A BINOMIAL MODEL OF ASSET AND OPTION PRICING WITH HETEROGENEOUS BELIEFS

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ABSTRACT. This paper provides a theoretical framework for pricing assets in a multi-period economy with heterogeneous beliefs. The stock price dynamics follow a binomial lattice structure. Agents are allowed to differ in their beliefs of the probability and asset return in each state of nature. By constructing a consensus belief, we examine the impact of heterogeneous beliefs on market equilibrium. Statically, divergence of opinions leads to lower risk premium, greater divergence of opinions regarding future return in the upstate (downstate) leads to lower (higher) expected return for the risky asset and the risk-free rate. Dynamically, we show that the consensus belief is a fair belief to price options since agents’ wealth share process is a martingale under the consensus belief. Furthermore, call option prices exhibit implied volatility skew when the optimistic (pessimistic) agent is also confident (doubtful) about future asset returns.

JEL Classification: G12, D84.

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1. INTRODUCTION

Since its advent in the 1970s, binomial models have been popular and widely used in the finance literature. The binomial model was first proposed by Cox, Ross and Rubinstein (1979) (CRR) which have subsequently become one of the most cited paper in the finance literature. At the time of its publication, economists were not conversant with the mathematical tools used to derive the Black-Scholes option pricing formula,
so the CRR paper re-derived the formula as a limit from the binomial model. Essentially, the binomial pricing model uses a “discrete-time binomial lattice (tree) framework” to model the dynamics of the underlying stock price. A binomial lattice can be characterized simply by the probability of an up move and size of the move in both the upstate and the downstate. When all three parameters are constant, with the appropriate specification, the binomial lattice in continuous time limit converges weakly to the Black-Scholes (BS) model. The binomial lattice provides a simple framework to model stock price dynamics and interest rate term structure due to its simplicity. The parameters can also be set for the binomial lattice to weakly converge to other popular diffusion models used in finance. Kloeden and Platen (1992) showed that the Euler scheme converges weakly to a diffusion process if one replaces the Wiener increment in the Euler scheme with a two-point distributed random variable, hence the resulting numerical scheme is a binomial lattice with time and state dependent upward and downward moves, which are equal likely to occur. Moreover, Nelson and Ramaswamy (1990) developed methods to construct a recombined binomial lattice for diffusion model to enhance computational efficiency and Hahn and Dyer (2008) applied the method to mean-reverting stochastic processes specifically for real option valuation. van der Hoek and Elliott (2006) present a text book treatment for binomial models and their application.

Binomial models have been employed to study the pricing of options in literature. Guidolin and Timmermann (2003) model the dividend growth rate as a binomial lattice with constant probability and rate changes in each state, the former is unknown whereas the latter is known. They also assume there is a representative agent with CRRA utility who updates his belief about the probability of a positive rate change as a Bayesian learner. They find that the stock price under Bayesian learning (BL) with incomplete information does not converge to the Black-Scholes model whereas the convergence occurs under complete information. Call prices under BL with certain priors exhibit implied volatility that resembles the market implied volatility observed with S&P500 index options in the given period. In another related paper, Guidolin and Timmermann (2007) characterize equilibrium asset prices under adaptive, rational and BL schemes in a model where dividends evolve on a binomial lattice. The properties of equilibrium stock and bond prices under learning are shown to differ significantly.
Our work is related to theirs in that we are also studying equilibrium asset pricing under a binomial framework with incomplete information. However, there are some key distinctions. First of all, we model the cum-price of the stock rather than its dividend process, secondly we have multiple agents with heterogeneous beliefs of the future cum-prices of the stock, who only consumes on the maturity date. This means that we do not impose any exogenous processes (dividend process) as inputs to our model and we do not model the agents’ learning process. Instead, we consider a disagreement model and assume that agents simply form different beliefs based on the same current information (currently observed stock price). We assume all agents maximize log-utility since in a market where stock returns are assumed to be independently distributed, the optimal portfolio under log-utility is the growth optimal portfolio (GOP), which outgrows any other portfolios. Second, it is also easier to obtain analytically tractable results under log-utility than the more general CRRA utility.

Empirically, it is commonly observed that the volatility inferred from option prices is neither constant with respect to the strike price nor time to maturity which violates the key assumption underlying the BS model, see for instance Rubinstein (1985, 1994) and Dumas et al (1998). Several pricing models have been proposed to overcome these problems. These include stochastic volatility models (Hull and White (1987), Wiggins (1987), Melino and Turnbull (1990), Heston (1993)), GARCH models (Duan (1995), Heston and Nandi (2000)) and models with jumps in the underlying price process (Merton (1976), Bates (1991)). By modifying the stochastic process followed by the underlying asset price, some of these models can be calibrated to the currently observed volatility surface. The problem is that the volatility surface is unstable over time and these models do not provide directly an economic explanation for this phenomenon observed in the option market. Another strand of literature proposes to solve the problem by assuming incomplete information, model uncertainty and rational learning. In Guidolin and Timmerman (2003, 2007), one representative Bayesian learner is assumed whereas others assume heterogeneous agents with different priors who learns rationally from observed quantities (David and Veronesi (2002), Buraschi and Jiltsov (2006) and Li (2007), Cao and Ou-Yang (2009)). David and Veronesi (2002) develop a continuous time model to study option pricing in which the dividend growth rate has two possible states, investors need to determine the current state of the growth rate. Buraschi and Jiltsov (2006) assume that investors observe the dividend process...
and also a signal that correlates with the growth rate, they use the model to explain
open interest in the option market since options are non-redundant in an incomplete
market. Li (2007) assumes investors have different time preferences as well as het-
erogeneous beliefs about the dividend process. Cao and Ou-Yang (2009) analyze the
effects of differences of opinion on the dynamics of trading volume in stocks and op-
tions. They find that difference in the mean and precision of the terminal stock payoff
impact differently on the trading of stocks and options. In general, models with un-
certainty and learning provide a better explanation to the observed implied volatilities
than models with an exogenous stock price process with respect to economic intuitions
and the fitting of the volatility surface. Other papers in a broader area of equilibrium
asset pricing with heterogeneous beliefs include Detemple and Murthy (1994), Zapa-
(2008), Berraday (2009), Chiarella, Dieci and He (2010a, 2010b).

