let $B = A + (\mu - \lambda)I$. Then $BB^t = \eta I + \mu J$. Suppose first that $B$ is singular. Then one of the eigenvalues of $\eta I + \mu J$ is zero: $\eta = 0$ or $\eta + \mu v = 0$. Since $\mu, v \geq 0$ this is possible only when $\eta = k - \mu + (\mu - \lambda)^2$ is 0. As $k + (\mu - \lambda)^2 \geq k \geq \mu$, $\mu = k + (\mu - \lambda)^2$ implies $k = \mu = \lambda$. This implies that $k = 0$. Since a graph with no edges is normal, we may thus assume that $B$ is non-singular, and the result follows from the lemma.

It follows that a normally regular digraph is both normal and regular, i.e., every vertex has in-degree $k$ and out-degree $k$. And the number of common in-neighbours of distinct vertices $x$ and $y$ is
\[ \begin{cases} 
\mu & \text{if } x \text{ and } y \text{ are non-adjacent,} \\
\lambda & \text{if either } x \rightarrow y \text{ or } y \rightarrow x, \text{ but not both,} \\
2\lambda - \mu & \text{if } x \leftrightarrow y.
\end{cases} \]

**Corollary 6.** The parameters of a normally regular digraph satisfy
\[ \mu v = \rho^2 - \eta. \] (5)

This equation is equivalent to the following
\[ 2k\lambda + (\nu - 2k - 1)\mu = k^2 - k \] (6)

This equation may also be obtained by counting in two ways the number of triples $(x, y, z)$ of vertices so that $x \rightarrow y \leftarrow z$ using the definition of a normally regular digraph and the fact that every vertex has in-degree $k$. 

9
From the theory of symmetric 2-designs we also have the Bruck-Ryser
type condition. It is based on the following general lemma from Beth, Jung-
nickel and Lenz [2]

**Lemma 7.** Suppose that \( N \) is a rational \( v \times v \) matrix satisfying the equation

\[
NN^t = (a - b)I + bJ
\]

for some integers \( a > b \) and \( v \) odd. Then the equation

\[
x^2 = (a - b)y^2 + (-1)^{(v-1)/2}bz^2
\]

has a solution \((x, y, z) \in \mathbb{Z}^3 \setminus \{(0, 0, 0)\}\).

For normally regular digraphs we have the following.

**Theorem 8.** Suppose that there exist an NRD\((v, k, \lambda, \mu)\).

- If \( v \) is even then \( \eta = k - \mu + (\mu - \lambda)^2 \) is a square.

- If \( v \equiv 1 \pmod{4} \) then the Diophantine equation \( x^2 - \mu y^2 = \eta z^2 \) has an
interger solution such that \( x, y, \) and \( z \) are not all zero.

- If \( v \equiv 3 \pmod{4} \) then the Diophantine equation \( x^2 + \mu y^2 = \eta z^2 \) has an
integer solution such that \( x, y, \) and \( z \) are not all zero.

**Proof** It follows from equation [2] that the determinant of \( \eta I + \mu J \) is a square. The
eigenvalues of this matrix are \( \eta + \mu = \rho^2 \) with multiplicity 1 and \( \eta \) with
multiplicity \( v - 1 \). For the equality we used equation [5]. Thus the result
follows when \( v \) is even.

For \( v \) odd, the theorem follows from equation [2] and the above lemma. \(\square\)

### 3 Complementary graphs and the parameters

The complement of a graph with adjacency matrix \( A \) is the graph with ad-
jacency matrix \( J - I - A \). The following theorem is proved by an easy computation.
Theorem 9. Let $A$ the adjacency matrix of a normally regular digraph with parameters $(v, k, \lambda, \mu)$. Then $J - I - A$ is the adjacency matrix of a normally regular digraph with parameters

$$(\overline{v}, \overline{k}, \overline{\lambda}, \overline{\mu}) = (v, v - k - 1, v - 2k + \lambda - 1, v - 2k + 2\lambda - \mu).$$

Note that $\overline{\eta} = \overline{k} - \overline{\mu} + (\overline{\mu} - \overline{\lambda})^2 = \eta$.

Two important cases, $\mu = k$ and $\mu = 0$, are considered in the next section. They are complementary.

Corollary 10. A normally regular digraph satisfies $\mu = k$ if and only if the complementary normally regular digraph satisfies $\mu = 0$.

Proof. If $\mu = 0$ then it follows from equation 6 that $2\lambda = k - 1$. And then $\overline{\mu} = v - 2k + 2\lambda - \mu = v - k - 1 = \overline{k}$.

If $\mu = k$ then it follows from equation 6 that $v = 3k - 2\lambda$. And then $\overline{\mu} = v - 2k + 2\lambda - \mu = 0$. \hfill \Box

We will now consider upper and lower bounds on the parameters $\mu$, $\lambda$ and $\lambda_2$. There exists normally regular digraphs for which $\lambda_2 = 2\lambda - \mu < 0$. But in that case there can not be any undirected edges and so the digraph is asymmetric. Note that Theorem 9 is still valid in this case.

Lemma 11. The parameters of an asymmetric normally regular digraph with $k \geq 1$ satisfy the following restriction:

$$k \geq 2\lambda + 1.$$

Proof. The number of edges in the subgraph spanned by the set $x^+$ of out-neighbours of a vertex $x$ is $k\lambda \leq \binom{k}{2}$. Thus $2\lambda \leq k - 1$. \hfill \Box

If a normally regular digraph is a tournament then $\mu$ can be chosen arbitrarily and if it is a complete undirected graph $K_v$ then $\mu$ and $\lambda$ can be chosen arbitrarily so that $\lambda_2 = 2\lambda - \mu = v - 2$. In all other cases we have $0 \leq \mu \leq k$.

Proposition 12. Suppose there exists an NRD$(v, k, \lambda, \mu)$ which is not a tournament or a complete graph. Then

$$0 \leq \mu \leq k,$$

and

$$\lambda_2 = 2\lambda - \mu \leq k - 1,$$

with equality if and only if $\mu = 0$. 11
Proof. Suppose that $\mu < 0$. Then the digraph does not have any pair of non-adjacent vertices. Since it is not a complete graph, $v - 2k - 1 > -k$. As the digraph is not a tournament, there exist undirected edges and $2\lambda - \mu = \lambda_2 \leq k - 1$. From equation 6 we have

$$k^2 - k = 2k\lambda + (v - 2k - 1)\mu < 2k\lambda - k\mu \leq k(k - 1),$$

a contradiction. Thus $\mu \geq 0$.

If $\mu = 0$ then by equation 6 $2\lambda = k - 1$ and so $\lambda_2 = k - 1$. If the digraph has an undirected edge then clearly $\lambda_2 \leq k - 1$. If $x \leftrightarrow y$ and $x$ and $y$ have $\lambda_2 = k - 1$ common out-neighbours then in the complementary graph they have $v - k - 1 = \overline{k}$ common out-neighbours. Thus $\overline{\mu} = \overline{k}$ and by Corollary 10 $\mu = 0$. So suppose that the digraph is asymmetric. Then by Lemma 11 $2\lambda \leq k - 1$ and so $2\lambda - \mu \leq k - 1$, with equality only if $\mu = 0$.

Suppose now that $\mu > k$. Let $(v, \overline{k}, \overline{\lambda}, \overline{\mu})$ be the parameters of the complementary normally regular digraph. Then by Theorem 9

$$v - 2\overline{k} + 2\overline{\lambda} - \overline{\mu} = \mu > k = v - \overline{k} - 1,$$

and so $2\overline{\lambda} - \overline{\mu} > \overline{k} - 1$, a contradiction. \qed

4 \hspace{1em} \mu = 0 \text{ or } \mu = k

By Theorem 10 normally regular digraphs with $\mu = 0$ and $\mu = k$ are complements of each other. We will therefore characterize normally regular digraphs with $\mu = 0$ and then get the case $\mu = k$ as a corollary.

4.1 \hspace{1em} \mu = 0

We will first characterize asymmetric normally regular digraphs with $\mu = 0$ and then generalize to digraphs with undirected edges.

A normally regular digraph with $\mu = 0$ need not be connected. However, each connected component will be a normally regular digraph with the same value of $k$ and $\lambda$. Thus we will only consider normally regular digraphs whose underlying undirected graph is connected. As each vertex has equal in- and out-degree this implies that the digraph is strongly connected. Thus there is a directed path from any vertex to any other vertex. A normally regular digraph with $\mu = 0$ may be a doubly regular tournament. Another possibility is that $k = 1$ and the digraph is a directed cycle.
Let $T$ be a tournament with adjacency $A$. Then $D(T)$ denotes the digraph with adjacency matrix

$$
\begin{pmatrix}
0 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
0 & 1 & & A & A^t & & & \\
& & & & & & & \\
0 & 1 & & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 \\
1 & 0 & & & & & & & & \\
& & & & & & & & & & \\
& & & & & & & & & & A^t \\
1 & 0 & & & & & & & &
\end{pmatrix}
$$

Thus if $T$ is a tournament with only one vertex then $D(T)$ is a directed cycle of length 4. In this section we consider a tournament with one vertex to be doubly regular.

