Components of efficiency evaluation in data envelopment analysis

Agha Iqbal Ali, Catherine S. Lerme, Lawrence M. Seiford

School of Management, The University of Massachusetts at Amherst, Amherst, MA 01003, USA
Carroll School of Management, Boston College, Chestnut Hill, MA 02167, USA
Industrial Engineering and Operations Research, The University of Massachusetts at Amherst, Amherst, MA 01003, USA

Abstract

This paper examines three essential components which comprise efficiency evaluation in data envelopment analysis. The three components are present in each DEA model and determine the implicit evaluation scheme associated with the model. These components provide a framework for classifying the various DEA models with respect to (i) the form of envelopment surface, (ii) the orientation, and (iii) the pricing mechanism implicit in the multiplier lower bounds. The discussion focuses on the standard DEA models, includes additional issues relating to efficiency evaluation, and is illustrated by a computational example.

Keywords: Data envelopment analysis; Efficiency; Framework

1. Introduction

Since the seminal paper by Charnes, Cooper and Rhodes in 1978, a variety of data envelopment analysis models has appeared in the literature as have numerous studies employing the technique (Banker, Charnes, Cooper, Swarts, Thomas, 1989; Seiford, 1990). Each of the various models for data development analysis (DEA) seeks to determine which of n decision making units (DMUs) determine an envelopment surface (or efficient frontier). Units that lie on (determine) the surface are deemed efficient in DEA terminology. Units that do not lie on the surface are termed inefficient and the analysis provides measures of their relative efficiency. These measures of relative efficiency depend upon the particular evaluation scheme implicit in the DEA model employed. As will be shown, the efficiency evaluation is determined by three components: envelopment surface, orientation, and multiplier lower bounds. As will be demonstrated, these components provide a unifying framework for classification of DEA models.
Three of the DEA models that are most often associated with the DEA methodology are the CCR, BCC and additive models. To introduce the components which form the framework developed in this paper we briefly examine these models. Our notation is as follows. We assume that there are \( n \) DMUs to be evaluated. Each DMU consumes varying amounts of \( m \) different inputs to produce \( s \) different outputs. Specifically, decision making unit \( l \), consumes amount \( x_{il} > 0 \) of input \( i \) and produces amount \( y_{rl} > 0 \) of output \( r \). Finally, in the model formulations, \( X_l \) and \( Y_l \) denote, respectively, the vectors of input and output values for DMU \( l \) while the \( s \times n \) matrix of outputs is denoted \( Y \) and the \( m \times n \) matrix of inputs is denoted \( X \).

The primal and dual linear programming statements for the (input oriented) CCR model are:

\[
\begin{align*}
\text{(CCR}_{IP}(Y_l, X_l)) & \\
\text{min} & & \theta - \varepsilon (1e + 1e) \\
\text{st.} & & Y_l - s = Y_l, \\
& & \theta X_l - X\lambda - e = 0, \\
& & \lambda \geq 0, \quad e \geq 0, \quad s \geq 0.
\end{align*}
\]

\[
\begin{align*}
\text{(CCR}_{ID}(Y_l, X_l)) & \\
\text{max} & & \mu Y_l \\
\text{s.t.} & & \nu X_l = 1, \\
& & \mu Y - \nu X \leq 0, \\
& & \mu \geq \varepsilon 1, \quad \nu \geq \varepsilon 1.
\end{align*}
\]

The primal and dual linear programming statements for the (input oriented) BCC model are:

\[
\begin{align*}
\text{(BCC}_{IP}(Y_l, X_l)) & \\
\text{min} & & \theta - \varepsilon (1s + 1e) \\
\text{st.} & & Y_l - s = Y_l, \\
& & \theta X_l - X\lambda - e = 0, \\
& & 1\lambda = 1, \\
& & \lambda \geq 0, \quad e \geq 0, \quad s \geq 0.
\end{align*}
\]

\[
\begin{align*}
\text{(BCC}_{ID}(Y_l, X_l)) & \\
\text{max} & & \mu Y_l + \omega \\
\text{s.t.} & & \nu X_l = 1, \\
& & \mu Y - \nu X + \omega 1 \leq 0, \\
& & \mu \geq \varepsilon 1, \quad \nu \geq \varepsilon 1.
\end{align*}
\]

The primal and dual linear programming statements of the additive model are:

\[
\begin{align*}
\text{(ADD}_{P}(Y_l, X_l)) & \\
\text{min} & & -(1s + 1e) \\
\text{s.t.} & & Y_l - s = Y_l, \\
& & -X\lambda - e = -X_l, \\
& & 1\lambda = 1, \\
& & \lambda \geq 0, \quad e \geq 0, \quad s \geq 0.
\end{align*}
\]

\[
\begin{align*}
\text{(ADD}_{D}(Y_l, X_l)) & \\
\text{max} & & \mu Y_l - \nu X_l + \omega \\
\text{s.t.} & & \mu Y - \nu X + \omega 1 \leq 0, \\
& & \mu \geq 1, \quad \nu \geq 1,
\end{align*}
\]

Each of the above models seeks to determine the efficiency of a particular DMU \((Y_l, X_l)\) with respect to the envelopment surface determined by the efficient (best practice) DMUs. The solution of a DEA model for DMU \((Y_l, X_l)\) results in a measure of efficiency and further identifies an efficient point \((\hat{Y}_l, \hat{X}_l)\) on the envelopment surface. As we shall see, efficiency evaluation and the location of this efficient (projected) point \((\hat{Y}_l, \hat{X}_l)\) are dependent on both the form of the envelopment surface and the evaluation system implicit in the particular DEA model. Furthermore, subtle differences in the mathematical formulations of the models can produce significant differences in one evaluation component yet leave another unchanged.