Our work contributes to the above literature in that we do not impose any exoge-
nous quantity (neither the stock price nor the dividend process) other than the current
wealth share of the agents and their subjective belief on the future evolution of the stock
price on the binomial lattice. Our binomial model is not restrictive in the sense that
probability of an up move and the size of price changes can be both time and state de-
pendent, which means that a wide range of stochastic models can be incorporated into
our framework. Unlike most papers in the asset pricing under incomplete information
literature, we allow disagreement in both model parameters and the model structure
in general so agents in our setting can believe in quite distinctive models for the stock
price dynamics. Since agents have different beliefs about the future stock price in each
state means they disagree on the state prices, hence the options are priced differently
by different agent. So the question is, what should be the fair price of the option? We
solve this problem by constructing a consensus belief (Jouini and Napp (2006, 2007)
and Chiarella et al (2010b, 2010a)), the consensus belief turns out to be a fair belief un-
der which every agent’s wealth share process is a martingale, hence we claim that the
option prices calculated under the consensus belief are fair. Heath and Platen (2006)
develops a unified framework to price options under real world probabilities using the
GOP as a numeraire. In an incomplete market, they show that option prices calculated
under real world probabilities are minimal prices since the corresponding replicating
portfolio is the cheapest. Our pricing formula shares some similarities in that we also
use the GOP as numeraire and real world probabilities, but all with respect to the consensus belief whereas in Heath and Platen (2006), the GOP is given exogenously. The minimal price for a call option in our model correspond to the expectation of the agent with the most pessimistic view of the future stock price, however the minimal price is not necessarily the fair price unless everyone shares the same view. Our work is also related to the market selection literature that concerns the survival of irrational or noise trader and their impact on the equilibrium price in the long run. This strand of literature was originated from the hypothesis of Friedman (1953) hypothesis that irrational trader do not survive in the long run and have no price impact in the long run. Under different model set ups, DeLong et al (1990, 1991) have found that noise trading can persist, Blume and Easley (2006) have shown that irrational traders can actually dominate the rational traders in an incomplete market, Sandroni (2000) and Dumas et al. (2009) show that irrational trader becomes extinct only after a long time, while Kogan et al. (2006) provide evidence that price impact and survivability are two different concepts, irrational traders can be very close to extinction yet still have a significant price impact. Most of their results are built on the assumption that there is at least one rational agent who knows the true or objective law of motion governing the exogenous process (dividend or endowment process). We on the other hand do not assume any of our agents have the objective belief regarding future stock prices, but simply argue that the consensus belief is a fair belief in the sense of the expected wealth share; while option is priced under the fair belief. We also show that agents with a relatively larger wealth share dominate the consensus belief which makes intuitive sense since wealthier agents are likely to be those who have been performing better than others in the past, whose beliefs are probably closer to the true or objective belief.

Our model investigates an economy with one risky asset (stock) and one riskless asset with positive and net-zero supply respectively. The stock price follows a binomial lattice that allows time and state dependent upward and downward moves and also probability of an up move. The agents are log-utility maximizers of their terminal wealth who form their subjective beliefs now about the probability and the size of the moves in each period from now to the terminal time or maturity date. Agents agree to disagree, the differences in opinion are due to the interpretation of the same information, our model concerns the equilibrium pricing of options and contingent
claims in an economy with disagreements about the evolution of the stock price. Our paper is organized as follows. Section 2 presents the binomial model that describes our economy. Section 3 defines a consensus belief and shows how the consensus belief can be constructed from investors’ subjective beliefs. We also define and identify a fair belief for pricing contingent claims in the market. Section 4 performs static analysis and studies the impact of heterogeneous beliefs on the equilibrium price of the risky asset and the risk-free rate in a single-period setting. In section 5, we develop a fair option pricing formula and use a numerical example to study the price distribution and the fair call prices under the consensus belief. Section 6 concludes.

2. SETUP OF THE ECONOMY AND PORTFOLIO STRATEGIES

We consider a simple economy with one risky and one riskless asset. Let time be discrete and finite, index by \( t = 0, 1, 2, \ldots, T \). The risky asset has one share available and the riskless asset is in zero net supply for all time \( t \). There are \( I \) agents in the economy, indexed by \( i = 1, 2, \ldots, I \). Agent \( i \)'s objective at time \( t = 0 \) is to maximize the quantity

\[
\mathbb{E}_0^i \left( U(\tilde{W}_i(T)) \right),
\]

where \( U(\cdot) \) is investor \( i \)'s utility function and \( \tilde{W}_i(T) \) is his/her portfolio’s terminal wealth at time \( T \) and \( \mathbb{E}_0^i \) denotes agent \( i \)'s expectation of the outcome of the market at time \( T \) conditional on the information available and him/her belief at time \( t = 0 \). We assume that all agents are log-utility maximizers, that is, \( U_i(x) = \ln(x) \) for all \( i \). Stock price \( S \) follow a multi-period Cox-Ross-Rubinstein model. This means, given information at time \( t \), the cum-price of the risky asset at time \( t + 1 \) has the following probability distribution,

\[
S(t + 1) = \begin{cases} 
S(t) \cdot u(t, t + 1), & p(t, t + 1); \\
S(t) \cdot d(t, t + 1), & 1 - p(t, t + 1)
\end{cases}
\]

with \( d(t, t + 1) < R_f(t) < u(t, t + 1) \), where \( R_f(t) = 1 + r_f(t) \) is the return of the riskless asset over the period \([t, t+1]\). Note that \( p(t, t + 1) \), \( u(t, t + 1) \) and \( d(t, t + 1) \) can vary with time. Agents’ beliefs about future asset returns are formed in the following way. Let \( \mathcal{B}_i := (p_i, u_i, d_i) \) denote agent \( i \)'s belief about the probability distribution of the future asset returns conditional on available information at time \( t = 0 \). \( p_i := (p_i(0), p_i(1), \ldots, p_i(T - 1))^T \) where \( p_i(t) \) denotes the value of \( p(t, t + 1) \) for \( t = \)}
0, 1, \cdots, T - 1 under agent \(i\)'s belief given the information at time \(t = 0\). Similarly, \(u_i := (u_i(0), u_i(1), \cdots, u_i(T - 1))^T\) and \(d_i := (d_i(0), d_i(1), \cdots, d_i(T - 1))^T\) where \(u_i(t)\) and \(d_i(t)\) denote the value of \(u(t, t + 1)\) and \(d(t, t + 1)\) for \(t = 0, 1, \cdots, T - 1\) under agent \(i\)'s belief given information at time \(t = 0\). Essentially, agents are provided with the same information at time \(t = 0\), however, each of them interprets the information differently and arrive at their subjective belief about the future distribution of asset returns. Let \(\omega_i(t)\) be the proportion of investor \(i\)'s wealth \(W_i(t)\), at time \(t\), invested in the risky asset and define the future return of the risky asset as

\[
R(t + 1) = \frac{S(t + 1)}{S(t)}, \quad r(t + 1) = R(t + 1) - 1,
\]

which is random at time \(t\). Then agent \(i\)'s objective in equation (2.1) becomes

\[
\max_{\{\omega_i(0), \omega_i(1), \cdots, \omega_i(T - 1)\}} \ln(W_i(0)) + \sum_{t=0}^{T-1} \mathbb{E}_0^0 \left[ \ln \left( R_f(t) + \omega_i(t)(R(t + 1) - R_f(t)) \right) \right].
\]

(2.2)

The optimization problem in (2.2) can be solved using *dynamic programming* or the *Martingale Approach*. Detailed solution to the problem under both methods can be found in Cvitanic and Zapatero (2004) Chapter 4. To ease the notations, we have suppressed the time indexes, all model parameters will correspond to the time period \([t, t + 1]\) unless otherwise stated.