**Theorem 13.** A connected digraph is an asymmetric normally regular digraph with $\mu = 0$ if and only if either

1. it is a directed cycle of length at least 5

2. it is a doubly regular tournament or

3. it is isomorphic to $D(T)$ for some doubly regular tournament $T$.

**Proof.** Suppose that $G$ is a connected asymmetric normally regular digraph with $\mu = 0$, $k \geq 2$ and that $G$ is not a tournament.

As $\mu = 0$ we get from equation 6 that $\lambda = \frac{k-1}{2}$. Let $x$ be a vertex of $G$. Then every vertex in $x^+$ has out-degree $\lambda$ in this subgraph and thus in-degree at most $k - 1 - \lambda = \lambda$. It follows that $x^+$ is a regular tournament. Similarly, $x^-$ is a regular tournament.

Since $G$ is strongly connected and it is not a tournament, there exist a vertex $y \in G$ so that $x$ and $y$ are non-adjacent and there is a path from $x$ to $y$ in $G$. We may choose $y$ so that the (directed) distance from $x$ to $y$ is minimal, i.e. $y$ is dominated by a vertex in $x^+$ or in $x^-$. Since $x$ and $y$ are non-adjacent and $\mu = 0$, $y$ is not dominated by any vertex in $x^-$, and similarly $y$ does not dominate any vertex in $x^+$. Thus $y$ is dominated by a vertex, say $v$, in $x^+$. Suppose there is a vertex $w$ in $x^+$ that does not dominate $y$. Since $x^+$ is a regular tournament it is strongly connected, so there is a directed path from $v$ to $w$ in $x^+$. On this path there are vertices $u$ and $u'$ so that $u \to u'$,
$u \to y$ but $u'$ does not dominate $y$. This is a contradiction to $\mu = 0$. Thus every vertex in $x^+$ dominates $y$. If another vertex $y'$, non-adjacent to $x$ was dominated by a vertex in $x^+$, it would be dominated by every vertex in $x^+$ and so $y$ and $y'$ have $k$ common in-neighbours, a contradiction. Thus every vertex in $x^+$ dominates $\lambda$ vertices in $x^-$. 

Now a vertex in $x^-$ dominated by a vertex in $x^+$ (which dominates $y$) must be adjacent to $y$, as $\mu = 0$. As above, $y$ then dominates every vertex of $x^-$, and every vertex in $x^-$ is dominated by $\lambda$ vertices in $x^+$. Also every vertex in $x^-$ dominates exactly $\lambda$ vertices in $x^+$. Thus $V(G) = \{x, y\} \cup x^+ \cup x^-$. Furthermore there is an enumeration of vertices $x^+ = \{v_1, \ldots, v_n\}$ and $x^- = \{v'_1, \ldots, v'_n\}$ such that $v'_i$ is the unique vertex non-adjacent to $v_i$ and vice versa.

If $v_i \to v_j$ then, since no vertex dominates both $v_j$ and $v'_j$, $v'_j \to v_i$. Similarly $v_j \to v'_i$ and $v'_i \to v'_j$. Thus the mapping $v_i \mapsto v'_i$ is an isomorphism.

We also see that $v'_i$ is a common out-neighbour of $v_i$ and $v_j$ if and only if $v'_i$ is a common in-neighbour of $v_i$ and $v_j$. Thus the number of vertices in $x^+$ dominating $v_i$ and $v_j$ plus the number of vertices in $x^+$ dominated by $v_i$ and $v_j$ is $\lambda - 1$. But since $x^+$ is a regular tournament it is normal (by Lemma 3) and thus these two numbers are both equal to $\frac{\lambda - 1}{2}$ and so $\lambda$ is odd, and $x^+$ is a doubly regular tournament, $\text{NRD}(k; \lambda, \frac{\lambda - 1}{2}, \cdot)$.

If on the other hand $G$ is a doubly regular tournament with degree $k = 2\lambda + 1$ and with vertex-set $\{x_1, \ldots, x_n\}$, $n = 2k + 1$, then we may construct a graph with vertex-set $\{v_0, \ldots, v_n, v'_0, \ldots, v'_n\}$ and edges

$$v_0 \to v_i \to v'_i \to v_0 \to v_i, \text{ for } 1 \leq i \leq n$$

and

$$v_i \to v_j \to v'_i \to v'_j \to v_i \text{ if } x_i \to x_j \text{ in } G, \text{ for } 1 \leq i, j \leq n.$$ 

It is easy to verify that this new graph is an $\text{NRD}(2n + 2, n, k, 0)$.

The smallest non-trivial example of the type of normally regular digraphs mentioned as possibility 3 is a Cayley graph of the quaternion group of order 8 (see Example[1]). In [18] and [24], it is investigated when normally regular digraphs of this type are Cayley graphs or vertex transitive.

We will now characterize normally regular digraphs with $\mu = 0$ and with undirected edges. We need a definition to describe the digraphs. Let $G$ be a digraph with vertices $x_1, \ldots, x_n$. Then we denote by $K_s(G)$ the digraph with vertex set partitioned in sets $V_1, \ldots, V_n$ of size $s$ where each $V_i$ induce an
complete undirected graph and furthermore for \( y \in V_i \) and \( z \in V_j \), \( y \rightarrow z \) if and only if \( x_i \rightarrow x_j \) in \( G \). If \( B \) is an adjacency matrix of \( G \) then an adjacency matrix of \( K_s(G) \) can be expressed using Kronecker products of matrices (see Hall [11]) as follows \( B \otimes J_s + I_n \otimes (J_s - I_s) = (B + I) \otimes J_s - I_{ns} \).

**Theorem 14.** Let \( \Gamma \) be a connected normally regular digraph with parameters \( (v, k, \lambda, 0) \), i.e., \( \mu = 0 \). Then for some number \( s \) there is an asymmetric normally regular digraph \( \Gamma' \) with parameters \( (\frac{v}{s}, \frac{k}{s} - 1 + \frac{\lambda}{s} - 1, 0) \) so that \( \Gamma \) is isomorphic to \( K_s(\Gamma') \).

Conversely, if \( \Gamma' \) is an asymmetric normally regular digraph with parameters \( (v, k, \lambda, 0) \) then \( K_s(\Gamma') \) is a normally regular digraph with parameters \( (sv, sk + s - 1 + s, s\lambda + s - 1, 0) \).

**Proof.** Consider a connected normally regular digraph with \( \mu = 0 \). We have that \( k = 2\lambda + 1 \). Then the number of common out-neighbours of \( x \) and \( y \), where \( x \leftrightarrow y \), is \( 2\lambda - \mu = k - 1 \).

Thus if \( x \leftrightarrow y \) then \( x \) and \( y \) have exactly the same set of out-neighbours (and the same set of in-neighbours) other than \( y \) and \( x \). In particular, if \( x \leftrightarrow y \leftrightarrow z \) then \( x \leftrightarrow z \).

It follows that the vertex set is partitioned in sets \( V_1, \ldots, V_m \), so that each \( V_i \) spans a complete subgraph and there are no undirected edges joining \( V_i \) and \( V_j \) for \( i \neq j \). If \( x \rightarrow y \) for some \( x \in V_i \) and \( y \in V_j \) then \( x \rightarrow y \) for every \( x \in V_i \) and \( y \in V_j \).

