A HOMOLOGICAL CHARACTERIZATION OF KRULL DOMAINS II

FANGGUI WANG AND LEI QIAO[†]

Abstract. In this paper, we introduce a new type of projective modules, called the weak w-projective module. By using this type of modules, we give a homological characterization of Krull domains. More precisely, it is shown that an integral domain is a Krull domain if and only if every submodule of a projective module is weak w-projective.

1. Introduction

Recall that a *Dedekind domain* is an integral domain in which each nonzero ideal is invertible, or equivalently, an integrally closed Noetherian domain of Krull dimension one. It is well known that a Dedekind domain can also be characterized in terms of homological algebra. More precisely, an integral domain is a Dedekind domain if and only if its global dimension is at most one (i.e., every submodule of a projective module over it is projective), if and only if every divisible module over it is injective. Thus, a natural question arises whether there exists the corresponding homological characterizations for Krull domains. In this direction, Nishi and Shinagawa showed in [10, Proposition 24] that for a module M over a Krull domain, M is injective if and only if it is divisorial and divisible. Also, they remarked that the property that every divisible divisorial module is injective can not characterize a Krull domain (see [10, Remark 6]). Recently, El Baghdadi et al. [2] have shown that the semi-divisorial (or w-module) concept is more suitable for a characterization of Krull domains by means of the property that divisibility implies injectivity. Namely, they prove that an integral domain is a Krull domain if and only if every divisible w-module over it is injective (see [2, Theorem 2.6]).

Now, we review some terminology related to the w-modules over commutative rings with zero divisors. All rings considered in this paper are assumed to be commutative and to have an identity element; in particular, R denotes such a ring. In the integral domain case, w-modules were called *semi-divisorial modules* in [5] and (in the ideal case) F_∞*-ideals* in [6], which have proved to be useful in the study of multiplicative ideal theory and module theory. In [20], the notion of w-modules was generalized to the ring with zero divisors. An ideal J of R is called a *Glaz-Vasconcelos ideal* (a *GV-ideal* for short) if J is finitely generated and the natural

Date: September 4, 2018.

Key Words: Krull domain; strong w-module; weak w-projective module; weak w-projective dimension; global weak w-projective dimension.

2010 Mathematics Subject Classification: 13F05; 13D07; 13D05; 13D30.

† Address correspondence to Dr. Lei Qiao, School of Mathematics and Software Science, Sichuan Normal University, Chengdu, Sichuan 610066, China; E-mail: lqiao@sicnu.edu.cn.

homomorphism $\varphi : R \to J^* = \mathrm{Hom}_R(J, R)$ is an isomorphism. Note that the set $\mathrm{GV}(R)$ of GV-ideals of R is a multiplicative system of ideals of R. Let M be an R-module. Define

$$\mathrm{tor}_{\mathrm{GV}}(M) := \{x \in M \mid Jx = 0 \text{ for some } J \in \mathrm{GV}(R)\}.$$

Thus $\mathrm{tor}_{\mathrm{GV}}(M)$ is a submodule of M. Now M is said to be *GV-torsion* (resp., *GV-torsionfree*) if $\mathrm{tor}_{\mathrm{GV}}(M) = M$ (resp., $\mathrm{tor}_{\mathrm{GV}}(M) = 0$). A GV-torsionfree module M is called a *w-module* if $\mathrm{Ext}^1_R(R/J, M) = 0$ for all $J \in \mathrm{GV}(R)$. Then projective modules and reflexive modules are both w-modules. In [14, Theorem 6.7.24], it was shown that all flat modules are w-modules. Also it is known that an GV-torsionfree R-module M is a w-module if and only if $\mathrm{Ext}^1_R(N, M) = 0$ for every GV-torsion R-module N (see [14, Theorem 6.2.7]). Let w-$\mathrm{Max}(R)$ denote the set of w-ideals of R maximal among proper integral w-ideals of R and we call $\mathfrak{m} \in w$-$\mathrm{Max}(R)$ a maximal w-ideal of R. Then every proper w-ideal is contained in a maximal w-ideal and every maximal w-ideal is a prime ideal. For any GV-torsionfree module M,

$$M_w := \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in \mathrm{GV}(R)\}$$

is a w-submodule of $E(M)$ containing M and is called the *w-envelope* of M, where $E(M)$ denotes the injective envelope of M. It is clear that a GV-torsionfree module M is a w-module if and only if $M_w = M$. It is worthwhile to point out that from a torsion-theoretic point of view, the notion of w-modules coincides with that of $\mathrm{tor}_{\mathrm{GV}}$-closed (i.e., $\mathrm{tor}_{\mathrm{GV}}$-torsionfree and $\mathrm{tor}_{\mathrm{GV}}$-injective) modules, where the torsion theory $\mathrm{tor}_{\mathrm{GV}}$ whose torsion modules are the GV-torsion modules and the torsionfree modules are the GV-torsionfree modules.

In a very recent paper [13], the first named author and Zhou have given a new homological characterization of Krull domains in terms of w-projective modules. The notion of w-projective modules appeared first in [15] when R is an integral domain and was extended to an arbitrary commutative ring in [19]. Recall that an R-module M is said to be a *w-projective module* if $\mathrm{Ext}^1_R(L(M), N)$ is a GV-torsion module for any torsionfree w-module N, where $L(M) = (M/\mathrm{tor}_{\mathrm{GV}}(M))_w$. Then it is shown that an integral domain R is a Krull domain if and only if every submodule of a finitely generated projective R-module is w-projective (see [13, Theorem 3.3]). However, they do not know whether the property that every submodule of an arbitrary projective module is w-projective can also characterize a Krull domain, and this question motivates us to seek an exact characterization of Krull domains which is similar to that of Dedekind domains.

In the present paper, we first introduce and study a new type of projective modules, called the weak w-projective module (see Section 2). Then we discuss, in Section 3, the weak w-projective dimension of modules and rings. Finally, in the last section, it is proved that an integral domain R is a Krull domain if and only if every submodule of a projective module is weak w-projective (see Theorem 4.3). In other words, we show that Krull domains are exactly the integral domains of global weak w-projective dimension at most one.