**Lemma 2.1.** Let \(\bar{u}_i = u_i(t, t + 1) - R_f(t)\) and \(\bar{d}_i = d_i(t, t + 1) - R_f(t)\) be the excess rate of return in the up and down states, respectively over the period \([t, t + 1]\) under investor \(i\)'s perspective. The solution to investor \(i\)'s multi-period optimization problem in equation (2.2) is given by

\[
\omega_i = R_f \frac{\bar{u}_i \ p_i + \bar{d}_i \ (1 - p_i)}{-\bar{u}_i \ \bar{d}_i}
\]

(2.3)

for \(t = 0, 1, \cdots, T - 1\).

Lemma 2.1 shows that agent \(i\) is able to determine the optimal proportion of his wealth to invest in the risky asset once the risk-free rate at time \(t\) is observed, also the optimal proportion only depend on agent \(i\)'s belief about the distribution of asset return in period \([t, t + 1]\). The intuition is that maximizing the logarithm of a portfolio’s terminal wealth is equivalent to maximizing the expected growth rate \(\mathbb{E}[\ln(1 + R_p(t + 1))]\) period by period, where \(R_p(t + 1)\) is portfolio’s rate of return from \(t\) to \(t + 1\). This is so called the short-sighted or *myopic behavior* of logarithmic utility, because
log-utility maximizers do not consider any future investment opportunities in their portfolio selections (see Cvitanic and Zapatero (2004), chapter 4).

3. CONSENSUS BELIEF AND MARKET EQUILIBRIUM

Since we have one unit of the risky asset available in the market and zero net supply for the riskless asset, in order for the market to clear, agents’ total dollar demand for the risky asset must equal to the aggregate market wealth at all times. This means the equilibrium price of the risky asset must equal to the aggregate market wealth at all times, that is,

$$\sum_{i=1}^{I} \omega_i(t) W_i(t) = W_m(t) = S(t), \quad t = 0, 1, \cdots, T - 1,$$

where $W_m(t) = \sum_{i=1}^{I} W_i(t)$ denote the aggregate market wealth at time $t$. We refer to equation (3.1) as the market clearing condition for our economy. Substituting equation (2.3) into the market clearing condition in (3.1) leads to the following expression involving the equilibrium risk free rate from time $t$ to $t + 1$,

$$\frac{1}{R_f} + \sum_{i} w_i \left( \frac{p_i}{d_i} + \frac{1 - p_i}{u_i} \right),$$

where $w_i = \frac{W_i(t)}{W_m(t)}$ is the wealth share of investor $i$ at time $t$. Equation (3.2) shows how to determine the equilibrium risk-free rate given beliefs of all the agents and their wealth share. Ideally one would like to aggregate agents’ heterogeneous beliefs and construct a consensus belief to determine the equilibrium risk-free rate. The aggregation of heterogeneous beliefs was also studied in Chiarella et al (2010a, 2010b) under a static mean-variance setting and Jouini and Napp (2006, 2007) in an intertemporal consumption setting. We will introduce a consensus belief for the CRR model.

**Definition 3.1.** A belief $\mathcal{B}_m := (p_m, u_m, d_m)$, defined by the probability of an up move and return of the risky asset in the up and down state respectively in period $[t, t+1]$ for $t = 0, 1, \cdots, T - 1$, is called a consensus belief if the asset price and the equilibrium risk-free rate $R_f$ under the heterogeneous beliefs is also that under the homogeneous belief $\mathcal{B}_m$.

The introduction of a consensus belief allows the transformation of a market with heterogeneous beliefs to a market under which all agents are identical in their beliefs. If the aggregate market invest as a sole log-utility maximizer, his belief at time $t = 0$
coincide with the consensus belief \( B_m \) and by the market clearing condition the risky asset must be the growth optimal portfolio. Intuitively, this is the most informed belief of the future because it takes into account every individual investor’s subjective belief at time \( t = 0 \) about the distribution of future asset returns over the entire period \([0, T]\).

The following Proposition 3.2 provides an implicit formula to compute the consensus belief and show that the risk-free rate and the consensus belief can be determined simultaneously.

**Proposition 3.2.**

(i) The consensus belief \( B_m := (p_m, u_m, d_m) \), is given by

\[
\begin{align*}
    p_m &:= (p_m(0), p_m(1), \ldots, p_m(T - 1))^T, \\
    u_m &:= (u_m(0), u_m(1), \ldots, u_m(T - 1))^T, \\
    d_m &:= (d_m(0), d_m(1), \ldots, d_m(T - 1))^T,
\end{align*}
\]

where in the time interval \([t, t+1]\) for \( t = 0, 1, \ldots, T - 1 \),

\[
p_m = \sum_i w_i p_i, \quad u_m = \bar{u}_m + R_f, \quad d_m = \bar{d}_m + R_f, \tag{3.3}
\]

and

\[
\bar{u}_m = \left( \sum_{i=1}^{I} w_i \frac{1 - p_i}{1 - p_m} \bar{u}_i^{-1} \right)^{-1}, \tag{3.4}
\]

\[
\bar{d}_m = \left( \sum_{i=1}^{I} w_i \frac{p_i}{p_m} \bar{d}_i^{-1} \right)^{-1}. \tag{3.5}
\]

(ii) The equilibrium risk free rate is given by

\[
\frac{1}{R_f} = \frac{1 - p_m}{d_m} + \frac{p_m}{u_m} = \mathbb{E}_t^m \left[ \frac{1}{R(t + 1)} \right]. \tag{3.6}
\]