Choose \( i \) so that \( |V_i| \geq |V_j| \), for all \( j \) and let \( s = |V_i| \). Let \( V_i^+ \) denote the set of out-neighbours outside \( V_i \) of vertices in \( V_i \). Then \( V_i^+ = V_{i_1} \cup \ldots \cup V_{i_{\ell}} \) for some \( i_1, \ldots, i_{\ell} \) and \( V_i^+ \) has size \( k - (s - 1) = 2\lambda + 2 - s \). In the subgraph spanned by \( V_i^+ \) every vertex has out-degree \( \lambda \). The average in-degree is also \( \lambda \). Thus the average number of undirected edges incident with a vertex is at least \( 2\lambda - (2\lambda + 2 - s - 1) = s - 1 \). By the maximality of \( |V_i| = s \), no vertex is incident with more than \( s - 1 \) undirected edges and so \( |V_{i_1}| = \ldots = |V_{i_s}| = s \). Since the graph is connected, repeated use of this argument shows that \( |V_1| = \ldots = |V_m| \). Consider a graph \( \Gamma' \) with vertices \( x_1, \ldots, x_m \) and edges \( x_i \rightarrow x_j \) if \( V_i \rightarrow V_j \). Then \( \Gamma' \) is a normally regular digraph with parameters \( (\frac{v}{s}, \frac{k - s + 1}{s}, \frac{\lambda - s + 1}{s}, 0) \), and \( \Gamma \) is isomorphic to \( K_s(\Gamma') \).
Example 3. If $\Gamma'$ in this proof is a normally regular digraph with parameters 
$(8t+8,4t+3,2t+1,0)$ then the parameters of $\Gamma$ are $(8t+8)s,(4t+4)s-1,(2t+2)s-1,0)$. Thus a normally regular digraph with parameters $(16,7,3,0)$ may appear with $(s,t) = (2,0)$ or with $(s,t) = (1,1)$ and so $s$ can not be determined from the parameters.

4.2 $\mu = k$

Theorem 15. A digraph $G$ is an asymmetric normally regular digraph with 
$\mu = k$ if and only if there is a number $s$ so that $G$ is obtained from a doubly-
regular tournament by replacing each vertex $x$ by a set $V_x$ of $s$ new vertices 
such that if $x \rightarrow y$ in the tournament then $u \rightarrow w$ for every $u \in V_x$ and 
$w \in V_y$. Then $s = k - 2\lambda = v - 2k$.

In other words a graph is an asymmetric normally regular digraph with 
$\mu = k$ if and only if it has an adjacency matrix which is the Kronecker 
product of an adjacency matrix of a doubly regular tournament and $J_s$.

Proof. If $G$ is an asymmetric normally regular digraph with $\mu = k$ then the 
complement $\overline{G}$ of $G$ is a connected normally regular digraph with $\mu = 0$ 
and with no pair of non-adjacent vertices. Then $\overline{G}$ is constructed as in 
Theorem 14 from an asymmetric normally regular digraph with $\mu = 0$ and 
with no pair of non-adjacent vertices. By Theorem 13 this is a doubly regular 
tournament. $\square$

5 Eigenvalues

If $A$ is the adjacency matrix of a normally regular digraph then we have the 
spectral decomposition for normal matrices

$$A = \sum_{\theta} \theta E_{\theta},$$

where the sum is over the eigenvalues of $A$, and $E_{\theta}$ is the matrix of the 
orthogonal projection on the corresponding eigenspace. As $A$ is real, the 
adjoint matrix (i.e., the complex conjugate of the transposed matrix) is $A^t$ 
and since orthogonal projections are self-adjoint we get

$$A^t = \sum_{\theta} \overline{\theta} E_{\theta}.$$
In particular, \( x \in \mathbb{C}^v \) is an eigenvector of \( A \) with eigenvalue \( \theta \) if and only \( x \) is an eigenvector of \( A^t \) with eigenvalue \( \bar{\theta} \).

In general it is not possible to compute the eigenvalues of a normally regular digraph from its parameters. We only know that the degree \( k \) is an eigenvalue and (if the graph is connected then) it has multiplicity 1. We now show that all other eigenvalues lie on a circle in the complex plane with centre \( \lambda - \mu \) and radius \( \sqrt{\eta} \).

**Theorem 16.** Suppose that \( \theta \neq k \) is an eigenvalue of an NRD\((v, k, \lambda, \mu)\).

Then
\[
|\theta - (\lambda - \mu)| = \sqrt{\eta}.
\]

**Proof.** Let \( A \) be the adjacency matrix of an NRD\((v, k, \lambda, \mu)\). Let \( x \in \mathbb{C}^v \) be an eigenvector for \( A \) with eigenvalue \( \theta \). Then \( x \) is eigenvector for \( A^t \) with eigenvalue \( \bar{\theta} \).

Thus \( (\theta + \mu - \lambda)(\bar{\theta} + \mu - \lambda) \) is an eigenvalue of \( (A + (\mu - \lambda)I)(A + (\mu - \lambda)I)^t = \eta I + \mu J \).

If \( \theta = k \) then all entries of \( x \) are equal and
\[
(k + \mu - \lambda)^2 = \eta + \mu v
\]
this is in fact equation \( 5 \).

If \( \theta \neq k \) then
\[
(\theta + \mu - \lambda)(\bar{\theta} + \mu - \lambda) = \eta
\]
or
\[
|\theta - (\lambda - \mu)| = \sqrt{\eta}.
\]

We now show that equation \( 7 \) characterizes normally regular digraphs. This theorem generalizes the well-known result that a connected regular undirected graph with exactly three eigenvalues is strongly regular, see \cite{7}.

**Theorem 17.** Suppose that \( G \) is a connected \( k \)-regular directed graph with a normal adjacency matrix \( A \). If there exist real numbers \( a \) and \( b \) so that every eigenvalue \( \theta \neq k \) satisfies \( |\theta - a| = b \) then \( G \) is a normally regular digraph with
\[
\lambda = a + \frac{(k-a)^2-b^2}{v} \quad \text{and} \quad \mu = \frac{(k-a)^2-b^2}{v} ,
\]
where \( v \) is the number of vertices, or else \( G \) is a strongly regular graph.
Proof. We can write $A = \sum_{i=1}^{m} \theta_i E_i$ where $\theta_1, \ldots, \theta_m$ are the eigenvalues of $A$ and $E_1, \ldots, E_m$ are the orthogonal projections on the corresponding eigenspaces. We may assume that $\theta_1 = k$ so that $E_1 = \frac{1}{v} J_v$. Then $A - aI = \sum_{i=1}^{m} (\theta_i - a) E_i$, $A^t - aI = \sum_{i=1}^{m} (\theta_i - a) E_i$ and so $(A - aI)(A^t - aI) = \sum_{i=1}^{m} (\theta_i - a) (\theta_i - a) E_i = (k-a)^2 E_1 + b^2 \sum_{i=2}^{m} E_i = (k-a)^2 I + b^2 (I - \frac{1}{v} J)$. This equation is equivalent to $AA^t = (b^2 - a^2 + \frac{(k-a)^2 - b^2}{v}) I + (a + \frac{(k-a)^2 - b^2}{v})(A + A^t) + \frac{(k-a)^2 - b^2}{v} (J - I - A - A^t)$. If $G$ is not undirected and not a tournament then clearly, $\lambda$ and $\mu$ are integers and then the theorem follows from Proposition 1. If $G$ is a tournament then $\lambda$ is an integer, there are infinitely many choices for $(a, b)$, and $\mu$ is arbitrary. If $G$ is undirected then it is strongly regular.

**Proposition 18.** Suppose that $A$ is the adjacency matrix of a connected NRD$(v, k, \lambda, \mu)$. Then

1. $k$ is an eigenvalue of multiplicity 1.
2. If $\theta$ is an eigenvalue of $A$ then $\overline{\theta}$ is an eigenvalue of the same multiplicity.
3. The spectrum of $A$ is completely determined by the spectrum of $A + A^t$ and the parameters $(v, k, \lambda, \mu)$.
4. If the digraph is not an undirected strongly regular graph then $A$ has at least one non-real eigenvalue.

Proof. 1. This is true for any connected $k$-regular digraph, see [7].
2. follows from the introduction to this section.
3. Suppose that $\tau$ is an eigenvalue of $A + A^t$ of multiplicity $m$. Since $A + A^t$ is symmetric, $\tau \in \mathbb{R}$. If $|\frac{\tau}{2} - (\lambda - \mu)| = \sqrt{\eta}$ then $\frac{\tau}{2}$ is an eigenvalue of $A$ of multiplicity $m$. Otherwise there are exactly two numbers $\theta$ and $\overline{\theta}$ on the circle with centre $\lambda - \mu$ and radius $\sqrt{\eta}$ so that $\theta + \overline{\theta} = \tau$. Then $\theta$ and $\overline{\theta}$ are eigenvalues of $A$ of multiplicity $\frac{m}{2}$.
4. If all eigenvalues of $A$ are real then since $A$ is normal it follows that $A$ is selvadjoint and thus symmetric. But $A$ has directed edges.

Remark. If $A$ is the adjacency matrix of a digraph $\Gamma$ without undirected edges then $A + A^t$ is the adjacency matrix of the underlying undirected graph of $\Gamma$, i.e., the graph obtained by replacing each directed edge by an undirected edge.
This seems to be all that we can say in general about the the spectrum of a normally regular digraph. But for \( \mu = 0 \) we can at least describe the spectrum for the most important class of normally regular digraphs.

