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Singularities Removed from the Lagrangian for Electromagnetic Fields when Sources of the Field Include Point Charges

Abstract

The textbooks that I learned from on classical mechanics have an unconvincing way of dealing with singularities in a Lagrangian describing an electromagnetic field, or both the field and particles in the field, when the particles are point sources. There probably are textbooks that give better treatments, and certainly somewhere in the history of literature there is a better treatment, but not in anything that I studied. Readers that have also not yet studied literature that gives a better treatment can find that treatment here. The method used here also produces a conformity between the Lagrangian for the field and the Lagrangian for the motion of particles in the field that facilitates the construction of a single Lagrangian for both particles and fields.

1. Introduction

The textbooks that I learned from on classical mechanics have an unconvincing way of dealing with singularities in a Lagrangian describing an electromagnetic field, or both the field and particles in the field, when the particles are point sources. There probably are textbooks that give better treatments, and certainly somewhere in the literature there is a better treatment, but not in anything that I studied. However, it turned out to be easier to figure out how to deal with this than perform a literature search for a better treatment. Readers that have also not yet studied literature that gives a better treatment can find that treatment here.

An illustration of the difficulty is one of the terms (shown in the text below) in the Lagrangian in the classical treatment. This term is a spatial volume integral of charge density multiplied by the scalar potential. The unconvincing treatment treats the charge density of a point charge in this integral as a Dirac delta function. The difficulty is seen by imagining the particle to have a greater than zero size. It becomes a point by taking the limit as the size approaches zero but we will postpone that limit until later. The potential includes the potential produced by this particle and is not nearly constant in the interior of the particle. Regardless of how small the particle is, the potential is not nearly constant in its interior so a delta function representation of the charge density used in this integral is invalid regardless of how small the particle is. Worse yet, the integral becomes infinite when taking the limit as the particle size shrinks to zero. However, consider another integral which is the spatial volume integral of the particle charge density multiplied by the external scalar potential, where external potential is created by all sources except the particle being investigated. The particle charge density can be represented by a delta function in an integral of charge density times external potential if the particle dimensions are much smaller than the spacing between particles, because the external potential is nearly constant in the particle interior. The unconvincing argument is to evaluate the original integral by replacing total potential by external potential, so that the delta function representation of the particle charge density can be used, without giving any justification as to why this replacement can be made. The goal of this report is to justify that replacement. This is done by subtracting from the Lagrangian the terms that becomes singular as the particle size approaches zero before taking the limit as the particle size approaches zero, and verifying that the new Lagrangian (which does not become singular in the limit) produces the same equations as the original Lagrangian that does become singular in the limit.

The method used here also produces a conformity between the Lagrangian for the field and the Lagrangian for the motion of particles in the field that facilitates the construction of a single Lagrangian for both particles and fields. Both kinds of Lagrangians are reviewed here in order to discuss that topic. We start (next section) with a review of the Lagrangian for particle motion. Gaussian units are used here for the electromagnetic field.

2. Lagrangian for Particles in a Given External Electromagnetic Field

We start by reviewing the Lorentz force acting on a particle which is given by

$$\boldsymbol{Force} = q\left[\boldsymbol{E}_E + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}_E \right]_{\boldsymbol{x}=\boldsymbol{X}(t)} \tag{2.1}$$

where q is the particle charge, c is the speed of light in a vacuum, bold face denotes the three-dimensional spatial components of vectors, $\boldsymbol{v}$ is the particle velocity vector, $\boldsymbol{E}$ is the electric field and $\boldsymbol{B}$ is the magnetic field. The subscript E to $\boldsymbol{E}$ and $\boldsymbol{B}$ emphasizes that these are external fields, i.e., do not include fields created by the particle under investigation. The subscript $\boldsymbol{x} = \boldsymbol{X}(t)$ to the square bracket emphasizes that the fields are evaluated at the particle location. The particle coordinates at time t are denoted $\boldsymbol{X}(t)$. This force equation does not include a self-force (the force producing the recoil of a particle from its own emitted electromagnetic radiation) but the self-force is assumed to be negligible throughout this analysis. The force equation can be expressed in terms of the scalar potential φ and vector potential $\boldsymbol{A}$, which are related to the electromagnetic fields according to