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2 The acronymic designation of the first two models arises from the authorship of the original articles (Charnes, Cooper, and Rhodes, 1978, and Banker, Charnes, and Cooper 1984). The additive model is developed in Charnes et al. (1985).

3 The assumption of positivity is for expositional convenience. There are several ways in which it can be relaxed. See, for example, Ali and Seiford (1990) and the discussion therein.
Examining the structure of the primal model formulations given above, we notice that the $CCR_{IP}$ and $BCC_{IP}$ models differ only in that the latter includes a 'convexity' constraint ($1\lambda = 1$). Again with respect to model structure, one observes that substitution of $\theta = 1$ (and $\epsilon = 1$) in the $BCC_{IP}$ model produces the additive model ($ADD_p$). This similarity in model structure points to an underlying framework within which these and other DEA models are embedded. As will be seen, the identification of such a framework provides a better understanding of the solution methodology and, also, aids in the interpretation of results.

The consequences of differences in formulation allow further comparisons between models. It can be easily shown that the envelopment surfaces for the BCC and additive DEA models are identical and thus the sets of DMUs determined to be efficient are exactly the same for both models. However, as we shall demonstrate, the two models differ in their implicit evaluation principles. In contrast, both the CCR and BCC models obey the same evaluation principles. However, the form (geometry) of the envelopment surface is different for these two models. The interrelationships between envelopment surface, orientation, and lower bound specification and their effect on efficiency evaluation are the subject of this paper.

The format for this paper is as follows. In Section 2, we discuss two underlying types of envelopment surfaces for data envelopment analysis. The notation introduced facilitates the subsequent discussion of different evaluation systems. The mathematical programming models introduced in this section are 'nonoriented models' and give rise to the non-oriented projections described in Section 3. Section 4 constrasts input and output orientations while Section 5 discusses the effect of changing the units of measurement of the data. Efficiency evaluation components are identified and illustrated with a computational study in Section 6. Concluding remarks are presented in Section 7.

2. Envelopment surfaces

The various models proposed in the DEA literature employ several different types of envelopment surfaces. Our discussion, however, will focus on the two basic piecewise linear envelopment surfaces commonly referred to as constant returns-to-scale (CRS) and variable returns-to-scale (VRS) surfaces. This is not in anyway restrictive.

One of the two types of envelopment surfaces (CRS or VRS) results from the particular DEA model employed, i.e., the pair of (dual) linear programs that effect data envelopment. For example the CCR model produces a CRS envelopment surface while the BCC model and the additive model produce a VRS envelopment surface. To facilitate our development of a framework in the sections to follow, we first examine the following general nonoriented envelopment models.

The primal and dual statements of the nonoriented constant returns-to-scale (CRS) model are:

$$\begin{align*}
CRS_{NP}(Y, X, u', v') & \\
\min & - (u's + v'e) \\
n & \lambda \geq 0, \quad e \geq 0, \quad s \geq 0.
\end{align*}$$

$$\begin{align*}
CRS_{ND}(Y, X, u', v') & \\
\max & \mu Y - vX \\
s.t. & \mu \geq u', \quad v \geq v'.
\end{align*}$$

---

4. The evaluation principles (system) for a particular DEA model define criteria that determine the manner in which projected points are determined for the inefficient DMUs. This is discussed in Sections 3 and 4.

5. For example, the multiplicative models (Charnes et al., 1982, 1983) produce piecewise log-linear or piecewise Cobb–Douglas envelopments. However these non-linearities can, by data transformations, be made linear. Similarly, although Seiford and Thrall (1990) propose four piecewise linear envelopment surfaces, two of their surface classifications result from hybrid combinations of our two basic surfaces. Extension of our results to the two additional hybrid surfaces is straightforward.
The primal and dual statements of the nonoriented variable returns-to-scale (VRS) model are:

\[
\begin{align*}
(VRS_{NP}(Y_l, X_l, u^l, v^l)) & : & \min & -(u^s + v^t\epsilon) \\
&s.t. & Yl - s = Y_l, & -XL - \epsilon = -X_l, & 1\lambda = 1, & \lambda \geq 0, & \epsilon \geq 0, & s \geq 0.
\end{align*}
\]

\[
\begin{align*}
(VRS_{ND}(Y_l, X_l, u^l, v^l)) & : & \max & \mu Y_l - \nu X_l + \omega \\
&s.t. & \mu Y - \nu X + 1\omega \leq 0, & \mu \geq u^l, & \nu \geq v^l.
\end{align*}
\]

We refer to the dual problem statements \((CRS_{ND} \text{ or } VRS_{ND})\) as multiplier or pricing models. The primal problems, on the other hand, can be characterized as envelopment or projection models. Of course, only one of these problems (primal or dual) needs to be solved; the solution to the other is easily obtained by the duality theory of linear programming.