(iii) State prices or the risk neutral probabilities of the up and down states at time \( t \) under individual subjective belief \( B_i \) and the consensus belief \( B_m \) are given by

\[
q_{i,u}(t) = \frac{-\bar{d}_i}{\bar{u}_i - \bar{d}_i}, \quad q_{i,d}(t) = \frac{\bar{u}_i}{\bar{u}_i - \bar{d}_i}, \quad i = 1, 2, \ldots, I, m. \tag{3.7}
\]

(iv) In equilibrium, given information at time \( t \), the stock price at time \( t \) can be written as

\[
S(t) = \frac{\mathbb{E}_t^Q_i(S(t + 1))}{R_f} = \frac{\mathbb{E}_t^i(Z S(t + 1))}{R_f}, \quad i = 1, 2, \ldots, I, m \tag{3.8}
\]
where
\[
Z = \begin{cases} 
\frac{q_u(t)}{p(t,t+1)}, & p(t,t+1); \\
\frac{q_d(t)}{1-p(t,t+1)}, & 1-p(t,t+1)
\end{cases}
\] (3.9)
is the Random-Nikodym derivative that change the probability measure from 
i to Q_i, and is often referred to as the “pricing kernel” in the asset pricing
literatures.

Proof of Proposition 3.2 is given in Appendix A. The consensus belief of the prob-
bability of an up move p_m is simply an arithmetic average of individual probability be-
liefs p_i weighted by their wealth shares w_i. This means that a wealthier investor has a
stronger impact on p_m. The consensus belief of the excess return in each state (\bar{u}_m, \bar{d}_m)
is a harmonic mean of individual beliefs of the excess returns (\bar{u}_i, \bar{d}_i) weighted by both
wealth shares and probabilities. A wealthier and relatively pessimistic (optimistic) in-
vestor would have a stronger impact on \bar{u}_m (\bar{d}_m). Furthermore, the consensus belief \mathcal{B}_m
can only be determined simultaneously with the risk-free rate given a path of the wealth
share of every agent in the market, w_i(t) for i = 1, 2, \cdots, I and t = 0, 1, \cdots, T - 1.
Equation (3.6) indicates relationship between the consensus belief and the risk-free rate,
the quantity R_f(t)/R(t + 1) is a martingale under the consensus belief. Agents
in this economy perceive different state prices, this is indicated by (3.7), hence op-
tion prices implied by each agent’s belief are also different. Equation (3.8) shows that
agents agree on the current observed asset price though they may have their distinctive
pricing kernels due to their different beliefs.

Each agent have their own set of state prices, hence prices of contingent claims
would differ under each agent’s subjective belief \mathcal{B}_i, i = 1, 2, \cdots, I and the consensus
belief \mathcal{B}_m. The question is which belief should one use for pricing contingent claims.\(^1\)
Next we define a “fair” belief to price contingent claims in our economy.

**Definition 3.3.** A belief \(\mathcal{B}^* := (p^*, u^*, d^*)\) given information at time \(t = 0\) is called
fair if and only if the wealth share of agent \(i\) for \(i = 1, 2, \cdots, I\) is a martingale under
the belief \(\mathcal{B}^*\), that is
\[
\mathbb{E}_t^*[w_i(t + 1)] = w_i(t)
\]
for \(t = 0, 1, \cdots, T - 1.\)

\(^1\)Any contingent claims other than the stock or the bond are redundant securities in our economy since
the market is complete in the sense that each agent can construct their optimal portfolio by investing in
the risk-free asset and the risky asset only.
The idea behind definition 3.3 is that under a fair belief, agent $i$’s wealth share is expected to remain at its current level from time $t$ to terminal time $T$ for all $i$. It is easy to see under the law of iterated expectations that $E_t^*[w_i(T)/w_i(t)] = 1$. The belief $B^*$ is fair in the sense that every agent will on average perform equally under this belief. For the rest of this section, we will identify in our economy those beliefs that are fair.

**Proposition 3.4.** The consensus belief $B_m$ is a fair belief, that is

$$E_t^m\left[\frac{w_i(t+1)}{w_i(t)}\right] = 1.$$

Proof of Proposition 3.4 is in Appendix B, it shows that the consensus belief is a fair belief under which any agent $i$’s wealth share is expected to remain the same in the next period. This is actually a very intuitive result since the consensus belief consists of beliefs from every agent, therefore, in equilibrium, the aggregate market does not expect anyone’s future wealth share to be more than their current wealth share levels. Next, we examine each agent $i$’s subjective belief $B_i$.

**Proposition 3.5.** Agent $i$’s subjective belief $B_i$ is fair if and only if $E_t^i[\frac{1}{R(t+1)}] = 1$ and

$$E_t^i\left[\frac{w_i(t+1)}{w_i(t)}\right] \geq 1.$$

for $t = 0, 1, \cdots, T-1$. Equality will hold if and only if $E_t^i[\frac{1}{R(t+1)}] = E_t^m[\frac{1}{R(t+1)}] = \frac{1}{R_f}$.

Proof of Proposition 3.5 is in Appendix C, it shows that agent $i$’s subjective belief is fair if and only if the discounted value of a dollar payoff by the stock under his/her expectation is the same as the zero-coupon bond price. Using the law of iterated expectation, one can see that $E_t^i\left[\frac{w_i(T)}{w_i(t)}\right] > 1$ if the condition in Proposition 3.5 is not satisfied for all $t \in [0, T-1]$. This indicates agent $i$ expects his/her market share to grow and perform better compare to other agents. The above analysis tells us that the consensus belief $B_m$ is a fair belief to price contingent claims in our economy and an agent $i$’s subjective belief is not fair unless the condition in Proposition 3.5 holds.

4. **Impact of Mean-Preserving Heterogeneous Beliefs**

In this section, we examine the impact of mean-preserving heterogeneous beliefs on the market consensus belief and the equilibrium risk-free rate. It is often believed
that effect from belief biases should cancel out if the “average” agent is unbiased. In our setting, the consensus belief $B_m$ rather than the average belief $\bar{B}$ in the market $\bar{B}$ correspond to the belief of the “average” agent. We want to see how the quantities such as the risk-free rate and risk premium differ under the consensus belief $B_m$ and the average belief $\bar{B}$ for an increasing level of divergence of opinions.

We first consider a static setting where agents’ wealth shares are equal and their beliefs of the asset return are uniformly distributed for both the upstate and downstate. We will focus on a single time period $[t, t + 1]$ where information at time $t$ is known.

