TRANSLATION INVARIANCE IN DATA ENVELOPMENT ANALYSIS

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Conditions are established under which DEA models are translation invariant. Specifically, an affine displacement does not alter the efficient frontier for models incorporating the convexity constraint. This affords a ready solution to the problems of scaling and the presence of zero values which arise in Data Envelopment Analysis.

Data Envelopment Analysis (DEA) • translation • invariance • affine displacement • efficiency • productivity

1. Introduction

Since the original paper by Charnes, Cooper and Rhodes [7] establishing Data Envelopment Analysis (DEA), a number of theoretical extensions have appeared in the literature (see Seiford [13]). Many of these extensions and the underlying models were originally proposed to overcome difficulties associated with the data encountered in the course of a particular study (see for example [3,4,5,9]). Examples of two such difficulties associated with data values encountered in DEA are the problems of scaling and the data measurements which have value zero.

DEA is computationally intensive and involves constraint matrices which are 100% dense. Furthermore, data frequently exhibits a wide range. These scaling problems cause numerical instability of the computations performed, usually require preprocessing of the data, and have been the driving force for the development of the units invariant models of DEA (e.g., [2,6,7,10] and also [1] for relations between these different models).

The second issue, of zero values, has been more difficult to address. As developed in Charnes, Cooper and Rhodes [7,8] the CCR model requires strict positivity of all of the input values and all of the output values. Charnes, Cooper and Thrall [11] describe an alternate approach which relaxes this positivity requirement. However, their treatment still requires each DMU to have at least one positive input and at least one positive output. As shown in Charnes, Cooper and Thrall [12] this relaxation of the original requirement of strict positivity of inputs and outputs requires the development of sophisticated classification theorems. While such an elaborate structure may be necessary for the CCR model, a simpler and more direct approach is developed for the DEA models considered in this paper. Furthermore, our approach does not require positivity of any inputs or outputs.

This paper resolves both of the above difficulties, scaling and zeros, by using an affine displacement of the data. It is shown that such a displacement does not alter the efficient frontier (or empirical production function) and hence the associated DEA model is translation invariant. In the following sections we establish conditions under which translation invariance holds in DEA.
2. Translation invariance for the additive model

In Charnes, Cooper, Golany, Seiford and Stutz [6], the additive model for obtaining and characterizing the empirical production function is described. We assume that \( n \) DMUs each consume varying amounts of \( m \) inputs in the production of \( s \) outputs. Let \( X_j \) and \( Y_j \) be, respectively, the observed (column) vector of the inputs and the outputs of the \( j \)-th DMU. To test an empirical "input–output" point, \((X_0, Y_0)\), for efficiency one solves

\[
\begin{align*}
\text{minimize} & \quad -e^T s^+ - e^T s^- \\
\text{subject to} & \quad Y\lambda - s^+ = Y_0, \\
& \quad -X\lambda - s^- = -X_0, \\
& \quad e^T\lambda = 1, \\
& \quad \lambda, s^+, s^- \geq 0, \\
\end{align*}
\]

where \( Y \) and \( X \) are matrices with column vectors \( Y_j \) and \( X_j \) respectively, and \( s^+ \) and \( s^- \) are slack vectors. \((X_0, Y_0)\) for DMU0 is efficient if and only if the above optimal value is zero.

We now establish that the solution to the above linear programming problem with translated data is exactly the same as the solution to the linear program with the original data.

Let the output measures \( y_{rj}, \ r=1,\ldots, s, \) be displaced by \( w_r, \ r=1,\ldots, s, \) and the input measures \( x_{ij}, \ i=1,\ldots, m, \) be displaced by \( z_i, \ i=1,\ldots, m. \) (We assume without loss of generality that \( w \) and \( z_i \) are nonnegative.) Let \( W \) be a matrix each column of which is \( w \) and let \( Z \) be a matrix each column of which is \( z. \) Then the linear programming problem for the translated data is given by

\[
\begin{align*}
\text{minimize} & \quad -e^T s^+ - e^T s^- \\
\text{subject to} & \quad \bar{Y}\lambda - s^+ = \bar{Y}_0, \\
& \quad -\bar{X}\lambda - s^- = -\bar{X}_0, \\
& \quad e^T\lambda = 1, \\
& \quad \lambda, s^+, s^- \geq 0, \\
\end{align*}
\]

where \( \bar{Y} = Y + W, \ \bar{X} = X + Z, \ \bar{Y}_0 = Y_0 + w, \) and \( \bar{X}_0 = X_0 + z. \)

**Theorem.** The additive model (1) is translation invariant. Specifically, (1) and (2) are equivalent problems.

**Proof.** Since \( \Sigma_i \lambda_i = 1, \) we have \( \bar{Y}\lambda = Y\lambda + w \) and \( \bar{X}\lambda = X\lambda + z. \) Thus (2) and (1) are equivalent constraint sets. \( \square \)

Thus for the additive model, affine displacement does not alter the efficient frontier and the classification of DMUs as efficient or inefficient (the objective function value) is invariant to translations of the data.

3. The BCC model

The Banker, Charnes and Cooper (BCC) model given in [2] is an extension of the CCR ratio model which allows increasing, decreasing, and constant returns to scale. The associated linear programming problem for DMU0 is given by

\[
\begin{align*}
\text{minimize} & \quad \theta - e \cdot e^T s^+ - e \cdot e^T s^- \\
\text{subject to} & \quad Y\lambda - s^+ = Y_0, \\
& \quad \theta X_0 - X\lambda - s^- = 0, \\
& \quad e^T\lambda = 1, \\
& \quad \lambda, s^+, s^- \geq 0. \\
\end{align*}
\]

For the BCC model DMU0 is efficient if and only if \( \theta^* = 1 \) and \( s^+ = s^- = 0 \) in the optimal solution to (3).

Employing the previous notation, the associated translated problem becomes

\[
\begin{align*}
\text{minimize} & \quad \theta - e \cdot e^T s^+ - e \cdot e^T s^- \\
\text{subject to} & \quad \bar{Y}\lambda - s^+ = \bar{Y}_0, \\
& \quad \theta \bar{X}_0 - \bar{X}\lambda - s^- = 0, \\
& \quad e^T\lambda = 1, \\
& \quad \lambda, s^+, s^- \geq 0. \\
\end{align*}
\]

**Theorem.** For the BCC model:

(a) DMU0 is efficient for (3) iff DMU0 is efficient for (4).

(b) DMU0 is inefficient for (3) iff DMU0 is inefficient for (4).

**Proof.** (a) When \( \theta^* = 1, \) (3) and (4) are equivalent. (See the proof of the previous theorem.)

(b) Statement (b) is logically equivalent to statement (a). \( \square \)
Thus for the BCC model, affine displacement of the data does not alter the efficient frontier and the classification of DMUs as inefficient or efficient is invariant to translation. However, the inefficiency scores (objective function values) obtained for the inefficient DMUs will be different when the data is translated.

4. Conclusion

It has been shown that, for both the additive and BCC models of Data Envelopment Analysis, affine displacement (translation) of the data values does not alter the efficient frontier. Thus, the classification of DMUs as efficient or inefficient is translation invariant. In the case of the BCC model, however, displacement can alter the actual inefficiency score.

The key to translation invariance in DEA lies in the convexity constraint. Data displacement would not be appropriate for models which do not include the convexity constraint.

We close with the observation that results identical to those presented above for the additive model hold for the multiplicative model described in Charnes, Cooper, Seiford and Stutz [10] with the translation applied to the transformed data. Specifically, one must apply the affine displacement to the logarithms of the original data.

References