These nonoriented programs directly address the underlying characterization of efficiency: A decision making unit, \(l\), is efficient if it lies on a facet-defining hyperplane of the envelopment surface; specifically, a hyperplane of the form \(\mu^l y - \nu^l x = 0\) for CRS envelopment or a hyperplane of the form \(\mu^l y - \nu^l x + \omega^l = 0\) for VRS envelopment.

The parameters \(u^l\) and \(v^l\) (introduced for notational convenience) are the specific lower bounds on the variables \(\mu\) and \(\nu\) in the dual programs. Our discussion in the present section is with respect to the general specification, \(u^l\) and \(v^l\), of these vectors. Further interpretation and effect of specific lower bounds are presented in Section 5.

Optimal values of variables for either of the primal nonoriented envelopment programs for DMU \(l\) are denoted by the \(s\)-vector \(s^l\), the \(m\)-vector \(e^l\), and the \(n\)-vector \(\lambda^l\). An optimal dual solution is given by the \(s\)-vector \(\mu^l\), the \(m\)-vector \(\nu^l\), and, for the VRS model, the variable \(\omega^l\).

Identification of the envelopment surface requires the solution of a linear programming model for each decision making unit \(l\). Each of the \(n\) sets of values given by \(\mu^l, \nu^l, (\omega^l), l = 1, 2, \ldots, n\), are the coefficients of hyperplanes that define facets of the envelopment surface.

The \(CRS\) envelopment surface consists of particular facets that result from the intersection of (supporting) hyperplanes in \(\mathbb{R}^{m+s}\) (which pass through the origin) and the convex polyhedral cone determined by the vectors \((Y_j, X_j), j = 1, \ldots, n\); a hyperplane \(\mu \cdot y - \nu \cdot x = \sum_{j=1}^{m+1} \mu_j y_j - \sum_{j=1}^{m} \nu_j x_j = 0\) intersects the cone in a facet of the CRS envelopment surface if and only if \(\mu Y_j - \nu X_j = \sum_{j=1}^{m+1} \mu_j y_j - \sum_{j=1}^{m} \nu_j x_j \leq 0\), for all \(j = 1, \ldots, n\) (with equality for at least one \(j\)).

The \(VRS\) envelopment surface consists of particular facets that result from the intersection of (supporting) hyperplanes in \(\mathbb{R}^{m+s}\) and the convex hull of the points \((Y_j, X_j), j = 1, \ldots, n\) in \(\mathbb{R}^{m+s}\); A hyperplane \(\mu \cdot y - \nu \cdot x + \omega = \sum_{j=1}^{m+1} \mu_j y_j - \sum_{j=1}^{m} \nu_j x_j + \omega = 0\) intersects the convex hull in a facet of the surface if an only if \(\mu Y_j - \nu X_j + \omega = \sum_{j=1}^{m+1} \mu_j y_j - \sum_{j=1}^{m} \nu_j x_j + \omega \leq 0\), for all \(j = 1, \ldots, n\) (with equality for at least one \(j\)).

Finally, the critical role of the vectors \((u^l, v^l)\) is worth emphasizing: In the primal formulation, they are the coefficients of the objective function and thus are a component of the evaluation mechanism. The result is that they determine the directions of the projection. We illustrate this in the following section although a complete discussion of the effect of lower bound specification is reserved for Section 5.

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6 For example, setting \(u^l = 1\) and \(v^l = 1\) in \(VRS_{NP}\) produces the additive model.

7 Alternate primal solutions can exist. Trivial alternate dual solutions always exist since each of the dual variables can be multiplied by a positive scalar to yield another optimal dual solution. Non-trivial alternate dual solutions can also exist. This can happen, for example, when the rows of the primal models are linearly dependent.
3. Non-oriented projections

For the primal envelopment problem for DMU \( l \), the optimal vector \( \lambda^l \) defines a point

\[
(\hat{Y}_l, \hat{X}_l) = \left( \sum_{j=1}^n \lambda^l_j Y_j, \sum_{j=1}^n \lambda^l_j X_j \right)
\]

on the envelopment surface. This point is a linear combination for the CRS model, or a convex combination for the VRS model, of units that lie on the envelopment surface. The point \((\hat{Y}_l, \hat{X}_l)\) is referred to as the projected point. The following theorem establishes that the projected point, in fact, lies on a face of the envelopment surface.

**Theorem.** Suppose \( \lambda^l \) is optimal for the primal problem. Then the point \((\hat{Y}_l, \hat{X}_l) = \left( \sum_{j=1}^n \lambda^l_j Y_j, \sum_{j=1}^n \lambda^l_j X_j \right)\) lies on a face of the envelopment surface determined by a supporting hyperplane defined by the optimal dual multipliers \( \mu^l, \nu^l, (\omega^l) \).