Any undefined notions or notation are standard, as in [11, 14, 12].

2. Strong w-modules and weak w-projective modules

Before we introduce the notion of weak w-projective modules we need to prepare a little. First, we introduce the concept of strong w-modules.

Definition. An GV-torsionfree module M is said to be a *strong w-module* if $\mathrm{Ext}_R^i(N, M) = 0$ for each integer $i \geqslant 1$ and for all GV-torsion modules N.

Clearly all strong w-modules are w-modules. But the converse is not true in general. For example, let R be a two-dimensional regular local ring with the maximal ideal $\mathfrak{m}$. Then it is easy to check that $\mathfrak{m}$ is a GV-ideal and that $\mathrm{Ext}_R^2(R/\mathfrak{m}, R) \neq 0$. So R, as a module over itself, is a w-module but not a strong w-module. Moreover, one can see that all GV-torsionfree injective modules are strong w-modules and that the class of all strong w-modules is closed under direct products.

Recall from [14] that a sequence $A \to B \to C$ of R-modules and R-homomorphisms is called a *w-exact sequence* if the sequence $A_\mathfrak{m} \to B_\mathfrak{m} \to C_\mathfrak{m}$ is exact over $R_\mathfrak{m}$ for any maximal w-ideal $\mathfrak{m}$ of R. The following basic facts on w-exact sequences may be found in [14, Proposition 6.3.4].

(i) A sequence $0 \to A \xrightarrow{f} B$ is w-exact if and only if $\ker(f)$ is GV-torsion.
(ii) A sequence $B \xrightarrow{g} C \to 0$ is w-exact if and only if $\mathrm{coker}(g)$ is GV-torsion.
(iii) A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is w-exact if and only if both $(\mathrm{im}(f)+\ker(g))/\mathrm{im}(f)$ and $(\mathrm{im}(f) + \ker(g))/\ker(g)$ are GV-torsion.

Next, we give some basic properties of strong w-modules that we shall use in the sequal.

Lemma 2.1. *Let $0 \to L \xrightarrow{f} F \xrightarrow{g} M \to 0$ be a w-exact sequence of R-modules and let N be a strong w-module over R. Then there is a long exact sequence of R-modules*

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_R(F, N) \longrightarrow \mathrm{Hom}_R(L, N) \\ &\longrightarrow \mathrm{Ext}_R^1(M, N) \longrightarrow \mathrm{Ext}_R^1(F, N) \longrightarrow \mathrm{Ext}_R^1(L, N) \longrightarrow \cdots \\ &\longrightarrow \mathrm{Ext}_R^n(M, N) \longrightarrow \mathrm{Ext}_R^n(F, N) \longrightarrow \mathrm{Ext}_R^n(L, N) \\ &\longrightarrow \mathrm{Ext}_R^{n+1}(M, N) \longrightarrow \cdots \end{aligned}$$

Proof. Set

$$A = \ker(f), \quad B_1 = \mathrm{im}(f), \quad B_2 = \ker(g), \quad C_1 = \mathrm{im}(g), \quad C_2 = \mathrm{coker}(g).$$

Then A, C_2, $(B_1 + B_2)/B_1$, and $(B_1 + B_2)/B_2$ are all GV-torsion modules. For any integer $k \geqslant 0$, since the sequence $0 \to A \to L \to B_1 \to 0$ is exact with A GV-torsion and N is a strong w-module, we have

$$\mathrm{Ext}_R^k(B_1, N) \cong \mathrm{Ext}_R^k(L, N).$$

Similarly, since $0 \to B_1 \to B_1 + B_2 \to B_1 + B_2/B_1 \to 0$ is an exact sequence with $(B_1 + B_2)/B_1$ GV-torsion,

$$\mathrm{Ext}_R^k(B_1 + B_2, N) \cong \mathrm{Ext}_R^k(B_1, N).$$

By the same argument,

$$\mathrm{Ext}_R^k(B_1 + B_2, N) \cong \mathrm{Ext}_R^k(B_2, N) \text{ and } \mathrm{Ext}_R^k(M, N) \cong \mathrm{Ext}_R^k(C_1, N).$$

Now by applying the functor $\mathrm{Hom}_R(-,N)$ to the exact sequence $0 \to B_2 \to F \to C_1 \to 0$, we obtain a long exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_R(C_1,N) \longrightarrow \mathrm{Hom}_R(F,N) \longrightarrow \mathrm{Hom}_R(B_2,N) \\ &\longrightarrow \mathrm{Ext}^1_R(C_1,N) \longrightarrow \mathrm{Ext}^1_R(F,N) \longrightarrow \mathrm{Ext}^1_R(B_2,N) \longrightarrow \cdots \\ &\longrightarrow \mathrm{Ext}^n_R(C_1,N) \longrightarrow \mathrm{Ext}^n_R(F,N) \longrightarrow \mathrm{Ext}^1_R(B_2,N) \\ &\longrightarrow \mathrm{Ext}^{n+1}_R(C_1,N) \longrightarrow \cdots \end{aligned}$$

Thus, by replacing $\mathrm{Ext}^k_R(C_1,N)$ with $\mathrm{Ext}^k_R(M,N)$, and $\mathrm{Ext}^k_R(B_2,N)$ with $\mathrm{Ext}^k_R(L,N)$ respectively in the long exact sequence above, where $k \geqslant 0$, we achieve the desired long sequence. $\square$

Let M and N be R-modules and let $f : M \to N$ be a homomorphism. Following [14], we say that f is a *w-monomorphism* (resp., *w-epimorphism, w-isomorphism*) if $f_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is a monomorphism (resp., an epimorphism, an isomorphism) for any maximal w-ideal $\mathfrak{m}$ of R.

Proposition 2.2.

(1) *Let N be a strong w-module and let $f : M \to M'$ be a w-isomorphism of R-modules. Then $\mathrm{Ext}^k_R(M,N) \cong \mathrm{Ext}^k_R(M',N)$ for all integers $k \geqslant 1$.*

(2) *Let $0 \to A \to B \to C \to 0$ be an exact sequence of R-modules where A is a strong w-module. Then B is a strong w-module if and only if so is C.*

Proof. (1) The proof is clear by applying Lemma 2.1 to the w-exact sequence $0 \to 0 \to M \to M' \to 0$.