$$\boldsymbol{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}, \qquad \boldsymbol{B} = \nabla\times\boldsymbol{A} \tag{2.2}$$

so the force equation becomes

$$\boldsymbol{Force} = q\left[-\nabla\varphi_E - \frac{1}{c}\frac{\partial \boldsymbol{A}_E}{\partial t} + \frac{1}{c}\boldsymbol{v}\times\left(\nabla\times\boldsymbol{A}_E\right) \right]_{\boldsymbol{x}=\boldsymbol{X}(t)} \tag{2.3}$$

where, as before, the subscript E emphasizes external potentials, i.e., do not include contributions from the particle under investigation. A relativistic treatment can be given but we simplify the analysis by using the nonrelativistic treatment that sets the force equal to mass times acceleration, so for a particle having mass m, the vector equation of motion is

$$m\frac{d^2\boldsymbol{X}(t)}{dt^2} = q\left[-\nabla\varphi_E - \frac{1}{c}\frac{\partial \boldsymbol{A}_E}{\partial t} + \frac{1}{c}\boldsymbol{v}\times\left(\nabla\times\boldsymbol{A}_E\right) \right]_{\boldsymbol{x}=\boldsymbol{X}(t)}. \tag{2.4}$$

The Lagrangian for a particle response in a given external electromagnetic field (external to the field created by the particle of interest) is designed so that Lagrange’s equations that vary the particle trajectory, with the fields held as fixed functions of particle location, reproduce the equation of motion (2.4). Such a Lagrangian is almost as easy to write down for a system of particles as for a single particle (because the equations of motion are the same for all particles) but the notation has fewer indices for a single particle, and a single particle produces simpler comparisons with results in a later section, so we consider the response of a single particle to an external electromagnetic field. Standard textbooks (e.g., Goldstein's "Classical Mechanics") give a very satisfactory treatment of this topic so the derivation of the Lagrangian need not be included here because it can be found in standard textbooks. This Lagrangian denoted $L_{particle}$ and applicable to nonrelativistic classical mechanics, is found in standard textbooks to be given by

$$L_{particle} = \frac{1}{2} m v^2 - q \left[\varphi_E - \frac{1}{c} \boldsymbol{v} \circ \boldsymbol{A}_E \right]_{\boldsymbol{x} = \boldsymbol{X}(t)} \tag{2.5}$$

3. Lagrangian for Electromagnetic Fields Created by Given Finite Current Densities and Finite Charge Densities

We start by writing Maxwell's equations in terms of the scalar and vector potentials. Maxwell's homogeneous equations become identities when written this way so the only nontrivial equations are the inhomogeneous equations expressed in terms of the potentials. These equations can be found in any standard textbook and are

$$\nabla^2 \varphi + \nabla \circ \dot{\boldsymbol{A}} = -4\pi \rho, \qquad \nabla^2 \boldsymbol{A} - \ddot{\boldsymbol{A}} - \nabla \left(\dot{\varphi} + \nabla \circ \boldsymbol{A} \right) = -\frac{4\pi}{c} \boldsymbol{j} \tag{3.1}$$

where ρ is the charge density, $\boldsymbol{j}$ is the current density vector, and we also define

$$x^0 \equiv ct, \qquad \dot{\varphi} \equiv \frac{\partial \varphi}{\partial x^0}, \qquad \dot{\boldsymbol{A}} \equiv \frac{\partial \boldsymbol{A}}{\partial x^0}, \qquad \ddot{\boldsymbol{A}} \equiv \frac{\partial \dot{\boldsymbol{A}}}{\partial x^0}. \tag{3.2}$$

The Lagrangian for the fields produced by given current and charge densities is designed so that Lagrange's equations that vary the fields, with current densities and charge densities held as fixed functions of time and space coordinates, reproduce (3.1). Standard textbooks (e.g., Goldstein's "Classical Mechanics") give a very satisfactory treatment of this topic when current densities and charge densities are finite, so the derivation of the Lagrangian need not be included here. This Lagrangian denoted L_{field} is found in standard textbooks to be given by

$$L_{field} = \frac{1}{8\pi} \int \left[\left(\nabla \varphi + \dot{\boldsymbol{A}} \right) \circ \left(\nabla \varphi + \dot{\boldsymbol{A}} \right) - \left(\nabla \times \boldsymbol{A} \right) \circ \left(\nabla \times \boldsymbol{A} \right) \right] d^3x + \int \left[\frac{1}{c} \boldsymbol{j} \circ \boldsymbol{A} - \rho \varphi \right] d^3x \tag{3.3}$$

where integrals denoted by a d^3x are spatial volume integrals.