**Proof.** From duality theory of linear programming, we know that for each \( j \) with \( \lambda^l_j > 0 \), the corresponding dual constraint is binding, i.e. \( \mu^l Y_j - \nu^l X_j + \omega^l = 0 \). Thus each decision making unit \( j \in \Lambda = \{ j \mid \mu^l Y_j - \nu^l X_j + \omega^l = 0 \} \) is efficient and lies on the hyperplane \( \mu^l y - \nu^l x + \omega^l = 0 \). This hyperplane defines a face (which may be a facet \(^8\)) of the envelopment surface with normal vector \((\mu^l, \nu^l)\). A sufficient, but not necessary, condition for membership in \( \Lambda \) is \( \lambda^l_j > 0 \). Since each point \( Y_j, X_j \), for which \( \lambda^l_j > 0 \) lies on the plane, we have

\[
\mu^l \left( \sum_{j=1}^n \lambda^l_j Y_j \right) - \nu^l \left( \sum_{j=1}^n \lambda^l_j X_j \right) + \omega^l = 0.
\]

Thus the point \((Y_l, X_l)\) is projected onto the point \((\hat{Y}_l, \hat{X}_l)\) on the hyperplane, \( \mu^l y - \nu^l x + \omega^l = 0 \). For the CRS model, it can be similarly shown that the projected point lies on the hyperplane \( \mu^l y - \nu^l x = 0 \).

The preceding theorem established that the efficient units (vertices) which comprise the projected point correspond to positive \( \lambda^l \) values. An alternate characterization of the projected point is available from the primal constraints.

\[
(\hat{Y}_l, \hat{X}_l) = \left( \sum_{j=1}^n \lambda^l_j Y_j, \sum_{j=1}^n \lambda^l_j X_j \right) = (Y_l + s, X_l - e^l).
\]

This characterization lends itself to referring to the vector \( s^l \) as the vector of output slacks and the \( m \)-vector \( e^l \) as the vector of excess inputs. The relationship between the values of output slacks and excess inputs and the relative prices is evident from the complementary slackness conditions from linear programming duality theory. Specifically

\[
\begin{align*}
\frac{\mu^l}{s^l} > 0 & \implies \mu^l_r = u^l_r, \quad r = 1, \ldots, s, \\
\frac{\nu^l}{e^l} > 0 & \implies \nu^l_i = v^l_i, \quad i = 1, \ldots, m.
\end{align*}
\]

Since positive values of the slack or excess variables correspond to multipliers which are at lower bound, the term \( \mu^l s^l + \nu^l e^l \) is exactly the objective value \( u^l s^l + v^l e^l \) at optimality. This establishes the earlier claim that these lower bounds \((u^l, v^l)\) directly effect the direction of projection. In Section 5, we

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\(^8\) A facet is a full dimensional face.
further examine the particular relationship between efficiency evaluations obtained for different specifications of the parameters $u^i$ and $v^j$.

4. Oriented projections

The previous section examined non-oriented projections; the additive model being the most frequent example appearing in the DEA literature of a non-oriented VRS model. In contrast, the DEA models examined next are orientable.

Input oriented models maximize the proportional decrease in the input vector (while remaining within the envelopment space). Clearly, a proportional decrease is possible until at least one of the excess input variables is reduced to zero. The input oriented models for CRS and VRS envelopment are stated below

\[
\begin{align*}
\text{(CRS}_{IP}\,(Y_i, X_i, u^i, v^j)) & \quad \min \theta - \varepsilon (u^i s + v^j e) \\
& \quad \text{s.t. } Y\lambda - s = Y_i, \\
& \quad \theta X_i - X\lambda - e = 0, \\
& \quad \lambda \geq 0, \quad e \geq 0, \quad s \geq 0. \\
\text{(VRS}_{IP}\,(Y_i, X_i, u^i, v^j)) & \quad \min \theta - \varepsilon (u^i s + v^j e) \\
& \quad \text{s.t. } Y\lambda - s = Y_i, \\
& \quad \theta X_i - X\lambda - e = 0, \\
& \quad 1\lambda = 1, \\
& \quad \lambda \geq 0, \quad e \geq 0, \quad s \geq 0.
\end{align*}
\]

Output oriented models maximize the proportional increase in the output vector possible while remaining within the envelopment space. Clearly, a proportional increase is possible until at least one of the output slack variables is reduced to zero. The output oriented models for CRS and VRS envelopment are stated below:

\[
\begin{align*}
\text{(CRS}_{OP}\,(Y_i, X_i, u^i, v^j)) & \quad \max \phi + \varepsilon (u^i s + v^j e) \\
& \quad \text{s.t. } \phi Y_i - Y\lambda + s = 0, \\
& \quad X\lambda + e = X_i, \\
& \quad \lambda \geq 0, \quad e \geq 0, \quad s \geq 0. \\
\text{(CRS}_{OD}\,(Y_i, X_i, u^i, v^j)) & \quad \min \nu X_i \\
& \quad \text{s.t. } \mu Y_i = 1, \\
& \quad -\mu Y + \nu X \geq 0, \\
& \quad \mu \geq \varepsilon u^i, \quad \nu \geq \varepsilon v^j.
\end{align*}
\]