**Theorem 19.** Suppose that \( T \) is a doubly regular tournament so that \( G = \mathcal{D}(T) \) is an NRD\((v, k, \lambda, 0)\). Then the eigenvalues of \( \mathcal{K}_s(G) \) are

\[
sk + s - 1, \quad -1, \quad s - 1 + is\sqrt{k}, \quad s - 1 - is\sqrt{k}
\]

with multiplicities

\[
1, \quad sv - k - 2, \quad \lambda + 1, \quad \lambda + 1.
\]

**Proof.** First we consider the eigenvalues of the adjacency matrix \( A \) of \( G \). Then \( A + A^t \) is the adjacency matrix of an imprimitive strongly regular graph. This graph has eigenvalues \( 2k, 0 \) and \( -2 \) with multiplicities \( 1, k + 1 \) and \( k \). Thus if \( \theta \neq k \) is an eigenvalue of \( A \) then \( \theta + \overline{\theta} \in \{0, -2\} \). By equation \( 7 \), \( |\theta - \lambda| = \lambda + 1 \), as \( \eta = k - \mu + (\mu - \lambda)^2 = 2\lambda + 1 + \lambda^2 \). If \( \theta + \overline{\theta} = -2 \) then \( \theta = -1 \). The multiplicity is \( k \). If \( \theta + \overline{\theta} = 0 \) then \( \theta = \pm i\sqrt{k} \). These two eigenvalues have multiplicity \( \frac{1}{2}(k + 1) = \lambda + 1 \).

Thus \( A + I \) has eigenvalues \( k + 1, 0 \) and \( 1 \pm i\sqrt{k} \). The matrix \( (A + I) \otimes J_s - I_{vs} \) is an adjacency matrix of \( \mathcal{K}_s(G) \). We first compute the eigenvalues of \( (A + I) \otimes J_s \) and then subtract \( 1 \).

For each eigenvector \( x \in \mathbb{C}^v \) of \( A + I \) with eigenvalue \( \theta \) we can replace each entry \( x_i \) with \( s \) entries equal to \( x_i \) to get an eigenvector in \( \mathbb{C}^{vs} \) of \( (A + I) \otimes J_s \) with eigenvalue \( s\theta \). Furthermore we can get \( v(s - 1) \) orthogonal eigenvectors with eigenvalue \( 0 \), by taking one of the blocks to be orthogonal to \( (1, \ldots, 1)^t \in \mathbb{C}^s \) and all other entries \( 0 \).

Note that the two non-isomorphic NRD\((16, 7, 3, 0)\) mentioned in Example 3 have different spectra.

### 6 Relation to association schemes

An asymmetric normally regular digraph may have the additional property that \( A^2 \) (where \( A \) is the adjacency matrix) can be expressed as linear combination of \( A, A^t, I \) and \( J \). In that case the digraph is related to an association scheme.
Definition 2. Let $X$ be a finite set and let $\{R_0, R_1, \ldots, R_d\}$ be a partition of $X \times X$. Then $\mathcal{X} = (X, \{R_0, R_1, \ldots, R_d\})$ is an association scheme with $d$ classes if the following conditions are satisfied

- $R_0 = \{(x, x) \mid x \in X\}$,
- for each $i \in \{0, \ldots, d\}$ there exists $i' \in \{0, \ldots, d\}$ such that $R_i' = \{(x, y) \mid (y, x) \in R_i\}$,
- for each triple $(i, j, k)$, $i, j, k \in \{0, \ldots, d\}$ there exist a number $p_{ij}^k$ such that for all $x, y \in X$ with $(x, y) \in R_k$ there are exactly $p_{ij}^k$ elements $z \in X$ so that $(x, z) \in R_i$ and $(z, y) \in R_j$.

If $i = i'$ for all $i$ then the association scheme is called symmetric, otherwise it is non-symmetric.

The relation $R_i$, $i = 1, \ldots, d$ can be considered as an undirected graph if $i = i'$ and as a directed graph if $i \neq i'$.

It is well-known that an undirected graph is strongly regular if and only if it is a relation of a symmetric association scheme with two classes. And a directed graph is a doubly regular tournament if and only if it is a relation of a non-symmetric association scheme with two classes.

For a general introduction to association schemes we refer to Bannai and Ito [1]. Goldbach and Classen [9] have studied non-symmetric association schemes with three classes and in [10] they describe the structure of non-symmetric association schemes with three classes that are imprimitive, i.e. at least one the graphs $R_1, R_2, R_3$ is disconnected. For tables of feasible parameter sets see [20].

Proposition 20. 1. If $(X, \{R_0, R_1, R_2, R_3\})$ is an association scheme with $1' = 2$ then $R_1$ is an asymmetric normally regular digraph.

2. If $(X, \{R_0, R_1, R_2, R_3, R_4\})$ is an association scheme with $1' = 2$ and $3' = 4$ then $R_1$ is an asymmetric normally regular digraph.

Proof. We prove case 2. Case 1 is similar. The graph $R_1$ is regular with degree $p_{12}^0$. Suppose that $x$ and $y$ are adjacent in $R_1$. We may assume that $(x, y) \in R_1$. Then the number of common out-neighbours of $x$ and $y$ is $p_{12}^0$. Suppose now that $x$ and $y$ are non-adjacent. We may assume that
$(x, y) \in R_3$, since otherwise $(x, y) \in R_4$ and then $(y, x) \in R_3$. Then the number of common out-neighbours of $x$ and $y$ is $p^3_{12}$. \hfill \Box

It follows from Proposition [18] that the adjacency matrix of a normally regular digraph has at least three distinct eigenvalues. The normally regular digraphs constructed from non-symmetric association schemes with 2, 3 or 4 classes have 3, 4 and 5 distinct eigenvalues, respectively.

We now consider normally regular digraphs where the number of distinct eigenvalues is either 3, 4 or 5, and try to construct association schemes.

In the following proofs it is easier to work with a reformulation of the definition of association schemes in terms of matrices.

**Proposition 21.** Let $R_0, \ldots, R_d$ be relations on a set $X$, with adjacency matrices $A_0, \ldots, A_d$. Let $\mathcal{A}$ be the vector space spanned by $\{A_0, \ldots, A_d\}$. Then $(X, \{R_0, \ldots, R_d\})$ is an association scheme if and only if $A_0 + \ldots + A_d = J$ and

- $I \in \{A_0, \ldots, A_d\}$, (say $I = A_0$),
- $A_i^t \in \{A_0, \ldots, A_d\}$, for all $i$, and
- $\mathcal{A}$ is closed under matrix multiplication.

In fact, $A_i A_j = \sum_k p^k_{ij} A_k$.

**Theorem 22.** Suppose that $\Gamma$ is a connected $\text{NRD}(v, k, \lambda, \mu)$ with exactly three distinct eigenvalues. Then either

- $\Gamma$ is an undirected strongly regular graph or
- $\Gamma$ is a doubly regular tournament and the eigenvalues are $k, -\frac{1}{2} + i\sqrt{\lambda + \frac{3}{4}}$ and $-\frac{1}{2} - i\sqrt{\lambda + \frac{3}{4}}$, with multiplicities 1, $k$ and $k$ respectively.

In both cases we have an association scheme with 2 classes.

**Proof.** Let $k$, $\theta$ and $\tau$ be the eigenvalues and let $E_k$, $E_\theta$ and $E_\tau$ be the orthogonal projections on the eigenspaces. Then

$$I = E_k + E_\theta + E_\tau,$$

$$J = vE_k$$

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\[ A = kE_k + \theta E_{\theta} + \tau E_{\tau}, \]

and the adjoint of \( A \) is

\[ A^t = kE_k + \bar{\theta} E_{\theta} + \bar{\tau} E_{\tau}. \]

It follows that \( I, J, A \) and \( A^t \) are linearly dependent, so there exists rational numbers \( a, b, c, d \) not all 0 so that

\[ aA + bA^t = c(J - I) + dI. \]

Clearly \( d = 0 \). If \( c = 0 \) then \( A = A^t \). Otherwise either \( A + A^t = J - I \) or \( A = A^t = J - I \). Thus \( I, A, A^t \) satisfy the properties required in Proposition 21.

Theorem 23. Suppose that \( \Gamma \) is a connected asymmetric NRD(\( v, k, \lambda, \mu \)) with exactly four distinct eigenvalues.

Then \( \Gamma \) is a relation of a non-symmetric association scheme with three classes.