(2) Let T be a GV-torsion R-module and let $k \geqslant 1$ be an integer. Then we consider the following two exact sequences of R-modules

$$\mathrm{Hom}_R(T,A) \to \mathrm{Hom}_R(T,B) \to \mathrm{Hom}_R(T,C) \to \mathrm{Ext}^1_R(T,A)$$

and

$$\mathrm{Ext}^k_R(T,A) \to \mathrm{Ext}^k_R(T,B) \to \mathrm{Ext}^k_R(T,C) \to \mathrm{Ext}^{k+1}_R(T,A).$$

Since A is a strong w-module, we obtain

$$\mathrm{Hom}_R(T,B) \cong \mathrm{Hom}_R(T,C) \quad \text{and} \quad \mathrm{Ext}^k_R(T,B) \cong \mathrm{Ext}^k_R(T,C).$$

Hence it follows that B is a strong w-module if and only if so is C. $\square$

Now we introduce a class of strong w-modules, which will be used to define the weak w-projective. Throughout this paper, $\mathcal{P}^\dagger_w$ denote the class of GV-torsionfree R-modules N with the property that $\mathrm{Ext}^k_R(M,N) = 0$ for all w-projective R-modules M and for all integers $k \geqslant 1$. Clearly, every GV-torsionfree injective R-module belongs to $\mathcal{P}^\dagger_w$.

Proposition 2.3.

(1) *Let $\{N_i\}_{i\in\Gamma}$ be a family of GV-torsionfree R-modules. Then $\prod_{i\in\Gamma} N_i \in \mathcal{P}^\dagger_w$ if and only if $N_i \in \mathcal{P}^\dagger_w$ for each $i \in \Gamma$.*

(2) *If $N \in \mathcal{P}^\dagger_w$, then $\mathrm{Ext}^k_R(T,N) = 0$ for all GV-torsion R-modules T and for all integers $k \geqslant 1$. Hence, every module in $\mathcal{P}^\dagger_w$ is a strong w-module.*

(3) *Let $0 \to A \to B \to C \to 0$ be an exact sequence of R-modules with $A \in \mathcal{P}^\dagger_w$. Then $B \in \mathcal{P}^\dagger_w$ if and only if so is C.*

Proof. (1) This follows easily from [11, Theorem 7.14] and the fact that the class of GV-torsionfree modules is closed under direct products.

(2) Since all GV-torsion module are w-projective, the proof is obvious.

(3) Let $A \in \mathcal{P}_w^\dagger$. Then for each GV-torsion R-module T, there exists an exact sequence of R-modules

$$0 = \mathrm{Hom}_R(T, A) \to \mathrm{Hom}_R(T, B) \to \mathrm{Hom}_R(T, C) \to \mathrm{Ext}_R^1(T, A).$$

By (2), $\mathrm{Ext}_R^1(T, A) = 0$. Therefore, we see that B is a GV-torsionfree module if and only if so is C. Moreover, for each w-projective R-module M and for each integer $k \geqslant 1$, we have

$$0 = \mathrm{Ext}_R^k(M, A) \to \mathrm{Ext}_R^k(M, B) \to \mathrm{Ext}_R^k(M, C) \to \mathrm{Ext}_R^{k+1}(M, A) = 0.$$

Thus, $\mathrm{Ext}^k(M, B) \cong \mathrm{Ext}_R^k(M, C)$, and so (3) is proved. □

To give a class of examples of modules in $\mathcal{P}_w^\dagger$ we recall the definition of w-Nagata rings (see [19]). Let M be an R-module. Write

$$M[x] := R[x] \bigotimes_R M = \left\{ \sum_i u_i x^i \;\middle|\; u_i \in M \right\}.$$

For any $\alpha \in M[x]$, we denote by $c(\alpha)$ the submodule of M generated by the coefficients of α and is called the *content* of α. If A is an $R[x]$-submodule of $M[x]$, then the subset $c(A)$ of all coefficients of elements in A is a submodule of M and is called the *content* of A.

In the following we set

$$S_w := \{f \in R[x] \mid c(f)_w = R\}.$$

It is easy to see that S_w is a multiplicative closed set of $R[x]$. Note that a finitely generated ideal J of R is a GV-ideal if and only if $J_w = R$ (see [20, Proposition 3.5]). From this, we have

$$S_w = \{f \in R[x] \mid c(f) \in \mathrm{GV}(R)\}.$$

For any R-module M, we set

$$R\{x\} := R[x]_{S_w}, \qquad M\{x\} := M[x]_{S_w} = R\{x\} \bigotimes_R M.$$

This type of rings was first introduced and studied by Nagata. So $R\{x\}$ is called a *w-Nagata ring* and $M\{x\}$ a *w-Nagata module.* For the Nagata ring (or Nagata module) relative to an arbitrary hereditary torsion theory, see [7].

Proposition 2.4.

(1) *Every $R\{x\}$-module, as an R-module, is in $\mathcal{P}_w^\dagger$.*
(2) *Let $\mathfrak{p}$ be a prime w-ideal of R. Then every $R_\mathfrak{p}$-module, as an R-module, is in $\mathcal{P}_w^\dagger$.*

Proof. (1) Let N be an $R\{x\}$-module and let M be a GV-torsion R-module. Then there exists an exact sequence of $R\{x\}$-modules $0 \to N \to E \to B \to 0$ with E injective over $R\{x\}$. By the the first and second parts of Exercise 5 on page 360 of [1], it is easy to see that E is also injective over R. Moreover, [14, Theorem 6.6.19(2)] says that N, E, B are all w-modules over R. Hence $\mathrm{Ext}_R^1(M, B) = 0$,

$\mathrm{Ext}_R^1(M,N)=0$ and $\mathrm{Ext}_R^{k+1}(M,N)\cong \mathrm{Ext}_R^k(M,B)$ for all integers $k\geqslant 1$. Thus, one can easily show, by induction, that N is a strong w-module.