\footnote{The labels for these models underscore the role of the envelopment surface in our efficiency evaluation framework. For CRS envelopment, CRS$_I$ and CRS$_O$ (with $u^i = 1, v^j = 1$) correspond to the input and output oriented CCR models (Charnes, Cooper and Rhodes, 1978). For VRS envelopment (with $u^i = 1, v^j = 1$) input and output orientation correspond to the BCC models (Banker, Charnes and Cooper, 1984).}
(VRS_{OP}(Y_t, X_t, u^i, v^i))
\begin{align*}
\text{max} & \quad \phi + \varepsilon (u^s + v^e) \\
\text{s.t.} & \quad \phi Y_t - Y \lambda + s = 0, \\
& \quad \lambda = 1, \quad \lambda \geq 0, \quad s \geq 0.
\end{align*}

(\text{VRS}_{OP}(Y_t, X_t, u^i, v^i))
\begin{align*}
\text{max} & \quad \nu X_t + \omega \\
\text{s.t.} & \quad \mu Y_t = 1, \\
& \quad -\mu Y + \nu X + \omega 1 \geq 0, \\
& \quad \mu \geq \varepsilon u^i, \quad \nu \geq \varepsilon v^i.
\end{align*}

Optimal primal variables for decision making unit, \( l \), are denoted \( \theta^l \) (or \( \phi^l \)), \( \lambda^l, s^l, e^l \). While the optimal value for \( \theta^l \) (or \( \phi^l \)) is unique there can be multiple optimal solutions for the other variables. We denote optimal dual variables by \( \mu^l, \nu^l \) and, where applicable, \( \omega^l \). For these oriented models, the point \((Y_t, X_t)\) is projected onto a face of the envelopment surface, i.e., a supporting hyperplane defined by the optimal dual multipliers \( \mu^l, \nu^l, (\omega^l) \).

By orienting the projection, for either a CRS or VRS envelopment surface one obtains different projected points for inefficient units. These projected points reflect the particular priority of the orientation. Input-orientation seeks a projected point such that the proportional reduction in inputs is maximized. Implicit in an input orientation is that the primary objective of the DMU being evaluated is to gain efficiency by reducing excess input consumption while continuing to operate with its current technology mix (characterized by the actual input ratios). Similarly, output-orientation seeks a projected point such that the proportional augmentation in outputs is maximized. In this situation, the primary objective is to reach efficiency by focusing on productivity gains while preserving the current output mix. Satisfaction of the primary objective, for either orientation, may not be sufficient to reach the envelopment surface, i.e. attain efficiency.

Oriented projections, hence, require first determining the maximum radial movement (intermediate projection) in the input or output direction. (In terms of the models stated above this corresponds to an optimal value for \( \theta^l \) or \( \phi^l \), respectively.) This maximum radial movement identifies an intermediate point \((Y_t, \theta^l X_t)\) for the input orientation or a point \((\phi^l Y_t, X_t)\) for the output orientation. This intermediate point reflects the primary emphasis of the input or output orientation.

However radial movement by itself is not sufficient to always guarantee that this intermediate point will lie on the efficient frontier. One must further apply a non-oriented projection to the intermediate point. The resulting projected point \((\bar{Y}_t, \bar{X}_t)\), which will lie on the efficient frontier, is expressed as \((Y_t + s^l, \theta^l X_t - e^l)\) for input orientations and as \((\phi^l Y_t + s^l, X_t - e^l)\) for output orientations.

The oriented models stated in this section accomplish the ‘total’ projection. I.e., the solution of these models produces a ‘final’ projected point \((\bar{Y}_t, \bar{X}_t)\) which lies on the efficient frontier. The preceding conceptual discussion in terms of an intermediate point only illustrates an essential nuance in the mechanism of efficiency evaluation for an oriented model. Although we will not do so, this conceptual development can be formalized as a two stage procedure characterized by the solution of first stage and second stage models. See Ali and Seiford (1993) for details.

5. Multiplier lower bounds and units of measurement

As described earlier, the manner in which the projection (onto the development surface) is effected in different DEA models depends not only on the orientation but also on the values of the vectors \( u^i, v^l \), i.e. the lower bounds on multipliers. Implicit in the choice of values for the vectors \( u^i, v^l \) are assumptions about marginal worths of outputs and inputs. We illustrate this dependence with three examples.

(1) Equal bounds

Lower bounds given by \( u^i_r = 1, \ r = 1, \ldots, s; \) and \( v^i_i = 1, \ i = 1, \ldots, m, \) implicitly assume that the marginal worth of each unit of the nonzero output slacks and nonzero excess inputs for a DMU are
identical (and equal to 1). Further, the marginal worths of nonzero output slacks for a particular output or nonzero excess inputs for a particular input are the same across all decision making units. This pricing mechanism implies that a unit of slack for any measure (input or output) is equated to a unit of waste, and that the objective function of the primal models simply accounts for global waste. This pricing mechanism therefore leaves unchanged the particular conversion system defined by the scale and units of measurement of the various inputs and outputs. For instance, if input 1 measures annual wages in millions of dollars, input 2 measures annual salaries in thousands of dollars and input 3 measures the number of hours worked in thousands of hours, then whenever there is slack in these inputs, one million dollars of annual wages is equivalent to one thousand dollars of annual salaries, and equivalent to one thousand hours of work. It follows that caution should be exercised in selecting units of measurement if the standard model is to be used without manipulation of the data. To fully ‘standardize’ the model and prevent the inappropriate aggregation of non-commensurable measures, some preprocessing of data may be required. Suggested approaches have involved scaling or normalizing the data using average, minimum, or maximum values. In our opinion, the use of averages should be discouraged due to the lack of justification/interpretation. Furthermore, the average values, themselves, are aggregates of both efficient and inefficient levels and therefore can only be representative of an inefficient theoretical DMU.