Proof. Let \( A \) be the adjacency matrix of \( \Gamma \). Let \( A = \text{span}\{I, A, A^t, J - I - A - A^t\} \). We need to show that \( A \) is closed under matrix multiplication. Let \( k, \tau, \theta \) and \( \bar{\theta} \) be the eigenvalues of \( A \). Let \( E_k, E_{\tau}, E_{\theta} \) and \( E_{\bar{\theta}} \), respectively, be the orthogonal projections on the corresponding eigenspaces. Then

\[ I = E_k + E_{\tau} + E_{\theta} + E_{\bar{\theta}}, \]

\[ J = vE_k, \]

\[ A = kE_k + \tau E_{\tau} + \theta E_{\theta} + \bar{\theta} E_{\bar{\theta}}, \]

and

\[ A^t = kE_k + \tau E_{\tau} + \bar{\theta} E_{\theta} + \theta E_{\bar{\theta}}. \]

Thus \( \{E_k, E_{\tau}, E_{\theta}, E_{\bar{\theta}}\} \) is a basis of \( A \). Since these projections are idempotents and the product of distinct projections is 0, \( A \) is closed under multiplication.

Remark. We did not use the fact that \( \Gamma \) is a normally regular digraph in the proof of Theorem 23. In fact we proved that if \( \Gamma \) is a regular connected graph without undirected edges whose adjacency matrix is normal and has exactly four distinct eigenvalues then \( \Gamma \) is a relation of a non-symmetric association scheme with three classes.
If an asymmetric normally regular digraph has exactly five distinct eigenvalues then it may have either three real eigenvalues and one pair of complex conjugate eigenvalues or else it has one real eigenvalues and two pairs complex conjugate eigenvalues. In the latter cases it seems likely that the graph is a relation of a non-symmetric association scheme with four classes. We can only prove the following.

**Proposition 24.** Suppose that $\Gamma$ is a connected asymmetric NRD($v, k, \lambda, \mu$) with exactly five distinct eigenvalues $k, \theta, \theta, \tau, \tau$. Then $\Gamma$ is an orientation of a strongly regular graph.

**Proof.** The underlying undirected graph is regular and has exactly three distinct eigenvalues $2k, \theta + \bar{\theta}$ and $\tau + \bar{\tau}$. Thus it is strongly regular, see [7].

Conversely, it follows from Proposition 18 that if a normally regular digraph is an orientation of a strongly regular graph then the number of distinct eigenvalues is either four or five.

### 7 Group divisible partitions

We start with the definition of two types of partitions of the vertex set.

Suppose that the vertex set of a normally regular digraph is partitioned in sets $V_1, \ldots, V_m$. Then we say that $V_1, \ldots, V_m$ is an equitable partition if there exists constants $c_{ij}, d_{ij}$ for $i, j \in \{1, \ldots, m\}$ so that for every vertex $x \in V_i$, $|x^+ \cap V_j| = c_{ij}$ and for every vertex $y \in V_j$, $|y^- \cap V_i| = d_{ij}$. If $|V_i| = |V_j|$ then $c_{ij} = d_{ij}$. We say that $C = (c_{ij})_{i,j=1,\ldots,m}$ is the quotient matrix of the equitable partition, see [7].

Let $G$ be an asymmetric NRD($v, k, \lambda, \mu$). Then we say that $G$ is group divisible if $G$ is a multipartite tournament, i.e., if $V(G)$ can be partitioned in sets $V_1, \ldots, V_r$ such that there is an edge between $x \in V_i$ and $y \in V_j$ if and only if $i \neq j$.

Since $G$ is regular the sets $V_i$ all have the same size, say $|V_i| = s = v - 2k$, for $i = 1, \ldots, r$. Then $v = rs$ and $k = \frac{1}{2}(r-1)s$. We assume that $s > 1$.

The adjacency matrix of a group divisible normally regular digraph with $\mu \neq \lambda$ is also the incidence matrix of a group divisible design, see [2].
Lemma 25. Let $G$ be an asymmetric NRD$(v, k, \lambda, \mu)$ with a partition $V_1, \ldots, V_r$ of the vertex set so that two vertices are adjacent if and only if they are in different cells.

Then $V_1, \ldots, V_r$ is an equitable partition.

Proof. For $x \in V_i$ let $c_{ij}(x) = |x^+ \cap V_j|$. We need to prove that $c_{ij}(x)$ does not depend on $x$. Let $x \in V_i$ and $y \in V_j$. We count the vertices in $S = \{z \mid x \to z \to y\}$ in two ways. The number of out-neighbours of $x$ outside $V_j$ is $k - c_{ij}(x)$. $\lambda$ of these out-neighbours are common out-neighbours of $x$ and $y$. The remaining $k - c_{ij}(x) - \lambda$ vertices are in $S$. Similarly, $y$ has $k - (|V_i| - c_{ji}(y)) = k - (s - c_{ji}(y))$ in-neighbours outside $V_i$. $k - (s - c_{ji}(y)) - \lambda$ vertices are in $S$. Thus $s = c_{ij}(x) + c_{ji}(y)$, for all $x \in V_i$. \qed

Proposition 26. Let $A$ be the adjacency matrix of a group divisible normally regular digraph. Then

- $A$ has either 4 or 5 distinct eigenvalues.
- If $A$ has 4 distinct eigenvalues then the graph is a relation of a non-symmetric imprimitive association scheme with three classes.
- If $A$ has 5 distinct eigenvalues then $r$ and $s$ are odd.

Proof. Since $A + A^t$ is a strongly regular graph it has three distinct eigenvalues. Then by Theorem 16 and Proposition 18, $A$ has either 4 or 5 eigenvalues.

It follows from Theorem 23 that if $A$ has 4 eigenvalues then the graph is a relation of a non-symmetric association scheme with three classes.

It follows from Proposition 18 that if $A$ has 5 distinct eigenvalues then eigenvalue $k$ has multiplicity 1 and the other 4 eigenvalues have pairwise the same multiplicity. Thus the number of vertices is odd, and so $r$ and $s$ are odd. \qed

I conjecture that if a group divisible normally regular digraph with an odd number of vertices exists then it satisfies $\mu = k$ and then by Theorem 15 it is a relation of a non-symmetric imprimitive association scheme with three classes.

Conjecture 1. Any group divisible normally regular digraph is a relation of a non-symmetric imprimitive association scheme with three classes.
Example 4. The parameters of a group divisible normally regular digraph must satisfy that $v - 2k$ divides $v$. This is satisfied by $(v, k, \lambda, \mu) = (16, 6, 2, 2)$. There are four asymmetric normally regular digraphs with these parameters. Two of these are group divisible and thus are relations of an association scheme.

One of these is a Cayley graph

$$Cay(\mathbb{Z}_4 \times \mathbb{Z}_4, \{(0,3), (1,3), (2,1), (3,0), (3,2), (3,3)\}).$$

The independent sets of vertices in this digraph are the cosets of the subgroup $\{(0,0), (0,2), (2,0), (2,2)\}$.

One of the normally regular digraphs with these parameters that is not group divisible has vertex set $\{a_i, b_i \mid i \in \mathbb{Z}_8\}$ and edges

$$a_i \to a_{i+1}, a_{i+2}, b_i, b_{i+1}, b_{i+4}, b_{i+6}, \quad i \in \mathbb{Z}_8,$$

$$b_i \to b_{i-1}, b_{i-2}, a_{i-2}, a_{i-3}, a_{i-5}, a_{i-7}, \quad i \in \mathbb{Z}_8.$$

Thus group divisibility is not determined by the parameters.

8 Combinatorial results for small $\lambda$

In this section we use combinatorial methods to prove non-existence for certain parameter sets where $\lambda$ is small. If $\lambda$ is so small that $2\lambda - \mu$ is negative then only the asymmetric case need to be considered.

Theorem 27. If there exist an asymmetric normally regular digraph with parameters $(v, k, \lambda, \mu)$ where $2\mu > k + \lambda$ then the graph is group-divisible and $v - 2k$ divides $v$.

Proof. Suppose that $G$ is an NRD$(v, k, \lambda, \mu)$. Let $x$ be a vertex in $G$. Let $y$ and $z$ be vertices in $V(G) - \{x\} - x^+ - x^-$. Then $x^+ \cap y^+$ and $x^+ \cap z^+$ each consist of $\mu$ vertices in the set $x^+$ of $k$ vertices. Thus $|y^+ \cap z^+| \geq 2\mu - k > \lambda$ and so $y$ and $z$ are nonadjacent. It follows that every vertex in $G$ belongs to a unique independent set of $v - 2k$ vertices and so $G$ is group divisible with $\frac{v}{v - 2k}$ groups. \qed

Corollary 28. Suppose that $2\lambda - \mu < 0$, $2\mu > k + \lambda$ and $v - 2k$ does not divide $v$ then an NRD$(v, k, \lambda, \mu)$ does not exist.
**Theorem 29.** If a normally regular digraph with \( \lambda = 0 \), \( \mu \neq k \) and \( \mu \geq 2 \) exists then \( k \geq 2\mu + \frac{1}{2} + \sqrt{2\mu + \frac{1}{4}} \).