Now assume that M is a w-projective R-module. Since the natural homomorphism $M\to L(M)$ is a w-isomorphism, it follows from [19, Proposition 2.3(1)] that $L(M)$ is a w-projective w-module. Thus by Proposition 2.2(1), we may assume without loss of generality that M is a w-module. Therefore, by [14, Theorem 6.7.9], we see that $\mathrm{Ext}_R^1(M,N)$ is a GV-torsion R-module. But note that $\mathrm{Ext}_R^1(M,N)$ is also a $R\{x\}$-module, and so it is a w-module (in particular, a GV-torsionfree module) over R. Consequently, $\mathrm{Ext}_R^1(M,N)=0$. Now the rest of this proof may be followed in much the same way as the previous paragraph.

(2) By using the fact that every $R_{\mathfrak{p}}$-module is a w-module over R (see [14, Proposition 6.2.18]), the proof is the similar to that of (1). □

Definition. Let M be an R-module. Then M is said to be a *weak w-projective module* if $\mathrm{Ext}_R^1(M,N)=0$ for all $N\in\mathcal{P}_w^\dagger$.

Clearly, all projective modules are weak w-projective. However, there is a weak w-projective module that is not projective. For example, let J be a GV-ideal of R with $J\neq R$. Then R/J is a GV-torsion R-module. By Proposition 2.3(2), every GV-torsion module is a weak w-projective module, and so R/J is weak w-projective. But R/J is not projective over R since $J\neq R$. Moreover, it is clear that every w-projective module is weak w-projective.

The next proposition collects some basic properties of weak w-projective modules.

Proposition 2.5.

(1) *The class of all weak w-projective modules is closed under arbitrary direct sums and under direct summands.*
(2) *An R-module M is weak w-projective if and only if* $\mathrm{Ext}_R^k(M,N)=0$ *for all $N\in\mathcal{P}_w^\dagger$ and for all $k>0$.*
(3) *Let $0\to A\to B\to C\to 0$ be a w-exact sequence of R-modules with C weak w-projective. Then A is weak w-projective if and only if so is B.*
(4) *If M is a weak w-projective R-module, then* $\mathrm{Hom}_R(M,N)$ *is a strong w-module for all $N\in\mathcal{P}_w^\dagger$.*

Proof. (1) This follows easily from [11, Theorem 7.13].

(2) Let M be a weak w-projective R-module and $N\in\mathcal{P}_w^\dagger$. Then the case $k=1$ is just the definition of weak w-projective modules. For any positive integer $k>1$, by Proposition 2.3(3), there is an exact sequence of R-modules

$$0\to N\to E_0\to E_1\to\cdots\to E_{k-2}\to C\to 0$$

where $E_0,\ldots,E_{k-2}$ are injective and GV-torsionfree and $C\in\mathcal{P}_w^\dagger$. Thus, we obtain $\mathrm{Ext}_R^k(M,N)\cong\mathrm{Ext}_R^1(M,C)=0$. The converse is trivial.

(3) Let $N\in\mathcal{P}_w^\dagger$. Then by Lemma 2.1 we have the exact sequence

$$\mathrm{Ext}_R^1(C,N)\to\mathrm{Ext}_R^1(B,N)\to\mathrm{Ext}_R^1(A,N)\to\mathrm{Ext}_R^2(C,N).$$

Since C is weak w-projective, $\mathrm{Ext}_R^1(B,N)\cong\mathrm{Ext}_R^1(A,N)$ by (2). It follows that A is weak w-projective if and only if so is B.

(4) Assume that M is a weak w-projective R-module and $N\in\mathcal{P}_w^\dagger$.

Let us first consider the special case when M is free, i.e., $M = \bigoplus_{j\in\Gamma} R$ for some index set Γ. Then

$$\mathrm{Hom}_R(M, N) \cong \prod_{j\in\Gamma} \mathrm{Hom}_R(R, N) \cong \prod_{j\in\Gamma} N.$$

Since $N \in \mathcal{P}_w^\dagger$ (in particular, N is a strong w-module), $\mathrm{Hom}_R(M, N)$ is also a strong w-module.

Now let M be arbitrary and T a GV-torsion R-module. Note that $\mathrm{Hom}_R(M, N)$ is a w-module ([20, Theorem 2.8]), and so $\mathrm{Ext}_R^1(T, \mathrm{Hom}_R(M, N)) = 0$. Now let $k > 0$ be an integer and let $0 \to A \to F \to M \to 0$ be an exact sequence of R-modules with F free. Then the weak w-projectivity of M gives an exact sequence

$$0 \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(F, N) \to \mathrm{Hom}_R(A, N) \to 0.$$

The middle term is a strong w-module as F is free. Therefore, we have

$$\mathrm{Ext}_R^k(T, \mathrm{Hom}_R(M, N)) \cong \mathrm{Ext}_R^{k-1}(T, \mathrm{Hom}_R(A, N)).$$

Note, by (3), that A is also weak w-projective. Thus, by induction on k, one can easily see that $\mathrm{Ext}_R^k(T, \mathrm{Hom}_R(M, N)) = 0$ for all $k > 0$, i.e., $\mathrm{Hom}_R(M, N)$ is a strong w-module. □

Corollary 2.6. *Let $0 \to L \xrightarrow{f} F \xrightarrow{g} M \to 0$ be a w-exact sequence of R-modules with F weak w-projective. Then for all $N \in \mathcal{P}_w^\dagger$ and all integers $k \geqslant 1$,*

$$\mathrm{Ext}_R^k(L, N) \cong \mathrm{Ext}_R^{k+1}(M, N).$$

Proof. This follows immediately from Proposition 2.5(2) and Lemma 2.1. □

Corollary 2.7. *If $f : M \to N$ is a w-isomorphism of R-modules, then M is weak w-projective if and only if so is N.*

Proof. This proof is a consequence of Proposition 2.5(3). □

Proposition 2.8. *If M is a weak w-projective R-module, then:*

(1) *$M\{x\}$ is projective over $R\{x\}$.*
(2) *$M_\mathfrak{p}$ is free over $R_\mathfrak{p}$ for all prime w-ideals $\mathfrak{p}$ of R.*

Proof. Assume that M is a weak w-projective R-module.

