Swing up Control of a Soft Inverted Pendulum with Revolute Base

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***Abstract*— In this paper, we introduce a novel soft robotic system, which is a soft inverted pendulum with a revolute joint at the base. This is an underactuated system as the revolute joint is not actuated. The soft body is hypothesized to be of constant curvature and it is actuated. Motivated by swing up controllers for classical underactuated systems, a switching based swing up and stabilization control of the proposed soft robot system is studied. We demonstrate that the swing up control guides the soft robot to the desired energy level, which is the upright position. Once the swing up is completed, the control is switched to an LQR controller to achieve the stabilization at the vertically upright pose. The switching occurs when the swing up control brings the robot inside the region of attraction for the LQR controller. The simulation results are depicted to illustrate the effectiveness of the proposed control approach for the soft inverted pendulum system.**

I. Introduction

The field of soft robotics has grown exponentially during the past two decades due to the possibility of expanded manipulation capabilities over existing rigid robots in complex, unstructured environments [1], [2]. Additionally, soft robots can potentially mitigate current safety risks associated with the rigid robots. The inspiration for soft robotics has been mainly due to the many examples from the nature, such as the agile environmental interactions of the elephant trunk and octopus tentacles [3]. Recently, research efforts have largely focused on developing modeling and control frameworks for soft robots [4], [5]. However, the main challenge has been to robustly control soft systems taking into account the highly underactuated nature of the soft robots. The study of control frameworks for underactuated soft robots can benefit from the corresponding development in underactuated rigid robots, which in the past four decades led to extensive advances, impacting industrial manipulators, wheeled and flying vehicles, and locomotion systems, among others.

As a first step toward understanding how the softness impacts the control performance of an underactuated soft inverted pendulum system, in this paper, we introduce a novel soft system inspired by the classical inverted pendulum - soft inverted pendulum with revolute base (SIPR). Essentially, this is a non-extensible soft bodied pendulum which is equipped with a revolute joint at the base. The soft body's curvature, which is hypothesized to be constant along its length, is actuated. This simplifying assumption made in this study allows the soft robot's curvature space to be fully actuated. However, the revolute joint is not actuated. Thus, the angular acceleration of the soft pendulum cannot be controlled directly making the SIPR system an underactuated mechanical system. It should be noted that the system considered in the current paper is different from the recently introduced soft inverted pendulum with affine curvature [6], wherein the base was considered to be fixed.

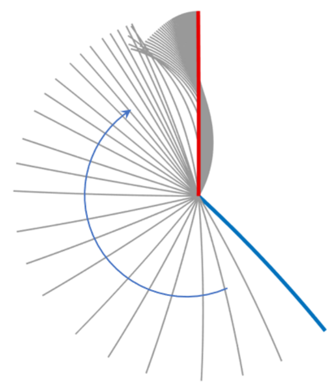

Fig. 1. Illustration of the swing up control of soft inverted pendulum with revolute base. Here the initial position is shown in blue and the final position in red. The intermediate positions are shown in grey.

We derive the dynamics of the SIPR system using the recent results of Della Santina et al., in [6], [7]. This system can be viewed as an extension of the prevailing rigid robot template models (e.g. acrobot and pendubot) for testing nonlinear control strategies. Although the structure of the dynamics of the SIPR system is similar to that of the classical acrobot [8], it is more complex and highly nonlinear. In view of the classical control problem for rigid underactuated robots, we will investigate the swing up control of the SIPR and the stabilization of the SIPR in its vertical upright position, as illustrated in Fig.1, initializing from below the horizontal level. To the best of the authors' knowledge, the system considered here is novel, and is a first provably correct controller development for the swing up problem for any soft robot.

In general, control of nonlinear underactuated systems, has been a challenging problem [9], [10]. This is more so, as the control algorithms developed for fully actuated robots cannot be directly utilized to control underactuated mechanical systems [9], [11]. In the literature, the researchers have considered specific underactuated systems such as the pendubot [12], acrobot [8], cart-and-pole system [13] and bipedal robots [14], and have developed

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controllers case-by-case for these systems. While all these are rigid systems, recently, an underactuated soft inverted pendulum with affine curvature has been analysed in [6], and an energy shaping control of a soft robot with in-plane disturbances has been developed in [15].