**Corollary 4.1.** Let there be $I$ investors with $w_i(t) = 1/I$ and $p_i = p$ for all $i$. Consider a benchmark belief $(u_o(t), d_o(t))$, and assume that investors’ subjective beliefs diverge from the benchmark belief uniformly. Let agent $i$’s belief be given by $(u_i(t), d_i(t)) = (u_o(t) + \tilde{\epsilon}_iu, d_o(t) + \tilde{\epsilon}_id)$ where $\tilde{\epsilon}_iu$ and $\tilde{\epsilon}_id$ are both i.i.d for agent $i$ with a bounded variance. Therefore agents’ divergence of opinions regarding the stock returns in both up and down states are i.i.d. As the number of agents approaches infinity, from (3.4) and (3.5) the consensus belief can be expressed implicitly as

$$\bar{u}_m = \left( \mathbb{E}[(u_o - R_f + \tilde{\epsilon}_u)^{-1}] \right)^{-1} \quad \bar{d}_m = \left( \mathbb{E}[(d_o - R_f + \tilde{\epsilon}_d)^{-1}] \right)^{-1}.$$  

where $\tilde{\epsilon}_u$ and $\tilde{\epsilon}_d$ have the same distribution as $\tilde{\epsilon}_iu$ and $\tilde{\epsilon}_id$ respectively.

In Corollary 4.1, it is clear that agents have heterogeneous beliefs regarding the future return of the risky asset with average belief $\bar{B}$ equal to the benchmark belief $B_o$. Now the question is whether this form of divergence of opinions have significant effect on the market consensus belief of asset return and the equilibrium risk-free rate. To answer this question, we let $(u_o, d_o) = (1.235, 0.905)$, $p = 0.5$ and $I = 5,000$, this means that all the agents agree on the fact that up and down states are equal likely to occur and each agent’s wealth share $w_i = 0.0002$ for all $i$. Moreover, we assume that $\tilde{\epsilon}_u \sim Unif(-\theta_u, \theta_u)$ and $\tilde{\epsilon}_d \sim Unif(-\theta_d, \theta_d)$. Next, we approximate the consensus belief $B_m$ and the risk-free rate $r_f$ by Monte-Carlo simulations with various combinations of the parameters $\theta_u$ and $\theta_d$. Figure 4.1 compares the consensus belief, risk-free rate and the risk premium with the benchmark for different combinations of $(\theta_u, \theta_d)$. In the special case of no divergence of opinions, that is when $(\theta_u = 0, \theta_d = 0)$, the consensus belief $B_m$ is the same as the benchmark belief $B_o$, the risk-free rate under the benchmark belief is 0.045 and the risk premium equals to 0.025.

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2The average belief is defined by $\bar{B} := (\frac{1}{I} \sum_{i=1}^I p_i, \frac{1}{I} \sum_{i=1}^I u_i, \frac{1}{I} \sum_{i=1}^I d_i)$
that risk premium to invest in the risky asset and more willing to invest at the risk-free rate. The fact free decreases from its benchmark value because the aggregate market is less willing impact on the consensus belief in the upstate, therefore agents who are pessimistic regarding future stock return in the upstate have a larger

\[ \theta \in [0, 0.2], \theta = 0 \]

\[ \theta = 0, \theta \in [0, 0.1] \]

\[ \theta \in [0, 0.1], \theta \in [0, 0.1] \]

**Figure 4.1**. Impact of divergence of opinion on the consensus belief

\[ B_m = (u_m, d_m), \text{ expected stock return } \mathbb{E}[R(t+1)] = p\ u_m + (1-p)\ d_m, \]

risk-free rate \( r_f = R_f - 1 \) and the risk premium \( \mathbb{E}[R(t + 1) - R_f] \).

Figure 4.1 (a1) shows that the consensus belief of expected stock return decreases as the level of divergence of opinion in the upstate \( \theta_u \) increases. This is because the agents who are pessimistic regarding future stock return in the upstate have a larger impact on the consensus belief in the upstate, therefore \( u_m \) decreases when \( \theta_u \) increases though agents’ beliefs \( u_i \) is a mean-preserving spread of the benchmark. The risk-free decreases from its benchmark value because the aggregate market is less willing to invest in the risky asset and more willing to invest at the risk-free rate. The fact that risk premium \( \mathbb{E}[R(t + 1) - R_f] \) is negatively related to divergence of opinions
sues that the reduction in the risk-free rate is less than that in the expected stock return under the consensus belief. Figure 4.1 (a2) shows that the consensus belief of stock return in the downstate $d_m$ increases as the level of divergence of opinion $\theta_d$ increases. This is because the optimistic agents who perceive a higher stock return in the downstate have a larger impact on the consensus belief in the downstate, therefore $d_m$ increases though agents’ beliefs $d_i$ is a mean-preserving spread of the benchmark. The risk-free rate increases from its benchmark value because the aggregate market is more willing to invest in the risky asset and less willing to invest at the risk-free rate. The fact that risk premium $\mathbb{E}[R(t+1) - R_f]$ is negatively related to divergence of opinions suggests that the increase in expected stock return is not enough compare to the increase in the risk-free rate. Finally, when we combine the divergence opinions in the upstate and downstate, Figure 4.1 (a3) shows the combined effect of $\theta_u$ and $\theta_d$ on the expected stock return, risk-free rate and the risk premium. It is clear that $\theta_u$ and $\theta_d$ have opposite effect on the expected stock return and risk-free rate, however they both have a negative effect on the risk premium.

**Remark 4.2.** Assume investors’ wealth are evenly distributed, then the aggregate market as a consensus agent believes that divergence of opinion regarding future asset return in the upstate (downstate) is negatively related to expected future stock return and the equilibrium risk-free rate. Higher divergence of opinions leads to lower risk premium under the consensus belief in both the upstate and downstate.

## 5. Option pricing under heterogeneous beliefs

We turn our focus to the pricing of options in this section. As discussed in the previous section, agents disagree on the state prices, therefore agents with different belief will price option differently. However, we claim that the fair price of an option should be computed under a fair belief. Next we present a fair option pricing formula under the consensus belief $B_m$ which is a fair belief.

**Proposition 5.1.** Given information at time $t$, the fair price of an option $V(t,S(t))$ with payoff function $H(T, S(T))$ is given by

$$V(t, S(t)) = S(t) \mathbb{E}_t^m(H(T, S(T))/S(T)).$$

**Proof.** We can always replicate the option with a portfolio that invests $\omega(t)$ in the risky asset and $1 - \omega(t)$ in the risk-free asset, this means that we can express the value of
the option at time $t+1$ as

$$V(t+1, S(t+1)) = V(t, S(t)) \left( R_f(t) + \omega(t)(R(t+1) - R_f(t)) \right).$$

Dividing by $S(t+1)$ on both side and taking expectation under the consensus belief yields

$$E^m_t \left( \frac{V(t+1, S(t+1))}{S(t+1)} \right) = \frac{V(t, S(t))}{S(t)} E^m_t \left( \omega(t) + (1 - \omega(t)) \frac{R_f(t)}{R(t+1)} \right).$$

Then using the law of iterated expectations, we get

$$\frac{V(t, S(t))}{S(t)} = E^m_t \left( \frac{V(t+1, S(t+1))}{S(t+1)} \right) = E^m_t \left( \frac{V(T, S(T))}{S(T)} \right)$$

and $V(T, S(T)) = H(T, S(T))$ and that completes the proof. \hfill \Box

In the following example, we will use the pricing formula developed in Proposition 5.1 to price European call options with different strikes written on the risky asset.