It is interesting that this Lagrangian reproduces (3.1) when the scalar potential and the three components of the vector potential are given independent variations even though a gauge condition might have been selected so that the four potentials are not really independent. The reason that no constraint between the potentials need be imposed when varying the potentials is that a requirement that must be satisfied by any allowed gauge condition is that it does not disturb the validity of (3.1). The potentials must satisfy (3.1) regardless of the selected gauge condition. The gauge condition can be imposed as a supplement to (3.1). One convenient choice is the Lorentz gauge given by

$$\dot{\varphi} + \nabla \circ \boldsymbol{A} = 0 \quad \text{(Lorentz gauge)}$$

so that (3.1) is equivalent to

$$\nabla^2 \varphi - \ddot{\varphi} = -4\pi\,\rho, \qquad \nabla^2 \boldsymbol{A} - \ddot{\boldsymbol{A}} = -\frac{4\pi}{c}\,\boldsymbol{j} \qquad \text{(Lorentz gauge)}.$$

However, the gauge condition is a matter of choice. The important field equations are (3.1).

4. Lagrangian for Electromagnetic Fields Created by Given Current Densities and Charge Densities when One or More Sources are Point Charges

We now consider a Lagrangian used to calculate fields from given currents and charges when one or more of the charges are point charges. Each of these charges or particles can be treated one at a time and we treat them one at a time by arbitrarily selecting one of the particles to be called the particle of interest. It has a charge denoted q_I. The analysis begins with the particle having a greater-than-zero size, with the interior having a finite charge density, but later takes the limit as the size shrinks to zero. This limit is postponed until after equations have been derived for which the limit makes sense.

The first step separates the particle of interest from all other sources of the electromagnetic field by defining the charge density ρ_I and current density $\boldsymbol{j}_I$ to be what are produced by the particle of interest, and defining the charge density ρ_E and current density j_E to be what are produced by all sources except the particle of interest. The total charge density and total current density are sums over sources so we can write (3.3) as

$$L_{field} = \frac{1}{8\pi}\int\left[\left(\nabla\varphi + \dot{\boldsymbol{A}}\right)\circ\left(\nabla\varphi + \dot{\boldsymbol{A}}\right) - \left(\nabla\times\boldsymbol{A}\right)\circ\left(\nabla\times\boldsymbol{A}\right)\right]d^3x + \int\left[\frac{1}{c}\boldsymbol{j}_E\circ\boldsymbol{A} - \rho_E\,\varphi\right]d^3x$$
$$+\int\left[\frac{1}{c}\boldsymbol{j}_I\circ\boldsymbol{A} - \rho_I\,\varphi\right]d^3x \qquad (4.1)$$

It is tempting to let the size of the particle of interest shrink to zero and represent its charge density in the last integral on the right side of (4.1) with a Dirac delta function. However, as already pointed out in the Introduction, the potential contained in this integral includes the potential produced by this particle and is not nearly constant in the interior of the particle. Regardless of how small the particle is, the potential is not nearly constant in the particle interior so a delta function representation of the charge density used in this integral is invalid regardless of how small the particle is. Worse yet, this limit makes the potential become singular at the same location where the charge density becomes singular, and the integral becomes infinite when taking this limit. To obtain an equation for which the limit makes sense, let φ_I and $\boldsymbol{A}_I$ be the potentials produced by the particle of interest. It is important to note that these are the actual potentials as opposed to trial functions to be varied in a variational procedure. Adding and subtracting terms on the right side of (4.1) gives

$$L_{field} = \frac{1}{8\pi}\int\left[\left(\nabla\varphi+\dot{\boldsymbol{A}}\right)\circ\left(\nabla\varphi+\dot{\boldsymbol{A}}\right)-\left(\nabla\times\boldsymbol{A}\right)\circ\left(\nabla\times\boldsymbol{A}\right)\right]d^3x+\int\left[\frac{1}{c}\boldsymbol{j}_E\circ\boldsymbol{A}-\rho_E\,\varphi\right]d^3x$$
$$+\int\left[\frac{1}{c}\boldsymbol{j}_I\circ\left(\boldsymbol{A}-\boldsymbol{A}_I\right)-\rho_I\left(\varphi-\varphi_I\right)\right]d^3x+\int\left[\frac{1}{c}\boldsymbol{j}_I\circ\left(\boldsymbol{A}_I\right)-\rho_I\left(\varphi_I\right)\right]d^3x \qquad (4.2)$$