Swing up control for the class of acrobots and gymnist robots using partial feedback linearization was proposed in [16], [8], [17]. In this method the inherent nonlinearities are cancelled before the control design. Lozano et al., have proposed control algorithms for the same class of robots considering the total energy of the system [13]. As this energy-shaping approaches does not require high gains, it can potentially preserve the compliance of soft systems in the closed loop system. Other potential methods to control such under-actuated systems include interconnection and damping assignment based control [18], [19]. Recently, an energy shaping method circumventing the solution of partial differential equations was introduced in in [20]. However, the underlying assumptions in their work, precluded the utilization for considered SIPR system.

The main contribution of the present work is in the swing up and upright stabilization for the introduced novel SIPR system. The swing up control algorithm proposed in the present paper closely follows the methodology outlined in [12] which was developed to control the well known Pendubot [21]. An energy based approach was used to develop the proposed control design. For stabilizing the soft pendulum in the vertical upright position, we switch to a linear quadratic regulator (LQR) when the swing up control guides the system to the region of attraction of the desired equilibrium. The simulation results are also presented illustrating the efficacy of the proposed control method for the novel SIPR system.

The rest of the paper is organized as follows. Section II introduces the soft robot model. The swing up control is developed in Section III. The LQR for stabilization is discussed in Section IV. Then, the simulation results are presented in Section V. Finally, in Section VI, the paper is summarized and future extensions are discussed.

II. The soft robotic system

In this section we will first discuss the kinematics of the of the considered soft inverted pendulum with base revolute joint (SIPR). Then, we will introduce the dynamic model of the system and discuss its properties such as the equilibrium points and the total energy of the system.

A. Kinematics

Let's consider the SIPR system, a planar non-extensible soft robot with constant curvature (CC) [22] that has a revolute joint at its base as shown in Fig.2. Let the length of the soft robot along the central axis be L and thickness D. Following the charaterization in [6], the positions along the central axis of the soft robot is parameterized by $s \in [0, 1]$ such that Ls is the arc length along the robot's central axis to the point s from the base. The lateral points at each position is parameterized by $d \in [-0.5, 0.5]$ such

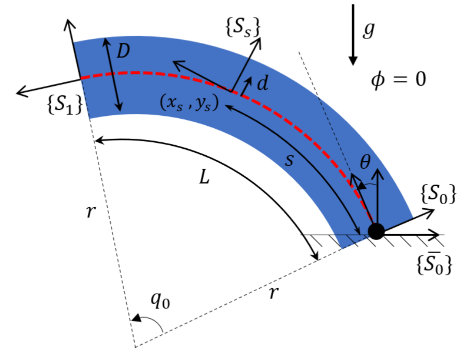


Fig. 2. Exaggerated illustration of the soft inverted pendulum with revolute joint at the base (SIPR)). The central axis is shown in dashed red line. The non-extensible length of the soft is L and the width is D.

that Dd is the lateral distance to the point d from the central axis. At each point s along the soft robot's body, we attach reference frames $\{S_s\}$ where as the base frame $\{\bar{S_0}\}$ is fixed in space. We can use these frames to describe any point on the soft robot along with the parameters (s, d).

Using the CC hypothesis, we select the configuration variables of this system as the degree of curvature of the soft robot $q_0(t) \in \mathbb{R}^1$ and the base rotation of the soft robot $\theta(t) \in \mathbb{S}^1$. Note that the degree of curvature $q_0(t)$ and the radius of curvature $r(t)$ has the relationship $q_0(t)r(t) = L$. Moreover, due to practical reasons we will assume that the material properties will only allow $q_0 \in [-n\pi, n\pi]$ for some finite $n > 0$. For concise representations, we will collect the configuration variables as $\mathbf{q}(t) = [q_0(t), \theta(t)]^\top$.