From Theorem [15] we know that an \( \text{NRD}(v, k, 0, k) \) is obtained from a directed triangle by replacing each vertex by \( k \) vertices.

**Proof** Suppose that \( G \) is an \( \text{NRD}(v, k, 0, \mu) \) with \( \mu \neq k \). Since \( 2\lambda - \mu < 0 \) any such normally regular digraph is asymmetric. In this proof we use notation \( U_x = V(G) - \{x\} - x^+ - x^- \) for a vertex \( x \) in \( G \). By equation [8] we have

\[
|U_x| = \frac{k(k-1)}{\mu} \tag{8}
\]

By Proposition [12] \( k > \mu \).

**Claim 1:** \( k > 2\mu \).

**Proof** Let \( x \) be any vertex in \( G \). By Theorem [27] \( U_x \) is an independent set if \( \mu < k < 2\mu \).

So suppose that \( k = 2\mu \) and \( k > 2 \). By equation [8] \( |U_x| = 2k - 2 \). Suppose that \( y, z \in U_x \) and \( y \) dominates \( z \). \( x^+ \cap y^+ \) and \( x^+ \cap z^+ \) are disjoint sets (as \( \lambda = 0 \)) of cardinality \( \mu \). Thus their union is \( x^+ \). \( z \) has \( \mu \) in-neighbours in \( x^- \), no in-neighbours in \( x^+ \) and thus \( \mu \) in-neighbours in \( U_x \). Let \( y' \in U_x \) be another vertex dominating \( z \). Then \( y' \) and \( z \) have no common out-neighbours in \( x^+ \), i.e. \( x^+ \cap (y')^+ = x^+ \cap y^+ \). Thus \( y \) and \( y' \) have at least \( \mu + 1 \) common out-neighbours, a contradiction.

Thus \( U_x \) is an independent set.

Let \( z \in x^+ \). Every vertex other than \( x \) dominating \( z \) belong to \( U_x \). As \( U_x \) is independent, no vertex dominates both \( z \) and \( y \in U_x \) and so \( z \) is adjacent to every vertex in \( U_x \) (but to no other vertex in \( x^+ \)). Thus \( z \) is adjacent to \( 2k - 1 - |U_x| \) vertices in \( x^- \). By equation [8] and \( \mu < k \leq 2\mu \), we have \( k > 2k - 1 - |U_x| \geq 1 \), and so there is a vertex \( w \in x^- \) adjacent to \( z \) and a vertex \( u \in x^- \) not adjacent to \( z \). Then \( w \) and \( z \) are adjacent vertices in \( U_u \), a contradiction.

**Claim 2:** \( k > 2\mu + 1 \).

**Proof** Suppose that \( k = 2\mu + 1 \) and let \( x \) be a vertex in \( G \). By equation [8] \( |U_x| = 2k \). Since \( G \) is regular and \( |U_x \cup \{x\}| > |x^+ \cup x^-| \), \( U_x \) cannot be an independent set. Let \( y, z \in U_x \) so that \( y \to z \). \( y \) and \( z \) have no common out-neighbours in \( x^+ \) so there a unique vertex \( w \in x^+ \) which is not dominated by \( y \) or \( z \). The in-neighbours of \( z \) are \( \mu \) vertices in \( x^- \), possibly \( w \), and at least \( \mu \) vertices in \( U_x \). Let \( y' \neq y \) be a vertex in \( U_x \) dominating \( z \). Since \( y' \) and
z have no common out-neighbours in $x^+$ and $y$ and $y'$ have only $\mu$ common out-neighbours, $y'$ dominates $w$. Since $\lambda = 0$, $w$ does not dominate $z$, and so $z$ has $\mu + 1$ in-neighbours in $U_x$. We have now shown that in the graph spanned by $U_x$ any vertex has in-degree either 0 or $\mu + 1$ and, by symmetry, it has out-degree either 0 or $\mu + 1$. Thus the in-neighbours of $z$ in $U_x$ has out-degree $\mu + 1$. Any two in-neighbours of $z$ in $U_x$ have at least $\mu - 1$ common out-neighbours in $(x^+ \cap y^+) \cup \{w\}$. Thus $z$ is their only common out-neighbour in $U_x$. Counting the vertices in $U_x$ we have

$$4\mu + 2 = |U_x| \geq 1 + (\mu + 1) + (\mu + 1)\mu,$$

and so $\mu = 2$.

Now $|U_x| = 10$ and we have at least 7 vertices of in-degree 3 in $U_x$ and, by symmetry, at least 7 vertices of out-degree 3. So there is a vertex with out-degree and in-degree 3. We may assume that $z$ is such a vertex. Then $z$ dominates a vertex which is also dominated by an in-neighbour of $z$. This is a contradiction to $\lambda = 0$. This proves claim 2.

Let $r = k - 2\mu$. Then $r \geq 2$, by Claim 2. By equation \ref{eq:mu}, $\mu$ divides $k(k-1) = (2\mu + r)(2\mu + r - 1) = \mu(4\mu + 4r - 2) + r(r-1)$. Thus $\mu$ divides $r(r-1)$ and so $r^2 - r = s\mu$, for some positive integer $s$. Then $r = \frac{1}{2} + \sqrt{s\mu + \frac{1}{4}}$.

If $s = 1$ then $\mu = r(r-1)$ and $k = 2\mu + r = r(2r-1)$. From equation \ref{eq:mu}, we see that $v = 1 + 2k + \frac{k(k-1)}{\mu} = 2k + 4r^2$ is even and so $\eta$ is a square, by Theorem \ref{thm:eta} But $\eta = k - \mu + \mu^2 = r^2((r - 1)^2 + 1)$ cannot be a square. Thus $s \geq 2$. This proves the theorem.

\[\square\]

9 Normally regular digraphs as quotient graphs

9.1 Subplane partition

Fossorier, Ježek, Nation and Pogel \cite{Fossorier2005} considered partition of a projective plane of order $n$ into subplanes $\pi_1, \ldots, \pi_v$ of order $q$, $v = \frac{n^2 + n + 1}{q^2 + q + 1}$. They say that such a partition is ordinary if for each pair $(i, j)$ either each point of $\pi_i$ is incident with a line of $\pi_j$ or no point of $\pi_i$ is incident with a line of $\pi_j$.

For an ordinary partition of a projective plane they consider the quotient graph with vertices $\pi_1, \ldots, \pi_v$ and an edge $\pi_i \rightarrow \pi_j$ if the points of $\pi_i$ are incident with lines of $\pi_j$. They proved that this quotient graph is what
they called an ordinary graph. This is a normally regular digraph in our terminology.

**Theorem 30** (Fossorier, Ježek, Nation and Pogel [6]). *If a projective plane of order \( n \) has an ordinary partition into projective planes of order \( q \) then the quotient graph is a normally regular digraph with \((v, k, \lambda, \mu) = (\frac{n^2+n+1}{q^2+q+1}, n - q, q^2, q^2 + q + 1)\).*

For a partition into Baer subplanes, i.e., \( n = q^2 \), the quotient graph is a complete undirected graph.

Theorem 33 below describes the special case of this theorem where we consider desarguesian planes. Theorem 31 may also be seen as a special case of Theorem 30 where \( q = 1 \).

### 9.2 Bipartite graphs of diameter 3

Delorme, Jørgensen, Miller and Pineda-Villavicencio [4] considered a similar quotient graph construction. In this paper we considered bipartite \( q + 1 \) regular graphs with diameter 3 and with \( 2(q^2 + q) \) vertices. (The largest possible bipartite \( q + 1 \) regular graph with diameter 3 has \( 2(q^2 + q + 1) \) vertices and it appears only as incidence graph of a projective plane of order \( q \).) In such graphs the vertices are partitioned into cycles of length 4. It is proved that the graph obtained by directing all edges from one bipartition class to the other and then identifying each 4-cycle to a vertex is a normally regular digraph with \((v, k, \lambda, \mu) = (\frac{q^2+q}{2}, q - 1, 0, 2)\).

This was our original motivation for studying normally regular digraphs.

### 10 Constructions

In this section we give a number constructions of families of normally regular digraphs. Most of these constructions use Cayley graphs of abelian groups.

#### 10.1 Asymmetric Cayley graph constructions

The first construction uses a partition of a projective plane into triangles. If a triangle is considered to be a “subplane” of order 1 then this is a special case of the construction in Theorem 30.
Theorem 31. Let $k$ be a multiple of 3 such that $k + 1$ is a prime power. Then there exists $S \subset \mathbb{Z}_v$, $v = \frac{k^2 + 3k + 3}{3}$ so that $\text{Cay}(\mathbb{Z}_v, S)$ is an asymmetric normally regular digraph with parameters $(v, k, 1, 3)$.