The last integral on the right side of (4.2) becomes singular when taking the limit as the size of the particle of interest shrinks to zero. However, when this Lagrangian is used to obtain the equations for the field, the total potentials φ and $\boldsymbol{A}$ are varied but with currents and charges held fixed, and the actual particle-of-interest potentials φ_I and $\boldsymbol{A}_I$ are also held fixed. Therefore, the last integral on the right side of (4.2) is not varied so the same equations for the fields are obtained from a Lagrangian that does not include that last integral. The new field Lagrangian, denoted $L_{field,new}$, and given by

$$L_{field,new} = \frac{1}{8\pi}\int\left[\left(\nabla\varphi+\dot{\boldsymbol{A}}\right)\circ\left(\nabla\varphi+\dot{\boldsymbol{A}}\right)-\left(\nabla\times\boldsymbol{A}\right)\circ\left(\nabla\times\boldsymbol{A}\right)\right]d^3x+\int\left[\frac{1}{c}\boldsymbol{j}_E\circ\boldsymbol{A}-\rho_E\,\varphi\right]d^3x$$
$$+\int\left[\frac{1}{c}\boldsymbol{j}_I\circ\left(\boldsymbol{A}-\boldsymbol{A}_I\right)-\rho_I\left(\varphi-\varphi_I\right)\right]d^3x \qquad (4.3)$$

produces the same field equations (3.1) as the original field Lagrangian (3.3) with the understanding that the particle-of-interest potentials φ_I and $\boldsymbol{A}_I$ are fixed functions while the total potentials φ and $\boldsymbol{A}$ are given independent variations in the variational procedure.

Interpreting the differences $\varphi - \varphi_I$ and $\boldsymbol{A} - \boldsymbol{A}_I$ as external potentials, we might expect that if the particle of interest is sufficiently small compared to its distances from other sources, its charge density and current densities in the last integral on the right side of (4.3) can be represented in terms of delta functions because the external potentials are nearly constant in the particle interior. When taking the limit as the size shrinks to zero, the delta function representation can be used provided only that the external potential is a continuous function of the spatial coordinates at the location of the particle of interest. However, there is one last issue to deal with. The potentials φ and $\boldsymbol{A}$ in the Lagrangian are trial functions to be varied in a variational procedure, they are not necessarily the actual potentials. The required continuity condition is made to be satisfied by imposing it as a condition for trial functions (potentials) to be admissible trial functions. Specifically, for trial functions φ and $\boldsymbol{A}$ to be admissible, they must make $\varphi - \varphi_I$ and $\boldsymbol{A} - \boldsymbol{A}_I$ continuous functions of the spatial coordinates at the location of the particle of interest. An equivalent statement of this condition of being admissible trial functions is that the trial potentials φ and $\boldsymbol{A}$ are expressible as $\varphi = \varphi_I + \varphi_E$ and $\boldsymbol{A} = \boldsymbol{A}_I + \boldsymbol{A}_E$ for some functions φ_E and $\boldsymbol{A}_E$ that are continuous at the location of the particle of interest. With the above continuity requirement enforced by a requirement on admissible trial functions, we can now take the limit as the size of the particle of interest shrinks to zero and use

$$\rho_I\left(t,\boldsymbol{x}\right)=q_I\,\delta^3\left(\boldsymbol{x}-\boldsymbol{X}(t)\right), \qquad \boldsymbol{j}_I\left(t,\boldsymbol{x}\right)=q_I\,\boldsymbol{v}(t)\,\delta^3\left(\boldsymbol{x}-\boldsymbol{X}(t)\right) \qquad (4.4)$$