The orientation $\alpha_s(t)$ of the reference frame $\{S_s\}$ at point s along the central axis with reference to the base frame $\{\bar{S_0}\}$, can be written as the sum of integral of the curvature and the base rotation,

$$\begin{aligned}\alpha_s(t) &= \theta(t) + \int_0^s q_0(t)\mathrm{d}l \\ &= \theta(t) + q_0(t)s.\end{aligned}$$

Thus the Cartesian coordinates $(x_{s,d}\,(t)\,, y_{s,d}\,(t))$ at a general point parameterized by (s, d) on the soft robot is given by,

$$\begin{aligned}x_{s,d}(t) &= Dd\cos\alpha_s(t) - L\int_0^s \sin\alpha_s(t)\mathrm{d}s \\ y_{s,d}(t) &= Dd\sin\alpha_s(t) + L\int_0^s \cos\alpha_s(t)\mathrm{d}s.\end{aligned}$$

B. Dynamics

The dynamics of the soft robotic system were derived using a similar approach as in [6], [7] assuming a uniform mass distribution $\rho(s, d) \equiv m$. In the following, where obvious, the time arguments will be suppressed in the expressions due to brevity and clear representation.

For the SIPR system, the inertia matrix $\mathbf{M}(q_0, \theta) \in \mathbb{R}^{2\times 2}$ is evaluated as,

$$\mathbf{M}(q_0, \theta) = \int_0^1 \int_{-0.5}^{0.5} m\nabla_q(x_{s,d}, y_{s,d})^\top \nabla_q(x_{s,d}, y_{s,d})\mathrm{d}d\mathrm{d}s$$

where ∇_q is the gradient operator $\nabla_q(\cdot) = \frac{\partial(\cdot)}{\partial q}$. Then the centrifugal and Coriolis terms matrix $\mathbf{C}(q_0, \theta, \dot{q}_0, \dot{\theta}) \in \mathbb{R}^{2\times 2}$ is evaluated using the standard Christoffel symbols [23].

The gravitational potential energy $P_{_g} \in \mathbb{R}$ of the SIPR system is given by,

$$P_{_g} = \int_0^1 \int_{-0.5}^{0.5} mg(x_{s,d} \sin(\phi) + y_{s,d}\cos(\phi)) \mathrm{d}d\mathrm{d}s$$

where ϕ defines the direction of the gravitational field which in this work $\phi = 0$. Therefore the gravity terms vetcor $\mathbf{G}(q_0, \theta) \in \mathbb{R}^2$ is evaluated as,

$$\mathbf{G}(q_0, \theta) = \nabla_q \left(P_{_g}\right).$$

Finally, adding the stiffness and damping terms, the complete dynamics are,

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{B}_{_\beta}\dot{\mathbf{q}} + \mathbf{K}_{_k}\mathbf{q} + \mathbf{G}(\mathbf{q}) = \begin{bmatrix} \tau \\ 0 \end{bmatrix} \quad (1)$$

where the stiffness $\mathbf{K}_{_k} = \left(\begin{smallmatrix} k & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and damping $\mathbf{B}_{_\beta} = \left(\begin{smallmatrix} \beta & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Note that, only the degree of freedom associated with the curvature has damping and stiffness, and the base rotation is free.

Remark 1: Since the SIPR dynamics are described as a Lagrangian system, they posses the following two fundamental properties [23], [24].

Property 1.1: $\mathbf{M}(\mathbf{q})$ is symmetric and positive definite and bounded- i.e there exist positive constants λ_a and λ_b such that $\lambda_a \leqslant \mathbf{M}(\mathbf{q}) \leqslant \lambda_b$.

Property 1.2: With $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ evaluated using Christoffel symbols, the matrix $\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}$ is skew symmetric.

Suppressing the arguments due to space and substituting for stiffness $\mathbf{K}_{_k}$ and damping $\mathbf{B}_{_\beta}$, we can re-write (1) as,

$$\begin{bmatrix} \ddot{q}_0 \\ \ddot{\theta} \end{bmatrix} = \mathbf{M}^{-1}\begin{bmatrix} \tau \\ 0 \end{bmatrix} - \mathbf{M}^{-1}\left(\mathbf{C}\begin{bmatrix} \dot{q}_0 \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} \beta\dot{q}_0 \\ 0 \end{bmatrix} + \begin{bmatrix} kq_0 \\ 0 \end{bmatrix} + \mathbf{G}\right)$$

and for concise representation defining $f_{11}, f_{12}, f_{21}, f_{22}$ accordingly,

$$\begin{aligned} \ddot{q}_0 &= f_{11}\tau + f_{12} \\ \ddot{\theta} &= f_{21}\tau + f_{22}. \end{aligned} \quad (2)$$

C. Equilibria

The soft robot model (1) has two equilibrium points for $\tau = 0$. One is $(q_o, \theta, \dot{q}_0, \dot{\theta}) = (0, 0, 0, 0)$ which is the unstable vertical upright equilibrium positions. The other $(q_o, \theta, \dot{q}_0, \dot{\theta}) = (0, \pi, 0, 0)$ is the bottom-down stable equilibrium position. In this work, we want to avoid the bottom-down position and reach the upright position.