(2) DMU specific bounds

Lower bounds given by \( u_r^l = 1/y_{rl}, \ r = 1, \ldots, s, \ v_i^l = 1/x_{il}, \ i = 1, \ldots, m, \) implicitly assume that the marginal values of non-zero output slack and excess input variables are not identical. These marginal values are consistent with an emphasis on proportional reduction of inputs and proportional augmentation of outputs for each DMU. These lower bounds are DMU specific, reflecting the fact that different techniques demand different input and output mixes. Consequently, different relative values are assigned to the various inputs and outputs.

(3) Barycentric bounds

Lower bounds given by the reciprocal coordinates of the barycenter of all units, namely

\[
 u_r^l = \frac{1}{\left( \sum_{l=1}^{n} y_{rl} \right)}, \quad r = 1, \ldots, s, 
\]

and

\[
 v_i^l = \frac{1}{\left( \sum_{l=1}^{n} x_{il} \right)}, \quad i = 1, \ldots, m, 
\]

possess different characteristics. They distinguish inputs and outputs in terms of a consistent relative value; i.e., the bounds are the same across all DMUs. Possible objections to this choice of lower bounds rest on uniformity across DMUs and artificiality. A unique set of lower bounds for all DMUs ignores the eventual reality of spatially separated markets which allow for different supply and demand conditions and equilibria and, hence, for multiple, or ranges of, relative prices. The barycenter of the points from which the lower bounds are derived corresponds to an artificial DMU that is definitely inefficient since it is interior to the polytope defined by the set of DMUs.

\[10\] While we discourage the use of averages to normalize the data, the use of the average as a substitute for a missing data value is frequently appropriate.
Table 1
Example data set

<table>
<thead>
<tr>
<th>DMU</th>
<th>Output 1</th>
<th>Output 2</th>
<th>Input 1</th>
<th>Input 2</th>
</tr>
</thead>
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<td>30</td>
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<td>30</td>
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<td>5</td>
<td>180</td>
<td>140</td>
<td>50</td>
<td>40</td>
</tr>
<tr>
<td>6</td>
<td>140</td>
<td>90</td>
<td>105</td>
<td>75</td>
</tr>
<tr>
<td>7</td>
<td>100</td>
<td>82</td>
<td>97</td>
<td>67</td>
</tr>
<tr>
<td>8</td>
<td>140</td>
<td>100</td>
<td>90</td>
<td>60</td>
</tr>
<tr>
<td>9</td>
<td>140</td>
<td>105</td>
<td>98</td>
<td>65</td>
</tr>
</tbody>
</table>

Each of the pricing mechanisms reflected in the above three lower bound specifications can lead to different projected points. This is a consequence of the fact that each mechanism reflects a different evaluation system in the underlying linear programming problem.

In typical linear programming problems, if a constraint is scaled (multiplied by a nonzero scalar), the solution is not altered. In data envelopment analysis, when the data is scaled (each input or output value is multiplied by a nonnegative scalar), the results can change. The manner in which projections and efficiency evaluations are altered under different units of measurement for data depends on the specification of the lower bounds.

As stated earlier, the term \( \mu^l s^l + v^l e^l \) is exactly \( \mu^i s^i + v^i e^i \) at optimality. This quantity obviously depends on the defined lower bounds as well as the data. Models with DMU specific bounds are units-invariant as are the models with barycentric bounds. Models with equal lower bounds \((u^l, v^l) = (1, 1)\) are not units-invariant (unless the data are unitless). If the data are not unitless, then the (obvious) dependence of \( l s + le \) on units of measurement alters both projected points and efficiency evaluation.

The reader will note that for the oriented models different lower bound specifications will not alter the value of \( \theta \) or \( \phi \). That is, both \( \theta \) and \( \phi \) are units invariant. This reflects the fact that \( \theta \) and \( \phi \) do not encompass the entire discrepancy between the observed and projected points. In other words, \( \theta \) and \( \phi \) fail to capture all of the information available in the model.