Proof When $\lambda = 1$ and $\mu = 3$, $\eta = k + 1$, which is assumed to be a prime power. By equation (5), $v = \eta^2 + \eta + 1$.

By Singer’s theorem [32] there exist a cyclic planar difference set of order $\eta$, i.e. a subset $D$ of $\mathbb{Z}_{3v}$ with $|D| = \eta + 1$ such that each non-zero element of $\mathbb{Z}_{3v}$ is a difference of exactly one ordered pair of elements in $D$. In particular there is a unique pair of difference $v$. By adding a constant to $D$ if necessary, we may assume that $v, 2v \in D$. Let $D' = D \setminus \{v, 2v\}$, and let $S \subseteq \mathbb{Z}_v$ be the numbers congruent to numbers in $D'$ modulo $v$.

As $v$ is not a difference in $D'$, $|S| = \eta - 1 = k$.

Suppose that $x$ and $-x$ are both in $S$. Then for some $a, b \in \{0, 1, 2\}$, $x + av, -x + bv \in D'$. Choose $i, j \in \{1, 2\}$ so that $a + b \equiv i + j (\text{mod } 3)$. Then we have two equal differences in $D$

$$(x + av) - iv = jv - (-x + bv)$$

a contradiction. Thus $S \cap -S = \emptyset$.

If $x \in \mathbb{Z}_v$ is congruent (mod $v$) to a difference of elements in $D$, one of which is $v$ or $2v$, then either $x \equiv a - iv$ or $x \equiv iv - a$ for $a \in D$ and $i \in \{1, 2\}$, i.e. $x$ or $-x \in S$. Conversely if $x \in S$ or $-x \in S$ then $x$ is in exactly to ways congruent (mod $v$) to a difference of two elements in $D$ one of which is $v$ or $2v$.

Let $x \in \mathbb{Z}_v \setminus (S \cup -S \cup \{0\})$. Then each of $x, x + v$ and $x + 2v$ can be written in exactly one way as a difference of elements in $D$, in fact in $D'$. Thus $x$ can be written in exactly three ways as a difference of elements in $S$.

If $x \in S \cup -S$ then only one of the three pairs of elements in $D$ whose difference is congruent to $x$ (mod $v$) is in $D'$. Thus $x$ can be written in exactly one way as a difference of elements in $S$.

Hence $\text{Cay}(\mathbb{Z}_v, S)$ is an NRD($v, k, 1, 3$) \qed

In the next theorem we construct a family of Cayley graphs of abelian but not necessarily cyclic groups. It is well-known that this digraph is one of the classes of a (so-called cyclotomic) association scheme with four classes.
Theorem 32. Suppose that $v$ is a prime power, $v \equiv 5 \mod 8$. Let $D$ denote the following subset of $GF[v]$:

$$D = \{x^4 | x \neq 0\}$$

Then the Cayley graph of the additive group of $GF[v]$ generated by $D$ is a normally regular digraph with $v = 4k + 1 = 8(\mu + \lambda) + 5$.

Proof. As $v \equiv 5 \mod 8$, the set $D$ has cardinality $k = \frac{v-1}{4}$ and $-1 \notin D$. Thus $D \cup -D$ is the set of squares in $GF[v]$. This means that the cosets of the subgroup (of the multiplicative group) $D$ are $D$, $-D$, $R$, and $-R$ for some set $R$. Let $D = \{1, q_2, \ldots, q_k\}$. Then $qq_i - q$, $2 \leq i \leq k$, $q \in D$ is the set of differences, we want to consider. For a fixed $i$, every element in the coset to which $q_i - 1$ belongs appears exactly once as a difference $qq_i - q$, $q \in D$. This means that if among the differences $q_2 - 1, \ldots, q_k - 1$, the number of elements in $D, -D, R$ and $-R$ are $\lambda_1, \lambda_2, \mu_1$ and $\mu_2$, respectively, then among all differences of distinct element of $D$ an element appears $\lambda_1, \lambda_2, \mu_1$, or $\mu_2$ times according to whether it belongs to $D, -D, R$, or $-R$. Since for every $x, x$ and $-x$ appears as a difference the same number of times, $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2$.

The only known infinite family of primitive non-symmetric association schemes with three classes is a family constructed by Liebler and Mena [26]. For every $s = 2^n$, they constructed a so-called distance regular digraph of girth 4 and degree $s(2s^2 - 1)$, as a Cayley digraph of $Z_4 \times \ldots \times Z_4$. Their graph is in fact an $NRD(4s^4, s(2s^2 - 1), 2s(s - 1), s(s - 1))$.

Some of the normally regular digraphs constructed in the next two subsections are also asymmetric.

10.2 Construction from desarguesian planes

We will now consider the subplane partition described in Section 9.1 for desarguesian projective planes.

Theorem 33. Let $q$ be a prime power and let $r \geq 2$ be an integer not divisible by 3. Let $v = \frac{q^{2r} + q^r + 1}{q^2 + q + 1}$. Then there exists a set $S \subset Z_v$ so that $Cay(Z_v, S)$ is a normally regular digraph with parameters $(v, q^r - q, q^2, q^2 + q + 1)$.
Proof. Let $\text{GF}[q^{3r}]$ be the field with $q^{3r}$ elements and with primitive element $\alpha$. Then $\text{GF}[q^r]$ and $\text{GF}[q^3]$ are subfields and their intersection is $\text{GF}[q]$ as $3$ does not divide $r$.

Let $\beta = \alpha^{(q^{3r} - 1)/(q^3 - 1)}$. Then $\beta$ is a primitive element of $\text{GF}[q^3]$ and $\beta \notin \text{GF}[q^r]$. Thus when $\text{GF}[q^{3r}]$ is considered as a $3$ dimensional vector space over $\text{GF}[q^r]$ then vectors $1$ and $\beta$ span a $2$ dimensional subspace $U$. Let $u = \frac{q^{2r+1}}{q^3 - 1} = q^{2r} + q + 1$ and let $D = \{i \in \mathbb{Z}_u \mid \alpha^i \in U\}$. Then by Singer’s theorem [32], $D$ is a planar difference set in $\mathbb{Z}_u$.

Similarly, we may consider $\text{GF}[q^3]$ as a $3$ dimensional vector space over $\text{GF}[q]$. In this space the vectors $1$ and $\beta$ span a $2$ dimensional subspace $W$. Let $w = \frac{q^3 + 1}{q - 1} = q^2 + q + 1$ and let $T = \{i \in \mathbb{Z}_w \mid \beta^i \in W\}$. Again $T$ is a planar difference set in $\mathbb{Z}_w$. As $\frac{q^3 - 1}{q - 1}$ and $w$ are coprime, multiplication by $\frac{q^3 - 1}{q - 1}$ is an automorphism of $\mathbb{Z}_w$ and so $T' = \{\frac{q^3 - 1}{q - 1}i \mid i \in T\}$ is a planar difference set. Then the set $T'' = \{\frac{q^3 - 1}{q - 1}vi \mid i \in T\}$ is a difference set in subgroup $\langle v \rangle$ of $\mathbb{Z}_u$.

This set satisfies $T'' = \{i \in \mathbb{Z}_u \mid \alpha^i \in W\} \subset D$, as $\beta = \alpha^{\frac{q^3 - 1}{q - 1}}v$.

If for some $x, y \in D$ the difference $x - y$ is a non-zero multiple of $v$ then $x, y \in T''$. Let $D' = D \setminus T''$. Let $S \subset \mathbb{Z}_u$ be the numbers congruent to numbers in $D'$ modulo $v$. As multiples of $v$ are not differences in $D'$, $|S| = q^2 - q$.

Let $g \in \mathbb{Z}_v, g \neq 0$. Then $g$ is congruent modulo $v$ to $q^2 + q + 1$ elements in $\mathbb{Z}_u$, each of which can uniquely be written as difference $x - y$ where $x, y \in D$. If $g \in S$ then exactly $q + 1$ of these differences satisfy $y \in T''$.

Thus the number of pairs $x, y \in S$ such that a nonzero element $g \in \mathbb{Z}_v$ can be written as $g = x - y$ is $\mu = q^2 + q + 1$ if $g \notin S \cup -S$, $\lambda = q^2$ if $g$ is in exactly one of the sets $S, -S$, and $2\lambda - \mu = q^2 - q - 1$ if $g \in S \cap -S$. \hfill $\square$

For $r = 2$ the graph constructed in this theorem is a complete graph with $v = \frac{q^4 + q^2 + 1}{q^2 + q + 1} = q^2 - q + 1$. If $r \geq 4$ is even then the projective plane of order $q^r$ has an ordinary partition in subplanes of order $q^2$ and the planes of order $q^2$ have an ordinary partition in subplanes of order $q$. Then the vertices of the normally regular digraph are partitioned in sets of size $q^2 - q + 1$ spanning complete subgraphs. Thus $S \cap -S$ contains all nonzero elements of the subgroup of order $q^2 - q + 1$.