D. Energy

Total energy of the SIPR system is given by,

$$\begin{aligned} E =& \frac{1}{2}\dot{\mathbf{q}}^\top\mathbf{M}\dot{\mathbf{q}} + \frac{1}{2}\mathbf{q}^\top\mathbf{K}_{_k}\mathbf{q} + P_{_g} \\ =& \frac{1}{2}\begin{bmatrix} \dot{q}_0 \\ \dot{\theta} \end{bmatrix}^\top \mathbf{M}\begin{bmatrix} \dot{q}_0 \\ \dot{\theta} \end{bmatrix} + \frac{1}{2}kq_0^2 \\ &+ \frac{Lgm}{q_0^2}\left(\cos\theta - q_0\sin\theta - \cos(q_0 - \theta)\right) \end{aligned} \quad (3)$$

Now, differentiating the total energy (3) with time and from (1) yields,

$$\begin{aligned} \dot{E} =& \dot{\mathbf{q}}^\top\mathbf{M}\ddot{\mathbf{q}} + \frac{1}{2}\dot{\mathbf{q}}^\top\dot{\mathbf{M}}\dot{\mathbf{q}} + \dot{\mathbf{q}}^\top\mathbf{K}_{_k}\mathbf{q} + \dot{\mathbf{q}}^\top\mathbf{G} \\ =& \dot{\mathbf{q}}^\top\left(\begin{bmatrix} \tau \\ 0 \end{bmatrix} - \mathbf{C}\dot{\mathbf{q}} - \mathbf{B}_{_\beta}\dot{\mathbf{q}} - \mathbf{K}_{_k}\mathbf{q} - \mathbf{G}\right) + \frac{1}{2}\dot{\mathbf{q}}^\top\dot{\mathbf{M}}\dot{\mathbf{q}} \\ &+ \dot{\mathbf{q}}^\top\mathbf{K}_{_k}\mathbf{q} + \dot{\mathbf{q}}^\top\mathbf{G} \\ =& \dot{\mathbf{q}}^\top\begin{bmatrix} \tau \\ 0 \end{bmatrix} - \dot{\mathbf{q}}^\top\mathbf{B}_{_\beta}\dot{\mathbf{q}} \\ =& \tau\dot{q}_0 - \beta\dot{q}_0{}^2 \end{aligned} \quad (4)$$

where we have used the skew symmetric property (1.2) of the Lagrangian systems.

III. Swing up control

In this section we will develop the swing up control to bring the SRIP system near the vertical upright position. To that end, we will use the energy based swing up control inspired by [12].

A. Control development

Recall that the control objective is to reach the top upright position $(q_o, \theta, \dot{q}_0, \dot{\theta}) = (0, 0, 0, 0)$. The total energy of the system in this configuration is,

$$E_d = \frac{mgL}{2}.$$

Now let us consider the following three conditions,

$$c1)\; E = E_d, \quad c2)\; q_0 = 0, \quad c3)\; \dot{q}_0 = 0. \quad (5)$$

If the above conditions $c1) - c3)$ are satisfied, then substituting L.H.S and R.H.S in (3) with explicit terms yields,

$$\frac{mgL}{2} = \frac{m(D^2 + 4L^2)\dot{\theta}^2}{24} + \frac{Lgm\cos\theta}{2}$$

resulting in the homoclinic orbit,

$$\alpha\dot{\theta}^2 = \frac{gL}{2}\left(1 - \cos\theta\right) \quad (6)$$

where $\alpha = \frac{m(D^2+4L^2)}{24}$. This will ensure that the soft pendulum will rotate clockwise or counter-clockwise until it reaches $(\theta, \dot{\theta}) = (0, 0)$. Therefore, for $(q_0, \dot{q}_0) = (0, 0)$, if the system can be driven to the above homoclinic orbit, it will solve the objective of "swinging up". Once the system reaches near the top upright position the control can be switched to a stabilizing controller. In this work we will consider an LQR controller for this purpose and it will be discussed subsequently in Section IV.