6. Computational illustration

Each DEA model determines an envelopment surface with which it evaluates efficiency. As stated earlier, a DMU is efficient if it lies on this efficient surface and inefficient otherwise. Thus the form of the envelopment surface affects efficiency evaluation. When unit \( l \) is evaluated we obtain (i) a valuation system, \( \mu^l, v^l, (\omega^l) \), that is representative of a supporting hyperplane of the envelopment surface, and (ii) a projected point \((\hat{Y}_l, \hat{X}_l)\) that lies on the face of the surface defined by this hyperplane. The efficiency evaluation is dependent upon the orientation and lower bound specification of the model as these clearly affect the location of the projected point. The interrelationship between these three components of efficiency (envelopment surface, orientation, multiplier lower bounds) is illustrated with the example

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11 DEA models are atypical linear programming problems. They involve constraint matrices that are 100% dense, are highly degenerate, and exhibit cycling. A discussion of efficient computational constructs which address these problems is given in Ali (1993).
dataset presented in Table 1. It consists of 11 DMUs each consuming 2 inputs and producing 2 outputs. Our discussion will focus entirely on the projected point.

Table 2 reports the projected point obtained for CRS and VRS envelopment surfaces with an input orientation. DMUs 1, 2, 4, and 5 are efficient in both cases. While intentional for our example, it is unusual to have identical sets of efficient DMU under the two models. However, note that even with identical sets of efficient DMUs the projected points are different for each inefficient DMU. In addition the distances between the observed and projected values are greater for the CRS surface as the VRS surface more closely fits the data.

Table 3 reports the projected points obtained for the input oriented, output oriented, and nonoriented VRS models with equal lower bounds. Of the seven inefficient points, only one (DMU 6) is projected to the same point in all three cases. The projected point for DMU 7 is the same for both the output oriented and nonoriented case. For all other inefficient DMUs different projected points are obtained.

As can be seen from the preceding tables the effect of the first two components (envelopment surface and orientation) on efficiency evaluation can be extensive. In contrast, the effect of the third component

Table 2

Effect of envelopment surface

<table>
<thead>
<tr>
<th>DMU</th>
<th>Input orientation, equal lower bounds</th>
<th>VRS envelopment surface, equal lower bounds</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Projected point ($\hat{y}_i, \hat{x}_i$) with CRS Envelopment surface</td>
<td>Projected point ($\hat{y}_i, \hat{x}_i$) with VRS Envelopment surface</td>
</tr>
<tr>
<td>1</td>
<td>(160.00, 100.00, 40.00, 30.00)</td>
<td>(160.00, 100.00, 40.00, 30.00)</td>
</tr>
<tr>
<td>2</td>
<td>(180.00, 70.00, 30.00, 60.00)</td>
<td>(180.00, 70.00, 30.00, 60.00)</td>
</tr>
<tr>
<td>3</td>
<td>(170.00, 112.00, 67.81, 29.16)</td>
<td>(174.88, 114.88, 69.75, 30.00)</td>
</tr>
<tr>
<td>4</td>
<td>(190.00, 130.00, 50.00, 70.00)</td>
<td>(190.00, 130.00, 50.00, 70.00)</td>
</tr>
<tr>
<td>5</td>
<td>(180.00, 120.00, 80.00, 30.00)</td>
<td>(180.00, 120.00, 80.00, 30.00)</td>
</tr>
<tr>
<td>6</td>
<td>(140.00, 82.00, 32.62, 41.94)</td>
<td>(170.00, 85.00, 35.00, 45.00)</td>
</tr>
<tr>
<td>7</td>
<td>(143.47, 90.00, 37.43, 26.73)</td>
<td>(161.00, 101.00, 42.00, 30.00)</td>
</tr>
<tr>
<td>8</td>
<td>(130.58, 82.00, 36.44, 24.29)</td>
<td>(161.27, 101.27, 44.54, 30.00)</td>
</tr>
<tr>
<td>9</td>
<td>(140.00, 91.00, 49.41, 24.71)</td>
<td>(160.00, 100.00, 40.00, 30.00)</td>
</tr>
<tr>
<td>10</td>
<td>(166.46, 105.00, 46.10, 30.73)</td>
<td>(165.00, 105.00, 47.73, 31.82)</td>
</tr>
<tr>
<td>11</td>
<td>(140.00, 88.00, 38.94, 25.83)</td>
<td>(162.62, 102.62, 45.23, 30.00)</td>
</tr>
</tbody>
</table>

Table 3

Effect of orientation

<table>
<thead>
<tr>
<th>DMU</th>
<th>VRS envelopment surface, equal lower bounds</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Input oriented</td>
</tr>
<tr>
<td>1</td>
<td>(160.00, 100.00, 40.00, 30.00)</td>
</tr>
<tr>
<td>2</td>
<td>(180.00, 70.00, 30.00, 60.00)</td>
</tr>
<tr>
<td>3</td>
<td>(174.88, 114.88, 69.75, 30.00)</td>
</tr>
<tr>
<td>4</td>
<td>(190.00, 130.00, 50.00, 70.00)</td>
</tr>
<tr>
<td>5</td>
<td>(180.00, 120.00, 80.00, 30.00)</td>
</tr>
<tr>
<td>6</td>
<td>(170.00, 85.00, 35.00, 45.00)</td>
</tr>
<tr>
<td>7</td>
<td>(161.00, 101.00, 42.00, 30.00)</td>
</tr>
<tr>
<td>8</td>
<td>(161.27, 101.27, 44.54, 30.00)</td>
</tr>
<tr>
<td>9</td>
<td>(160.00, 100.00, 40.00, 30.00)</td>
</tr>
<tr>
<td>10</td>
<td>(165.00, 105.00, 47.73, 31.82)</td>
</tr>
<tr>
<td>11</td>
<td>(162.62, 102.62, 45.23, 30.00)</td>
</tr>
</tbody>
</table>
Table 4
Effect of multiplier lower bounds