(1) Let N be an $R\{x\}$-module and let $0 \to A \to F \to M \to 0$ be an exact sequence of R-modules with F free. Then $N \in \mathcal{P}_w^\dagger$ by Proposition 2.4(1), and so $\mathrm{Ext}_R^1(M, N) = 0$. Now, let us consider the following commutative diagram with exact rows.

$$\begin{tikzcd}
\mathrm{Hom}_{R\{x\}}(F\{x\}, N) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}_{R\{x\}}(A\{x\}, N) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Ext}^1_{R\{x\}}(M\{x\}, N) \arrow[r] \arrow[d] & 0 \\
\mathrm{Hom}_R(F, N) \arrow[r] & \mathrm{Hom}_R(A, N) \arrow[r] & \mathrm{Ext}^1_R(M, N) \arrow[r] & 0
\end{tikzcd}$$

By the Adjoint Isomorphism, the first two vertical maps are isomorphisms. Thus, we have

$$\mathrm{Ext}^1_{R\{x\}}(M\{x\}, N) \cong \mathrm{Ext}^1_R(M, N) = 0.$$

So $M\{x\}$ is projective over $R\{x\}$.

(2) The proof is similar to that of (1). □

Recall that an R-module M is said to be *of finite type* if there is a w-exact sequence of R-modules $F \to M \to 0$, where F is finitely generated free, and to be *of finitely presented type* if there is a w-exact sequence of R-modules $F_1 \to F_0 \to M \to 0$, where F_1 and F_0 are finitely generated free (see [14]). Also recall from [8] that an R-module M is called *w-flat* if $M_\mathfrak{m}$ is flat over $R_\mathfrak{m}$ for all maximal w-ideals $\mathfrak{m}$ of R.

Corollary 2.9. *The following statements are equivalent for an R-module M.*

(1) *M is a weak w-projective module of finite type.*
(2) *M is a w-projective module of finite type.*
(3) *M is a w-flat module of finitely presented type.*

Proof. (1) $\Rightarrow$ (2) Assume that (1) holds. Then it follows from [19, Proposition 3.9(3)] and Proposition 2.8(1) that $M\{x\}$ is finitely generated projective over $R\{x\}$. Hence, by [19, Theorem 3.11], M is w-projective.

(2) $\Rightarrow$ (1) is clear.

(2) $\Leftrightarrow$ (3) This follows from [19, Theorem 2.19] and [14, Theorem 6.7.23]. □

Recall that a nonzero (fractional) ideal I of an integral domain R (with quotient field K) is called *w-invertible* if $(II^{-1})_w = R$, where $I^{-1} = \{r \in K \mid rI \subseteq R\}$. It was proved in [13, Theorem 2.7] that a nonzero (fractional) ideal of an integral domain is w-projective if and only if it is w-invertible. In fact, we also have the following corollary.

Corollary 2.10. *An ideal of an integral domain is weak w-projective if and only if it is w-projective, if and only if it is w-invertible.*

Proof. It suffices by Corollary 2.9 to show that every weak w-projective ideal of an integral domain is of finite type. For this, let I be a weak w-projective ideal of an integral domain R. Then by Proposition 2.8, $I\{x\}$ is projective over $R\{x\}$, and so it is finitely generated. Thus, [19, Proposition 3.9(3)] says that I is of finite type. □

As a consequence of Proposition 2.8(2), we have the following corollary.

Corollary 2.11. *Every weak w-projective module is a w-flat module.*

It was shown in [19, Proposition 2.10] that if I is a nonzero nil ideal of R, then I is never w-projective. In fact, we have:

Proposition 2.12. *If I is a nonzero nil ideal of R, then I is never weak w-projective.*

Proof. By using Proposition 2.8(2), the proof is essentially the same as that given for [19, Proposition 2.10]. □

3. The weak w-projective dimension of modules and rings

We now introduce the notion of weak w-projective dimension as follows.

Definition. If M is an R-module, then $\text{w.}w\text{-pd}_R M \leqslant n$ (w.w-pd abbreviates *weak w-projective dimension*) if there is a w-exact sequence of R-modules

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0, \qquad (\star)$$

where each P_i is a weak w-projective module. The w-exact sequence $(\star)$ is called a *weak w-projective w-resolution* of length n of M. If no such finite w-resolution exists, then $\mathrm{w}.w\text{-pd}_R M = \infty$; otherwise, define $\mathrm{w}.w\text{-pd}_R M = n$ if n is the length of a shortest weak w-projective w-resolution of M.

Clearly, an R-module M is weak w-projective if and only if $\mathrm{w}.w\text{-pd}_R M = 0$, and $\mathrm{w}.w\text{-pd}_R M \leqslant \mathrm{pd}_R M$, where $\mathrm{pd}_R M$ denotes the classical projective dimension of M.

Proposition 3.1. *The following statements are equivalent for an R-module M.*

(1) $\mathrm{w}.w\text{-pd}_R M \leqslant n$.
(2) $\mathrm{Ext}_R^{n+k}(M, N) = 0$ *for all* $N \in \mathcal{P}_w^\dagger$ *and for all* $k > 0$.
(3) $\mathrm{Ext}_R^{n+1}(M, N) = 0$ *for all* $N \in \mathcal{P}_w^\dagger$.
(4) *If* $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ *is an exact sequence, where* $P_0, P_1, \ldots, P_{n-1}$ *are projective R-modules, then* P_n *is weak w-projective.*
(5) *If* $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ *is a w-exact sequence, where* $P_0, P_1, \ldots, P_{n-1}$ *are weak w-projective R-modules, then* P_n *is weak w-projective.*
(6) *If* $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ *is an exact sequence, where* $P_0, P_1, \ldots, P_{n-1}$ *are weak w-projective R-modules, then* P_n *is weak w-projective.*
(7) *If* $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ *is a w-exact sequence, where* $P_0, P_1, \ldots, P_{n-1}$ *are projective R-modules, then* P_n *is weak w-projective.*
(8) *If* $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ *is a w-exact sequence, where* $P_0, P_1, \ldots, P_{n-1}$ *are weak w-projective w-modules over R, then* P_n *is weak w-projective.*