We conjecture that these are the only elements in $S \cap -S$.  

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Conjecture 2. Let $S$ be as in Theorem 33.

- If $r$ is odd then $S \cap -S = \emptyset$.
- If $r$ is even then $S \cap -S$ consists of the nonzero elements of the subgroup of order $q^2 - q + 1$.

The proof of Theorem 33 is an algorithm for computing the set $S$. The above conjecture is based on computations of $S$ for the following values of $(q, r)$: (2,4), (2,5), (2,7), (2,8), (2,10), (3,4), (3,5), (4,4), (4,5), (5,4).

Example 5. 1. For $q = 2$ and $r = 4$, we get

\[ S = \{7, 13, 14, 17, 19, 23, 26, 28, 29, 31, 34, 35, 37, 38\} \in \mathbb{Z}_{39}. \]

This gives a normally regular digraph with parameters $(39, 14, 4, 7)$. In this particular case, the normally regular digraph is isomorphic to the graph constructed in Corollary 37, with $(s, t) = (0, 3)$.

2. For $q = 2$ and $r = 5$, we get

\[ S = \{11, 17, 21, 22, 25, 29, 31, 34, 42, 43, 44, 45, 49, 50, 58, 62, 68, 81, 84, 86, 88, 90, 91, 97, 98, 100, 116, 121, 124, 136\} \in \mathbb{Z}_{151}. \]

This gives an asymmetric normally regular digraph with parameters $(151, 30, 4, 7)$.

10.3 Product constructions

In this section we give two constructions of normally regular digraphs that are not asymmetric. They are products involving doubly regular tournament and conference graphs. In some cases they are Cayley graphs.

Theorem 34. Let $T$ be a doubly regular tournament with $4t + 3$ vertices and let $K_{2t+1}$ be the complete graph of order $2t + 1$. Then the cartesian product with vertex set $V(T) \times V(K_{2t+1})$ and edge set \{(x, u) \rightarrow (y, u) \mid x \rightarrow y \text{ in } T\} \cup \{(x, u) \leftrightarrow (x, v) \mid u \leftrightarrow v \text{ in } K_{2t+1}\}$ is a normally regular digraph with parameters $((4t + 3)(2t + 1), 4t + 1, t, 1)$.

Proof. Let $(x, u)$ and $(y, v)$ be vertices in the cartesian product. If $x = y$ and $u \neq v$ then $(x, u)$ and $(y, v)$ are joined by an undirected edge and their common out-neighbours are the remaining $2t - 1$ vertices of the form $(x, w)$.
If \( x \neq y \) and \( u = v \) then \((x, u)\) and \((y, v)\) are joined by a directed edge and their common out-neighbours are the vertices \((z, u)\) where \( z \) is a common out-neighbour of \( x \) and \( y \) in \( T \). There are \( t \) such vertices. If \( x \neq y \) and \( u \neq v \) then \((x, u)\) and \((y, v)\) are non-adjacent. We may assume \( x \rightarrow y \) in \( T \). Then \((y, u)\) is the unique common out-neighbour of \((x, u)\) and \((y, v)\).

If \( T \) is a Paley tournament then the above construction is a Cayley graph.

**Corollary 35.** Let \( \mathbb{F} \) be the field of \( 4t + 3 \) elements and let \( Q \) be the set of non-zero squares in \( \mathbb{F} \). Let \( \mathbb{Z}_{2t+1} \) be the cyclic group of order \( 2t + 1 \). Let \( G \) be the direct product \( \mathbb{F} \times \mathbb{Z}_{2t+1} \) and let \( S = \{(d, 0) \mid d \in Q\} \cup \{(0, z) \mid z \neq 0\} \).

Then \( \text{Cay}(G, S) \) is a normally regular digraph with parameters \(((4t + 3)(2t + 1), 4t + 1, t, 1)\).

**Theorem 36.** Let \( H \) be a conference graph with \( 4t + 1 \) vertices and let \( T \) be a doubly regular tournament with \( 4s + 3 \) vertices. Let \( \Gamma \) be the graph with vertex set \( V(H) \times V(T) \) and with edge set

\[
\{(u, x) \rightarrow (v, y) \mid \text{either } u \leftrightarrow v \text{ and } x \rightarrow y, \text{ or } u \not\leftrightarrow v \text{ and } x \leftarrow y\}
\]

\[
\cup \{(u, x) \leftrightarrow (u, y) \mid u \in V(H), x, y \in V(T)\}.
\]

Then \( \Gamma \) is a normally regular digraph with parameters

\[
((4t + 1)(4s + 3), (4t + 2)(2s + 1), 4ts + 3s + t + 1, (2t + 1)(2s + 1)).
\]

**Proof.** Suppose that \((u, x) \rightarrow (v, y)\) but \((u, x) \not\rightarrow (v, y)\) in \( \Gamma \). Then either \( u \leftrightarrow v \) and \( x \rightarrow y \), or \( u \) and \( v \) are non-adjacent and \( x \leftarrow y \). Suppose that \( u \leftrightarrow v \) and \( x \rightarrow y \). Let \((w, z)\) be a common out-neighbour of \((u, x)\) and \((v, y)\). Then \( w \) and \( z \) satisfy one of the following six cases.

\[
w = u, y \rightarrow z,
\]

\[
w = v, x \rightarrow z, z \neq y,
\]

\[
u \leftrightarrow w \leftrightarrow v, x \rightarrow z, y \rightarrow z,
\]

\[
u \not\leftrightarrow w \not\leftrightarrow v, x \leftarrow z, y \leftarrow z,
\]

\[
u \leftrightarrow w \not\leftrightarrow v, x \rightarrow z, y \leftarrow z,
\]

\[
u \not\leftrightarrow w \leftrightarrow v, x \leftarrow z, y \rightarrow z.
\]
The number of vertices \((w, z)\) in each case are \(2s + 1, 2s, ts, (t - 1)s, ts\) and \(t(s + 1)\), respectively. The case where \(u\) and \(v\) are non-adjacent and \(x \leftarrow y\) is similar. Thus \(\lambda = 4ts + 3s + t + 1\).

The parameters \(k\) and \(\mu\) are easy to compute. Now suppose that \((u, x) \leftrightarrow (v, y)\). Then \(u = v\). A common out-neighbour \((w, z)\) of \((u, x)\) and \((u, y)\) is of one of the following three cases.

\[
w = u, z \neq x, y,
\]
\[
w \leftrightarrow u, x \rightarrow z, y \rightarrow z,
\]
\[
w \not\leftrightarrow u, x \leftarrow z, y \leftarrow z.
\]

The number of vertices of each type is \(4s + 1, 2ts\) and \(2ts\), respectively. This adds up to \(4ts + 4s + 1 = 2\lambda - \mu\).

If the conference graph and the doubly regular tournament in this theorem are both of Paley type then \(\Gamma\) is a Cayley graph.

**Corollary 37.** Let \(\mathbb{F}\) and \(\mathbb{E}\) be finite fields of order \(4t + 1\) and \(4s + 3\), respectively. Let \(Q_\mathbb{F}\) and \(Q_\mathbb{E}\) be the sets of non-zero squares in \(\mathbb{F}\) and \(\mathbb{E}\), respectively. Let \(R_\mathbb{F} = \mathbb{F} \setminus (Q_\mathbb{F} \cup \{0\})\) and \(R_\mathbb{E} = \mathbb{E} \setminus (Q_\mathbb{E} \cup \{0\})\). Let \(S = (Q_\mathbb{F} \times Q_\mathbb{E}) \cup (R_\mathbb{F} \times R_\mathbb{E}) \cup (\{0\} \times (\mathbb{E} \setminus \{0\}))\). Then \(\text{Cay}(\mathbb{F} \times \mathbb{E}, S)\) is a normally regular digraph with parameters \((4t+1)(4s+3), (4t+2)(2s+1), 4ts+3s+t+1, (2t+1)(2s+1))\).

Note that if \((4t+1) - (4s+3) = \pm 2\) then we get the difference set with Hadamard parameters constructed by Stanton and Sprott \[33\] by adding \((0, 0)\) to \(S\) if \((4t+1) - (4s+3) = 2\) or by taking the complement of \(S\) if \((4t+1) - (4s+3) = -2\).

**References**