<table>
<thead>
<tr>
<th>DMU</th>
<th>VRS envelopment surface, input orientation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Equal lower bounds</td>
</tr>
<tr>
<td>1</td>
<td>(160.00, 100.00, 40.00, 30.00)</td>
</tr>
<tr>
<td>2</td>
<td>(180.00, 70.00, 30.00, 60.00)</td>
</tr>
<tr>
<td>3</td>
<td>(174.88, 114.88, 69.75, 30.00)</td>
</tr>
<tr>
<td>4</td>
<td>(190.00, 130.00, 50.00, 70.00)</td>
</tr>
<tr>
<td>5</td>
<td>(180.00, 120.00, 80.00, 30.00)</td>
</tr>
<tr>
<td>6</td>
<td>(170.00, 85.00, 35.00, 45.00)</td>
</tr>
<tr>
<td>7</td>
<td>(161.00, 101.00, 42.00, 30.00)</td>
</tr>
<tr>
<td>8</td>
<td>(161.27, 101.27, 44.54, 30.00)</td>
</tr>
<tr>
<td>9</td>
<td>(160.00, 100.00, 40.00, 30.00)</td>
</tr>
<tr>
<td>10</td>
<td>(165.00, 105.00, 47.73, 31.82)</td>
</tr>
<tr>
<td>11</td>
<td>(162.62, 102.62, 45.23, 30.00)</td>
</tr>
</tbody>
</table>

Table 5
Effect of units of measurement

<table>
<thead>
<tr>
<th>DMU</th>
<th>VRS envelopment surface, input orientation, equal lower bounds</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Unscaled data scale factor (1, 1, 1, 1)</td>
</tr>
<tr>
<td>1</td>
<td>(160.00, 100.00, 40.00, 30.00)</td>
</tr>
<tr>
<td>2</td>
<td>(180.00, 70.00, 30.00, 60.00)</td>
</tr>
<tr>
<td>3</td>
<td>(174.88, 114.88, 69.75, 30.00)</td>
</tr>
<tr>
<td>4</td>
<td>(190.00, 130.00, 50.00, 70.00)</td>
</tr>
<tr>
<td>5</td>
<td>(180.00, 120.00, 80.00, 30.00)</td>
</tr>
<tr>
<td>6</td>
<td>(170.00, 85.00, 35.00, 45.00)</td>
</tr>
<tr>
<td>7</td>
<td>(161.00, 101.00, 42.00, 30.00)</td>
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<tr>
<td>8</td>
<td>(161.27, 101.27, 44.54, 30.00)</td>
</tr>
<tr>
<td>9</td>
<td>(160.00, 100.00, 40.00, 30.00)</td>
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<tr>
<td>10</td>
<td>(165.00, 105.00, 47.73, 31.82)</td>
</tr>
<tr>
<td>11</td>
<td>(162.62, 102.62, 45.23, 30.00)</td>
</tr>
</tbody>
</table>

is less pronounced, particularly in an oriented model. For the input oriented VRS model results of Table 4, we see that the projections obtained for DMUs 3, 7, 8, 9, and 11 are different for the two sets of multiplier lower bounds.

Finally, Table 5 illustrates the effect of units of measurement when using equal lower bounds. The input oriented VRS model with equal lower bounds is solved with the data as in Table 1 and after scaling the first output by 100 and the first input by 1000. (The results are only presented for the equal lower bounds case as the other two lower bound specifications are units invariant and produce equivalent projections under scaling.) DMUs 3, 7, 8, and 11 have different projected points when scaled. Thus scaling the data, performing the analysis, and rescaling the data does not produce the same results as performing the analysis on the original data.

7. Conclusions

Each of the DEA models evaluates efficiency on the basis of three essential components: a form of envelopment surface, orientation, and relative tradeoffs implicit in the multiplier lower bounds.
(i) For each type of envelopment surface different projected points (for inefficient units) are obtained for different evaluation systems, i.e. by effecting a particular orientation and/or by employing a particular pricing mechanism.

(ii) Models that are units-invariant produce efficiency evaluations and projected points that do not depend on the units of measurement of the data. This property stems from the selection and definition of multiplier lower bounds which adjust to (mirror) changes in units of measurement.

(iii) The suitability of a particular model for an application should be gauged with respect to the choice of the form of envelopment and the evaluation system. The evaluation systems considered in this paper allow only for lower bounds on the relative prices.

(iv) Extensions to the basic DEA models considered in this paper either alter the form of the surface or change the evaluation system. For example, both are altered for nondiscretionary variables as only a subset of the input variables or only a subset of the output variables are used for effecting orientation (Banker and Morey, 1986); when additional restrictions are placed on the multipliers, the evaluation system, and, possibly, the envelopment surface are altered (Charnes, Cooper, Huang and Sun, 1990).

References