Proof. (1) $\Rightarrow$ (2) We prove (2) by induction on $n \geqslant 0$. For the case $n = 0$, M is a weak w-projective module. Then (2) holds by Proposition 2.5(2). If $n > 0$, then there is a w-exact sequence $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$, where each P_i is a weak w-projective R-module. Set $K_0 = \ker(P_0 \to M)$. Then both $0 \to K_0 \to P_0 \to M \to 0$ and $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to K_0 \to 0$ are w-exact, and $\mathrm{w}.w\text{-pd}_R K_0 \leqslant n-1$. By induction, $\mathrm{Ext}_R^{n-1+k}(K_0, N) = 0$ for all $N \in \mathcal{P}_w^\dagger$ and all $k > 0$. Thus, it follows from Corollary 2.6 that $\mathrm{Ext}_R^{n+k}(M, N) = 0$.

(2) $\Rightarrow$ (3) Trivial.

(3) $\Rightarrow$ (4) Let $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ be an exact sequence of R-modules with $P_0, P_1, \ldots, P_{n-1}$ projective, and write $K_0 = \ker(P_0 \to M)$ and $K_i = \ker(P_i \to P_{i-1})$, where $i = 1, \ldots, n-1$. Then $K_{n-1} = P_n$. Since all $P_0, P_1, \ldots, P_{n-1}$ are projective, $\mathrm{Ext}_R^1(P_n, N) \cong \mathrm{Ext}_R^{n+1}(M, N) = 0$ for all $N \in \mathcal{P}_w^\dagger$. Hence, P_n is a weak w-projective module.

(4) $\Rightarrow$ (1) Obvious.

(3) $\Rightarrow$ (5) Let $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ be a w-exact sequence of R-modules with $P_0, P_1, \ldots, P_{n-1}$ weak w-projective, and set $L_n = P_n$ and $L_i = \mathrm{im}(P_i \to P_{i-1})$, where $i = 1, \ldots, n-1$. Then both $0 \to L_{i+1} \to P_i \to L_i \to 0$ and $0 \to L_1 \to P_0 \to M \to 0$ are w-exact sequences. By using Corollary 2.6 repeatedly, we will see that

$$\mathrm{Ext}_R^1(P_n, N) \cong \mathrm{Ext}_R^{n+1}(M, N) = 0$$

for all $N \in \mathcal{P}_w^\dagger$. Thus, P_n is a weak w-projective module.

(5) $\Rightarrow$ (6) $\Rightarrow$ (4) and (5) $\Rightarrow$ (8) $\Rightarrow$ (7) $\Rightarrow$ (4) are obvious. $\square$

Definition. The *global weak w-projective dimension* of a ring R is defined by

$$\text{gl.w.}w\text{-dim}(R) = \sup\{\text{w.}w\text{-pd}_R M \mid M \text{ is an } R\text{-module}\}.$$

Proposition 3.2. *The following statements are equivalent for R.*

1. $\text{w.}w\text{-pd}_R M \leqslant n$ *for all R-modules M, that is,* $\text{gl.w.}w\text{-dim}(R) \leqslant n$.
2. $\text{w.}w\text{-pd}_R R/I \leqslant n$ *for all ideals I of R.*
3. $\text{id}_R N \leqslant n$ *for all $N \in \mathcal{P}_w^\dagger$.*

Consequently, the global weak w-projective dimension of R is also determined by the formulas:

$$\begin{aligned} \text{gl.w.}w\text{-dim}(R) &= \sup\{\text{w.}w\text{-pd}_R R/I \mid I \textit{ is an ideal of } R\} \\ &= \sup\left\{\text{id}_R N \mid N \in \mathcal{P}_w^\dagger\right\} \end{aligned}$$

Proof. (1) $\Rightarrow$ (2) Trivial.

(2) $\Rightarrow$ (3) Let $N \in \mathcal{P}_w^\dagger$. Then by Proposition 2.3(3), there exists an exact sequence of R-modules

$$0 \to N \to E_0 \to E_1 \to \cdots \to E_{n-1} \to E_n \to 0$$

where $E_0, E_1, \dots, E_{n-1}$ are GV-torsionfree and injective and $E_n \in \mathcal{P}_w^\dagger$. Thus, it follows from (2) that $\text{Ext}_R^1(R/I, E_n) \cong \text{Ext}_R^{n+1}(R/I, N) = 0$, and so E_n is injective. Therefore, $\text{id}_R N \leqslant n$.

(3) $\Rightarrow$ (1) Let M be an R-module and let

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

be an exact sequence of R-modules, where $P_0, P_1, \dots, P_{n-1}$ are projective. Then for each $N \in \mathcal{P}_w^\dagger$, $\text{Ext}_R^1(P_n, N) \cong \text{Ext}_R^{n+1}(M, N) = 0$, and so P_n is weak w-projective. Hence, $\text{w.}w\text{-pd}_R M \leqslant n$. $\square$

Following [18], for an R-module M, we denote by $w\text{-fd}_R M$ the w-flat dimension of M, and we use the notation $w\text{-w.gl.dim}(R)$ to denote the w-weak global dimension of R. As a consequence of Corollary 2.11, we have the following proposition.

Proposition 3.3. *Let M be an R-module. Then $w\text{-fd}_R M \leqslant \text{w.}w\text{-pd}_R M$. Consequently, $w\text{-w.gl.dim}(R) \leqslant \text{gl.w.}w\text{-dim}(R)$.*

Recall from [14] that a ring R is said to be *w-coherent* if every finite type ideal of R is of finitely presented type, and is said to be *w-Noetherian* if every ideal of R is of finite type. The w-Noetherian domain is also called the strong Mori domain (for short, SM domain) in [16].

Proposition 3.4. *Let R be a w-coherent ring and let M be an R-module of finitely presented type. Then*

1. *There is a w-exact sequence of R-modules*

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0,$$

where all P_i are finitely generated projective.