**Example 5.2.** Assume there are two agents ($i = 1, 2$) whose $B_i$ is characterized by

$$u_i(t) = 1 + \mu_i \Delta + \sigma_i \sqrt{\Delta},$$

$$d_i(t) = 1 + \mu_i \Delta - \sigma_i \sqrt{\Delta},$$

$$p_i(t) = 0.5,$$

$$w_i(t) = 0.5,$$

where $t = 0, 1, \cdots, T-1$, $\Delta = (T-t)/n$ and $n$ is the number of trading period from time 0 to $T$.

In Example 5.2, both agents agree that the stock price is always equally likely to move up or down and the relative price changes are constant from time 0 to $T$. However, they disagree on the size of the relative price changes, which are characterize by the parameters $\mu_i$ and $\sigma_i$. These two parameters can be interpreted as agent $i$’s belief of the expected return and volatility of the stock per annum, compounded $n$ periods a year. Obviously, as the number of trading period $n$ approaches infinity, the stock price dynamics under agent $i$’s belief converges weakly to the stochastic differential equation (SDE) (see Kloeden and Platen (1992) and Nelson and Ramaswamy (1990))

$$dS(t)/S(t) = \mu_i \, dt + \sigma_i \, dW(t) \quad (5.1)$$
where $W(t)$ is the Wiener process.

**Remark 5.3.** In Example 5.2, when $\mu_i = \mu$ and $\sigma_i = \sigma$ for $i = 1, 2$ and $n$ approaches infinity, the price of a call option on the risky asset is given by the Black-Scholes formula and the instantaneous risk-free rate is constant and given by $\mu - \sigma^2$.

The proof of Remark 5.3 is in Appendix D, it suggests that as the number of trading period $n \to \infty$, the risk-free rate is constant and depend on expected return and volatility of the stock, furthermore, call option prices are given by the Black-Scholes formula using $r_f = \mu - \sigma^2$. Next we will consider the case where agents differ in their beliefs of the growth rate and the volatility of the stock. More specifically, we will assume that $(\mu_1, \mu_2) = (\mu_o + \delta_\mu, \mu_o - \delta_\mu)$ and $(\sigma_1, \sigma_2) = (\sigma_o + \delta_\sigma, \sigma_o - \delta_\sigma)$ such that $\mu_o$ and $\sigma_o$ are the arithmetic average beliefs of the expected return and the volatility of future stock returns respectively. Note that $\delta_\mu > 0$ indicates that agent 1 is relatively more optimistic than agent 2 since he/she believes in a higher expected return, while $\delta_\sigma > 0$ indicates that agent 1 is less confident than agent 2 in the sense that he/she believes in a higher volatility. We obtain distribution of the log stock price under the consensus belief at time $T$ and compute call option prices via Monte Carlo simulation using Proposition 5.1. The benchmark belief is given by $(\mu_o, \sigma_o) = (0.07, 0.225)$. Time to maturity is $T = 0.25$, time increment is set to $\Delta = 0.00025$. We use Monte Carlo simulation for evaluation of the option prices because under the consensus belief $B_m$, the binomial tree for future stock prices is non-recombining though it is recombining under both agents’ beliefs. Furthermore, it is computationally very expensive to generate a path for the stock price, because for each path in this example, we need to numerically solve for the risk-free rate at each step. Therefore, we use the Black-Scholes option price as a control variate to reduce the variance of the simulated option payoff. This is because the option payoff under the Black-Scholes model at time $T$ is strongly correlated with the one under the consensus belief, correlation is estimated to be close to 1. Using this technique, we are able to reduce the standard deviation associated with the option payoff by up to ten times. After obtaining the call option prices under the consensus belief, we then calculate the implied volatilities for these prices by finding a $\sigma_{imp}$ such that

$$C_m(K) = BS(S(0), \sigma_{imp}, r_f, T)$$

\footnote{Non-recombining means that an up move follow by a down move is not the same as a down move follow by an up move. For a non-recombining tree, there are $2^{1000}$ possible values for the stock price after 1000 steps.}
where \( K \) is the strike price, \( S(0) \) is the current stock price, \( T \) is the time to maturity and \( r_f \) is the current risk-free rate. \( C_m(K) \) is the fair price of a call option with strike \( K \).

Figure 5.1 shows the distribution of the log stock price at maturity (\( \ln(S_T) \)) under agent 1 and 2’s subjective beliefs (\( B_1 \) and \( B_2 \)) and under the consensus belief (\( B_m \)). Table 5.1 shows the sample statistics and Figure 5.2 shows the implied volatilities for different strike prices. If we normalize the current stock price to 1, then \( \ln(S_T) \) measure the continuous return in the period \([0,T]\), it is normally distributed with mean 
\[
(\mu_i - \frac{1}{2}\sigma_i^2)T
\]
and standard deviation \( \sigma_i\sqrt{T} \) under the subjective belief of agent \( i \). If we interpret the expected log price as the growth rate of the stock, then agents can agree on the expected stock return but perceive different growth rates. Fig. 5.1 (b1) and Tab. 5.1 (c1) demonstrate the case when agents agree on the volatility but disagree on the expected stock return. In this case, the growth rate under the consensus belief \( B_m \) is between that of both agents but closer to that of agent 1’s (the more optimistic agent), the volatility is close to the common belief, moreover the distribution of \( \ln(S_T) \) is approximately normal under \( B_m \) since skewness is close to zero and kurtosis close to 3. Fig. 5.2 (d1) shows that the implied volatility is almost flat with respect to the strike, this means that the call prices are consistent with the BS formula. Fig. 5.1 (b2) and Tab. 5.1 (c2) shows that when agents agree on the expected return but disagree on the volatility, \( \ln(S_T) \) becomes negatively skewed under the consensus belief \( B_m \), the growth rate and volatility under \( B_m \) is closer to agent 2’s belief, who is the more confident agent. Furthermore, the market perceives a higher growth rate than both agents. Fig. 5.2 (d2) shows that the implied volatility exhibit a positive skewness consistent with the observed pattern in option markets. The intuition is that since agent 2 perceives a higher growth rate he/she has a larger wealth share and dominate the consensus belief in the upper part of the binomial lattice which matters more for pricing out of the money (OTM) call options (calls with strikes above the current spot price). This means that the OTM call prices would reflect more of agent 2’s belief of stock volatility. Since agent 2 perceives a lower stock volatility OTM call prices has a lower implied volatility than at the money (ATM) and in the money (ITM) call prices. As one moves gradually towards the lower part of tree, agent 1’s belief becomes more and more dominant in determining the consensus belief, therefore implied volatility also increases as the strike prices decreases. Fig. 5.1 (b3) and Tab. 5.1 (c3) show...
that when agent 1 is more optimistic and confident, the consensus belief of the growth rate and volatility of the stock is closer to those under agent 1’s belief, the distribution of \(\ln(S_T)\) is negatively skewed and Fig. 5.2 (d3) indicates that the call prices exhibit volatility skew. The intuition is similar to the previous case. Fig. 5.1 (b4) and Tab. 5.1 (c4) show that when agent 1 is more optimistic but less confident about future stock returns, the growth rate and volatility under consensus belief is closer to those under agent 2’s belief who is less optimistic and more confident. The distribution of \(\ln(S_T)\) is positively skewed and Fig. 5.2 (d4) indicates the OTM call prices has higher implied volatilities than ATM and ITM call prices. The intuition is that agent 1 who is more optimistic dominates the consensus belief in the upper part of the tree as previously discussed, however agent 1 perceives a larger volatility than agent 2, therefore implied volatility is positively related to the strike price.