together with

$$\varphi_E \equiv \varphi-\varphi_I, \qquad \boldsymbol{A}_E \equiv \boldsymbol{A}-\boldsymbol{A}_I \qquad (4.5)$$

to write (4.3) as

$$L_{field,new} = \frac{1}{8\pi}\int\left[\left(\nabla\varphi+\dot{\boldsymbol{A}}\right)\circ\left(\nabla\varphi+\dot{\boldsymbol{A}}\right)-\left(\nabla\times\boldsymbol{A}\right)\circ\left(\nabla\times\boldsymbol{A}\right)\right]d^3x+\int\left[\frac{1}{c}\boldsymbol{j}_E\circ\boldsymbol{A}-\rho_E\,\varphi\right]d^3x$$
$$-q_I\left[\varphi_E-\frac{1}{c}\boldsymbol{v}(t)\circ\boldsymbol{A}_E\right]_{\boldsymbol{x}=\boldsymbol{X}(t)} \qquad (4.6)$$

Note that if the only goal is to derive the equations for the fields, the simplest analysis postpones taking the limit as the size of the particle of interest shrinks to zero until the last step. That is, we start with the Lagrangian L_{field} given by (3.3), use that to derive the field equations (3.1), and then as the last step we take the limiting case to replace the right sides of (3.1) according to (4.4). The only motive for the more cumbersome steps that take the limit inside the Lagrangian to produce (4.6) is the application in the next section.

5. Lagrangian for Both the Fields and the Particle of Interest

We start by defining a Lagrangian, denoted $L_{field\&particle}$, and then investigate what its usefulness is. The definition is

$$L_{field\,\&\,particle} = \left\{\frac{1}{8\pi}\int\left[\left(\nabla\varphi+\dot{\boldsymbol{A}}\right)\circ\left(\nabla\varphi+\dot{\boldsymbol{A}}\right)-\left(\nabla\times\boldsymbol{A}\right)\circ\left(\nabla\times\boldsymbol{A}\right)\right]d^3x+\int\left[\frac{1}{c}\boldsymbol{j}_E\circ\boldsymbol{A}-\rho_E\,\varphi\right]d^3x\right\}$$
$$+\left\{-q_I\left[\varphi_E-\frac{1}{c}\boldsymbol{v}(t)\circ\boldsymbol{A}_E\right]_{\boldsymbol{x}=\boldsymbol{X}(t)}\right\}+\left\{\frac{1}{2}mv^2\right\} \qquad (5.1)$$

The usefulness is as follows. Suppose the goal is to derive the field equations with sources of the field given and fixed. The far-right curly bracket on the right side of (5.1) is held constant when varying trial field functions so only the first two curly brackets are relevant, and these are the same terms in the field Lagrangian in (4.6) that correctly describes the field (recall the qualifiers, the particle field is given and there is a constraint on admissible trial functions to satisfy a continuity condition). Now suppose the goal is to derive the equations of motion for the particle of interest when the field is given and fixed (external charge densities and current densities are also given and fixed). The first curly bracket on the right side of (5.1) is held constant when varying trial particle trajectories so only the last two curly brackets are relevant, and these are the same terms in the particle Lagrangian (2.5) that correctly describes the (nonrelativistic) equations of motion for the particle. Therefore, this same Lagrangian can be used to derive both, the equations for the field and the equations of motion for a particle responding to the field.

6. Conclusions

Nothing in this paper is claimed to be original but one of the steps might be unfamiliar to some readers (it was new to me) because at least some popular textbooks omit that discussion. This step replaced the field Lagrangian given by (4.2) with the one given by (4.3) and gave justification for that replacement. This replacement not only removes singularities but also produces a conformity between the field Lagrangian and particle Lagrangian. That conformity was utilized to construct the single Lagrangian given by (5.1) for both particles and fields. Versions of this equation can be found in standard textbooks (e.g., Goldstein's "Classical Mechanics"). The distinction between (5.1) and versions found elsewhere in the literature is the clear distinction in (5.1) between total potentials and external potentials. Versions of this equation found elsewhere in the literature replace φ_E and $\boldsymbol{A}_E$ on the right side of (5.1) with φ and $\boldsymbol{A}$, producing singularities that are not discussed.