2. $w\text{-fd}_R M = \text{w.}w\text{-pd}_R M$.

Proof. (1) Since M is of finitely presented type, there exists a w-exact sequence of R-modules $P_0 \xrightarrow{f} M \to 0$ with P_0 finitely generated projective. Hence, the w-coherence of R implies that $A := \ker(f)$ is of finitely presented type. Thus, continuing in this way, we can obtain the desired w-exact sequence.

(2) It follows from (1), Corollary 2.9 and Proposition 3.3. □

If we denote the classical global dimension of a ring R by gl.dim(R), then:

Proposition 3.5.

(1) *If R is a w-coherent ring, then*

$$w\text{-w.gl.dim}(R) = \sup\{\text{w.}w\text{-pd}_R M\}$$

where M runs over all R-modules of finitely presented type.

(2) *If R is a w-Noetherian ring, then*

$$\text{gl.w.}w\text{-dim}(R) = w\text{-w.gl.dim}(R) = \text{gl.dim}(R\{x\}).$$

Proof. (1) Immediate from [18, Proposition 3.3] and Proposition 3.4.

(2) By (1) and Proposition 3.2, we have gl.w.w-dim$(R) = w$-w.gl.dim(R). Since R is w-Noetherian, $R\{x\}$ is Noetherian (see [14, Theorem 6.8.8]). Hence, it follows from [18, Proposition] and [14, Corollary 3.9.6] that

$$\text{gl.w.}w\text{-dim}(R) = w\text{-w.gl.dim}(R) = \text{w.gl.dim}(R\{x\}) = \text{gl.dim}(R\{x\}).$$

□

Proposition 3.6. *If R is a SM domain which is not a field, then*

$$\text{gl.w.}w\text{-dim}(R[x_1, \ldots, x_n]) = \text{gl.w.}w\text{-dim}(R).$$

Proof. This follows from Proposition 3.5(2), [17, Theorem 1.13] and [18, Theorem 4.7]. □

4. Rings with global weak w-projective dimension less than or equal to one

Proposition 4.1. *The following statements are equivalent for a ring R.*

(1) *R is semisimple.*
(2) *Every cyclic R-module is projective.*
(3) *Every R-module is weak w-projective, that is,* gl.w.w-dim$(R) = 0$.

Proof. By using Proposition 2.9, the proof is essentially the same as that given for [19, Theorem 3.15]. □

Throughout the rest of this article, R will denote an integral domain. Recall that an R-module D is called *h-divisible* if it is an epic image of an injective R-module. Similarly, we can define the divisible module relative to $\mathcal{P}_w^\dagger$ as follows:

Definition. An R-module D is said to be *$\mathcal{P}_w^\dagger$-divisible* if it is isomorphic to E/N where E is a GV-torsinfree injective R-module and $N \in \mathcal{P}_w^\dagger$ is a submodule of E.

It is obvious that every $\mathcal{P}_w^\dagger$-divisible R-module is h-divisible. Moreover, it follows from Proposition 2.3(3) that every $\mathcal{P}_w^\dagger$-divisible R-module is in $\mathcal{P}_w^\dagger$.

It was shown in [4, VII, Proposition 2.5] that an R-module M has projective dimension at most one if and only if $\mathrm{Ext}_R^1(M,D)=0$ for all h-divisible R-modules D. Our next proposition is a w-theoretic analogue of this result.

Proposition 4.2. *Let M be an R-module. Then* $\mathrm{w}.w\text{-}\mathrm{pd}_R M \leqslant 1$ *if and only if* $\mathrm{Ext}_R^1(M,D)=0$ *for all $\mathcal{P}_w^\dagger$-divisible R-modules D.*

Proof. Let $\mathrm{w}.w\text{-}\mathrm{pd}_R M \leqslant 1$ and let D be a $\mathcal{P}_w^\dagger$-divisible R-module. Then there is an exact sequence of R-modules $0 \to N \to E \to D \to 0$ with E injective and $N \in \mathcal{P}_w^\dagger$. Hence, $\mathrm{Ext}_R^1(M,D) \cong \mathrm{Ext}_R^2(M,N)=0$.

Conversely, let $N \in \mathcal{P}_w^\dagger$. Then there exists an exact sequence of R-modules $0 \to N \to E \to D \to 0$ with E GV-torsionfree injective. Therefore, D is $\mathcal{P}_w^\dagger$-divisible, and so $\mathrm{Ext}_R^2(M,N) \cong \mathrm{Ext}_R^1(M,D)=0$. Thus, $\mathrm{w}.w\text{-}\mathrm{pd}_R M \leqslant 1$. □

Theorem 4.3. *The following statements are equivalent for an integral domain R.*

(1) *R is a Krull domain.*
(2) *Every divisible w-module over R is injective.*
(3) *Every h-divisible w-module over R is injective.*
(4) *Every divisible strong w-module over R is injective.*
(5) *Every h-divisible strong w-module over R is injective.*
(6) *Every $\mathcal{P}_w^\dagger$-divisible R-module is injective.*
(7) *Every submodule of a finitely generated projective R-module is w-projective.*
(8) *Every submodule of a projective R-module is weak w-projective.*
(9) *Every w-submodule of a projective R-module is weak w-projective.*
(10) *Every submodule of a weak w-projective R-module is weak w-projective.*
(11) $\mathrm{gl.w}.w\text{-}\mathrm{dim}(R) \leqslant 1$.
(12) $\mathrm{gl.dim}(R\{x\}) \leqslant 1$, *i.e., $R\{x\}$ is a Dedekind domain.*

Proof. (1) ⇔ (2) ⇔ (3) See [2, Theorem 2.6].

(1) ⇔ (7) See [13, Theorem 3.3].

(8) ⇔ (9) ⇔ (10) ⇔ (11) This follows easily from Proposition 3.1.

(2) ⇒ (4) ⇒ (5) ⇒ (6) Trivial.

(1) ⇔ (12) See [3, Corollary 4.19].