In order to examine the term structure of the implied volatilities, we calculate the implied volatilities from the fair prices of call option with time to maturity \(T \in [0.25, 1.00]\) for \(\delta_\mu = 0.05\) and \(\delta_\sigma = -0.075\). Figure 5.3 shows that the implied volatility surface flattens out as time to maturity increases, this is consistent with the observed

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Table 5.1. Impact of heterogeneous beliefs in the growth rate \((\mu)\) and the volatility \((\sigma)\) of future stock returns on the distribution of the log stock price at maturity \((\ln(S_T))\) for \(T = 0.25\). The first and second columns in each table provide the first 4 moments of the distribution of \(\ln(S_T)\) under the subjective beliefs of agent 1 and 2 respectively and the third column of each table correspond to that under the consensus belief \(B_m\).
patterns from market data. This indicates that our fair option prices can mimic important features exhibited by the market data of option prices though Example 5.2 is only a very simple specification of our model. We expect the volatility surface generated from fair option prices under the consensus belief to exhibit even richer patterns with a more general specification.

Example 5.2 can be generalized to take into account other popular stochastic processes for modelling stock prices dynamics. In general, if the belief of agent $i$ is
characterized by
\[ u_i(t) = 1 + \mu_i(t, Y_t) \Delta + \sigma_i(t, Y_t) \sqrt{\Delta}, \]
\[ d_i(t) = 1 + \mu_i(t, Y_t) \Delta - \sigma_i(t, Y_t) \sqrt{\Delta}, \]
\[ p_i(t) = 0.5. \]

where \( t = 0, 1, \cdots, T - 1 \), \( Y_t \) is the stock price at time \( t \) and \( \Delta = T/n \) is the time increment, then as \( n \) approaches infinity, the above characterization implies that agent \( i \) believes that the stock price dynamics is describe by the SDE,
\[ dS(t)/S(t) = \mu_i(t, S(t)) \, dt + \sigma_i(t, S(t)) \, dW(t) \]

Therefore, in principle our option pricing formula can take into account not only disagreement in model parameters, but also differences in the model structure. This is an
advantage since the number of option pricing models has exploded with the recent advances in mathematical finance, so it is reasonable to assume that quantitative analysts can use quite distinctive models for modelling stock price dynamics. Furthermore, we can also accommodate currently observed values of other explanatory variables of stock returns. If we denote $\phi(t)$ as the value of a set of explanatory variable at time $t$, then agent $i$’s belief can be written as

$$u_i(t) = 1 + \mu_i(t, Y_t, \phi_t) \Delta + \sigma_i(t, Y_t, \phi_t) \sqrt{\Delta},$$
$$d_i(t) = 1 + \mu_i(t, Y_t, \phi_t) \Delta - \sigma_i(t, Y_t, \phi_t) \sqrt{\Delta},$$
$$p_i(t) = 0.5,$$

which weakly converges to the SDE as $\Delta \to 0$ given by

$$dS(t)/S(t) = \mu_i(t, S(t), \phi(t)) \, dt + \sigma_i(t, S(t), \phi(t)) \, dW(t).$$

6. Conclusion

In this paper, we provide an aggregation method of heterogeneous beliefs within a multi-period binomial lattice framework. The heterogeneity is characterized by the differences in agents’ beliefs about the probability of an up move in each period and also the relative size of the price changes in each period. Agents are bounded rational in the sense that they invest in the growth optimal portfolio based on their own subjective belief. To analyze the impact of heterogeneity, we introduce the concept of a consensus
belief, which relates our heterogeneous market to an equivalent homogeneous market. The consensus belief is basically a wealth weighted average of agents’ subjective beliefs and can be determined simultaneously with the risk-free rate. Through various numerical examples, we examined the impact of heterogenous beliefs on the equilibrium risk-free rate, the equity risk premium and option prices. By static analysis, given that agents’ wealth shares are equal and agree on the probability, the market expects a lower (higher) future return when the divergence of opinions are greater regarding future stock return in upstate (downstate), the risk-free is negatively (positively) related to divergence of opinion about future return in the upstate (downstate) and risk premium is negatively related to divergence of opinions in general. Dynamically, we found that the consensus belief is a fair belief in that every agent’s wealth share is a martingale under the consensus belief. Agents’ subjective beliefs are not fair unless their perceived present value of a dollar payoff by the stock in the next period is equal to the zero-coupon bond price, agent i’s wealth share process is a sub-martingale under his/her own subjective belief and expected to grow in the future. Options prices calculated under the consensus belief are called fair prices. We demonstrate that the implied volatilities of fair call prices can exhibit “volatility skew” observed in real data when agents disagree only on the volatility of of future stock returns or when the optimistic agent is also confident about future stock returns. Finally, our binomial model can take into agents’ disagreements in both model parameters and the model structure. Future avenues of research include incorporating disagreements in model structures into option pricing and compare fair option prices implied by agents’ beliefs and those option prices observed in the market to quantify the level of mispricing. It would be also interesting to extend the current model to include multiple assets including the bond and currency markets.

**APPENDIX A. PROOF OF PROPOSITION 3.2**

We start from equation (3.2) which is given below

\[
\frac{1}{R_f} + \sum_i w_i(t) \left( \frac{p_i}{d_i} + \frac{1 - p_i}{\bar{u}_i} \right) = 0
\]

On the one hand, if every investor has identical belief about the future return in the upstate and downstate respectively in the period \([t, t + 1]\), i.e \(p_i = p_m, \bar{u}_i = \bar{u}_m\) and \(d_i = d_m\) for all \(i\). Then it is obvious that equation (3.2) becomes

---

4Analyst forecasts can act as a proxy for agents’ belief about future asset returns, the risky asset can be taken to be a stock index.
\[
\frac{1}{R_f} + \frac{p_m}{d_m} + \frac{1-p_m}{u_m} = 0 \quad (A.1)
\]
Solving equation (A.1) for \( R_f \) leads to relationship between the consensus belief and the the risk-free rate in (3.6). Next it is obvious that in order for \( B_m(t) \) to be the consensus belief, the following must hold in every period \([t, t+1] \) for \( t = 0, 1, \ldots, T \),
\[
\begin{align*}
\frac{p_m}{d_m} &= \sum_i w_i \frac{p_i}{d_i}, \\
\frac{p_m}{u_m} &= \sum_i w_i \frac{1-p_i}{u_i},
\end{align*} \quad (A.2) \quad (A.3)
\]
which leads to equations (3.4) and (3.5). When investors agree on the future return in each state, then the consensus belief must reflect this common belief. This means that \( u_i = u_o \Rightarrow u_m = u_o \) and \( d_i = d_o \Rightarrow d_m = d_o \). This fact gives us the expression for the consensus probability belief \( p_m = \sum_i w_i p_i \).

To prove (iv), we simply substitute equation (3.9) into the right hand side of equation (3.8), then we find that under the belief of a particular agent \( i \),
\[
[S(t)/R_f(t)](q_{i,u}(t) \ u_i(t) + q_{i,d}(t) \ d_i(t)) = S(t)
\]
Since \( q_{i,u}(t) \ u_i(t) + q_{i,d}(t) \ d_i(t) = R_f(t) \), we find that the relation hold for all agent \( i \), hence it must also hold for the consensus investor \( m \).

**APPENDIX B. PROOF OF PROPOSITION 3.4**

The wealth for individual \( i \) at time \( t+1 \) is given by
\[
W_i(t+1) = W_i(t)(\omega_i(t) \ R(t+1) + (1-\omega_i(t)) \ R_f)
\]
Dividing the aggregate market wealth \( W_m(t+1) \) on both side and using the fact that \( W_m(t+1) = W_m(t)R(t+1) \),
\[
w_i(t+1) = w_i(t) \left( \omega_i(t) + (1-\omega_i(t)) \frac{R_f}{R(t+1)} \right)
\]
. Taking expectation under the consensus belief on both side leads to
\[
\mathbb{E}_t^m \left[ \frac{w_i(t+1)}{w_i(t)} \right] = \omega_i(t) + (1-\omega_i(t))\mathbb{E}_t^m \left[ \frac{R_f}{R(t+1)} \right] \quad (B.1)
\]
By Proposition 3.2 (ii), \( \mathbb{E}_t^m [R_f/R(t+1)] = 1 \) and this completes the proof.
APPENDIX C. PROOF OF PROPOSITION 3.5

Similar to the proof of Proposition 3.4, we have equation (B.1), however, expectation will be taken under agent $i$’s belief,

\[ E_i \left[ \frac{w_i(t+1)}{w_i(t)} \right] = \omega_i(t) + (1 - \omega_i(t)) E_i \left[ \frac{R_f}{R(t+1)} \right]. \tag{C.1} \]

Using Lemma 2.1 to expand above expression leads to

\[ E_i \left[ \frac{w_i(t+1)}{w_i(t)} \right] - 1 = \left[ R_f (p_i \, d_i + (1 - p_i) \, u_i) - u_i \, d_i \right]^2 \geq 0. \]

Equality holds if and only if the numerator is zero, that is

\[ R_f (p_i \, d_i + (1 - p_i) \, u_i) = u_i \, d_i \]

\[ \Rightarrow \frac{p_i}{u_i} \, d_i + \frac{(1 - p_i)}{d_i} = \frac{1}{R_f} \]

Hence we must have \( E_i \left[ \frac{1}{R(t+1)} \right] = E_t \left[ \frac{1}{R(t+1)} \right] = \frac{1}{R_f} \).

APPENDIX D. PROOF OF REMARK 5.3

Since both agents have identical beliefs about the distribution of future asset returns, the consensus belief must coincide with the homogeneous belief, that is

\[ u_m(t) = 1 + \mu \, \Delta + \sigma \, \sqrt{\Delta} \]
\[ d_m(t) = 1 + \mu \, \Delta - \sigma \, \sqrt{\Delta} \]
\[ p_m(t) = 0.5 \]

for \( t = 0, 1, \cdots , T - 1 \). By Proposition 3.2 (ii), the price of a zero-coupon bond is given by

\[ \frac{1}{R_f} = \frac{1}{2} \left[ \frac{1}{1 + \mu \, \Delta + \sigma \, \sqrt{\Delta}} + \frac{1}{1 + \mu \, \Delta - \sigma \, \sqrt{\Delta}} \right] = \frac{1 + \mu \, \Delta}{(1 + \mu \, \Delta)^2 - \sigma^2 \Delta}. \]

Let \( r \) be the continuous compounded risk-free rate, we have from above that

\[ R_f = e^{r \Delta} = 1 + \frac{\mu \Delta - \sigma^2 \Delta}{1 + \mu \, \Delta}. \]

Therefore the instantaneous risk-free rate \( r_f \) is given by

\[ r_f = \lim_{\Delta \to 0} \frac{1}{\Delta} \ln \left[ 1 + \mu \Delta - \frac{\sigma^2 \Delta}{1 + \mu \, \Delta} \right] \]

By applying \( L'Hôpital's \) rule we obtain,

\[ r_f = \mu - \lim_{\Delta \to 0} \frac{\sigma^2}{(1 + \mu \Delta)^2 (1 + \mu \Delta - \frac{\sigma^2 \Delta}{1 + \mu \, \Delta})} = \mu - \sigma^2. \]
REFERENCES


Friedman, M. (1953), The case of flexible exchange rates, Univ. Chicago Press. in Essays in Positive Economics.