(6) ⇒ (11) Let M be an R-module and D a $\mathcal{P}_w^\dagger$-divisible R-module. Then by (6), D is injective, and so $\mathrm{Ext}_R^1(M,D)=0$. Hence, Proposition 4.2 says that $\mathrm{w}.w\text{-}\mathrm{pd}_R M \leqslant 1$. Thus, (11) holds.

(8) ⇒ (1) Let I be a nonzero ideal of R. Then by (8), I is weak w-projective. Therefore, Corollary 2.10 implies that I is w-invertible. Thus, the proof completed by the fact that a domain is a Krull domain if and only if every nonzero ideal over it is w-invertible (see [16, Theorem 5.4]). □

To close this section, we give an example of a weak w-projective module that is not w-projective. However, the following lemma is needed.

Lemma 4.4. *Let M be a GV-torisonfree R-module. Then M is projective if and only if $\mathrm{Ext}_R^1(M,N)=0$ for all w-modules N over R.*

Proof. Assume that M is a GV-torsionfree R-module such that $\mathrm{Ext}_R^1(M, N) = 0$ for all w-modules N over R. Then there exists an exact sequence of R-modules

$$0 \to K \to P \to M \to 0$$

where P is a projective module. Since M is GV-torsionfree, K is a w-module, and so $\mathrm{Ext}_R^1(M, K) = 0$. But [11, Theorem 7.11] says that the exact sequence is split. Hence, M is projective. The converse is clear. □

We now offer the promised example. In fact, this example also shows that not all submodules of a projective module over a Krull domain are w-projective.

Example 4.5. Let F be an uncountable field and $R = F[x, y]$ a polynomial ring over F in two indeterminates x and y. Let Q be the quotient field of R. Then by [9, Theorem 2], $\mathrm{pd}_R Q = 2$. Let us consider the following exact sequence of R-modules

$$0 \to K \to P \to Q \to 0$$

with P a projective module. Then K is not projective. Since R is a unique factorization domain (of course, it is a Krull domain), K is weak w-projective by Theorem 4.3. Also note that K is a w-module because Q is GV-torsionfree as an R-module. Next we show that K is not w-projective, either. Indeed, if not, then it follows from [14, Theorem 6.7.9] that

$$\mathrm{Ext}_R^1(K, N) \cong \mathrm{Ext}_R^2(Q, N)$$

is a GV-torsion R-module for all w-modules N over R. But note that $\mathrm{Ext}_R^2(Q, N)$ is also a Q-module, and so it is GV-torsionfree over R. Thus, $\mathrm{Ext}_R^1(K, N) = 0$. However, Lemma 4.4 implies that K is projective, which is a contradiction.

Acknowledgments

The authors would like to thank the referee for suggestions that resulted in several significant improvements. This work was partially supported by NSFC (Nos. 11671283 and 11701398), NSFC (Tianyuan Fund for Mathematics, No. 11626162), the Scientific Research Fund of Sichuan Normal University (No. 16QP02), and the Scientific Research Fund of Sichuan Provincial Education Department (No. 17ZB0362).

References

[1] H. Cartan and S. Eilenberg, *Homological algebra*, (Princeton University Press, Princeton, 1956).

[2] S. El Baghdadi, H. Kim and F. Wang, Divisible is injective over Krull domains, *Comm. Algebra* **44** (2016) 4294–4301.

[3] S. El Baghdadi, M. Fontana, G. Picozza, Semistar Dedekind domains. *J. Pure Appl. Algebra* **193** (2004) 27–60.

[4] L. Fuchs and L. Salce, *Modules over non-Noetherian domains*, Mathematical Surveys and Monographs, Vol. 84 (American Mathematical Society, Providence, RI, 2001).

[5] S. Glaz and W. V. Vasconcelos, Flat ideals II. *Manuscripta Math.* **22** (1977) 325–341.

[6] J. R. Hedstrom and E. G. Houston, Some remarks on star-operations, *J. Pure Appl. Algebra* **18** (1980) 37–44.

[7] P. Jara, Nagata rings, *Front. Math. China* **10** (2015) 91–110.

[8] H. Kim and F. Wang, On LCM-stable modules, *J. Algebra Appl.* **13** (2014) 1350133, 18 pp.

[9] I. Kaplansky, The homological dimension of a quotient field, *Nagoya Math. J.* **27** (1966) 139–142.

[10] M. Nishi and M. Shinagawa, Codivisorial and divisorial modules over completely integrally closed domains (I), *Hiroshima Math. J.* **5** (1975) 269–292.

[11] J. J. Rotman, *An Introduction to Homological Algebra*, (Academic Press, New York, 1979).

[12] B. Stenström, *Rings of Quotients*, (Springer-Verlag, Berlin, 1975).

[13] F. Wang and D. Zhou, A homological characterization of Krull domains, *Bull. Korean Math. Soc.* **55** (2018) 649–657.

[14] F. Wang and H. Kim, *Foundations of Commutative Rings and Their Modules*, Algebra and Applications, Vol. 22 (Springer, Singapore, 2016).

[15] F. Wang, On w-projective modules and w-flat modules, *Algebra Colloq.* **4** (1997) 111–120.

[16] F. Wang and R. L. McCasland, On w-modules over strong Mori domains. *Comm. Algebra* **25** (1997) 1285–1306.

[17] F. Wang and R. L. McCasland, On strong Mori domains. *J. Pure Appl. Algebra* **135** (1999) 155–165.

[18] F. Wang and L. Qiao, The w-weak global dimension of commutative rings, *Bull. Korean Math. Soc.* **52** (2015) 1327–1338.

[19] F. Wang and H. Kim, Two generalizations of projective modules and their applications, *J. Pure Appl. Algebra* **219** (2015) 2099–2123.

[20] H. Yin, F. Wang, X. Zhu and Y. Chen, w-modules over commutative rings, *J. Korean Math. Soc.* **48** (2011) 207–222.

(Fanggui Wang) School of Mathematics and Software Science, Sichuan Normal University, Chengdu, Sichuan 610066, China

E-mail address: `wangfg2004@163.com`

(Lei Qiao) School of Mathematics and Software Science, Sichuan Normal University, Chengdu, Sichuan 610066, China

E-mail address: `lqiao@sicnu.edu.cn`