In view of the control objective for the "swing up", let's define the error in energy $\tilde{E} = E - E_d$, error in degree of curvature $\tilde{q_0} = q_0 - 0 = q_0$ and error in rate of change of degree of curvature $\dot{\tilde{q_0}} = \dot{q}_0 - 0 = \dot{q}_0$. Following [12], we shall seek a Lyapunov based control by selecting the Lyapunov function candidate,

$$V(q, \dot{q}) = \frac{k_e\tilde{E}^2}{2} + \frac{k_d\dot{q_0}^2}{2} + \frac{k_p{q_0}^2}{2} \quad (7)$$

with $k_e, k_d, k_p > 0$ constant gains. Then differentiating V we get,

$$\begin{aligned}\dot{V} &= k_e\tilde{E}\dot{E} + k_d\dot{q}_0\ddot{q}_0 + k_p q_0\dot{q}_0 \\ &= k_e\tilde{E}(\tau\dot{q}_0 - \beta\dot{q}_0{}^2) + k_d\dot{q}_0\ddot{q}_0 + k_p q_0\dot{q}_0 \\ &= \dot{q}_0\left(k_e\tilde{E}(\tau - \beta\dot{q}_0) + k_d\ddot{q}_0 + k_p q_0\right).\end{aligned}$$

Let us now choose the control to satisfy,

$$\begin{aligned}-\dot{q}_0 &= \left(k_e\tilde{E}(\tau - \beta\dot{q}_0) + k_d\ddot{q}_0 + k_p q_0\right) \\ &= \left(k_e\tilde{E}(\tau - \beta\dot{q}_0) + k_d(f_{11}\tau + f_{12}) + k_p q_0\right) \\ &= \tau(k_e\tilde{E} + k_d f_{11}) + (-k_e\tilde{E}\beta\dot{q}_0 + k_d f_{12} + k_p q_0)\end{aligned} \tag{8}$$

and therefore let,

$$\tau = \frac{\left(-\dot{q}_0 + k_e\tilde{E}\beta\dot{q}_0 - k_d f_{12} - k_p q_0\right)}{(k_e\tilde{E} + k_d f_{11})}. \tag{9}$$

This will yield,

$$\dot{V} = -\dot{q}_0^2 \le 0$$

which is negative semi-definite. The stability for this selection of the Lyapunov function candidate will be proved using the Lasalle's invariance principle subsequently under Theorem 1.

Here the control law (9) will have no singularities provided that,

$$(k_e\tilde{E} + k_d f_{11}) \neq 0$$

which is satisfied for some $\epsilon > 0$,

$$|\tilde{E}| \le \left(\frac{k_d}{k_e} - \epsilon\right)|f_{11}|. \tag{10}$$

Since f_{11} is the $(1,1)$ element of $\mathbf{M}^{-1}$ (inverse of the inertia matrix) and as the inertia matrix $\mathbf{M}$ is bounded (property 1.1), f_{11} is bounded.

Also note that the control will produce no action if the soft robot is in either of the equilibrium positions. Hence, in order to exclude being stuck in the undesirable bottom-down equilibrium, we require,

$$|\tilde{E}| < |E_d - E_{bottom}| = mgL. \tag{11}$$

Therefore, taking both the constraints (10) and (11) into account we need,

$$|\tilde{E}| < c = \min\left(mgL, \left(\frac{k_d}{k_e} - \epsilon\right)|f_{11}|\right) \tag{12}$$

Since $\dot{V} \le 0$, the Lyapunov function candidate V is non increasing and therefore the condition for $\tilde{E}$ will hold if the initial conditions are such that $V(0) \le c^2/2$ for $k_e \ge 1$.

B. Stability analysis

Here we present the main result in Theorem 1 and use similar arguments as in [12], [13] to prove the stability.

Theorem 1: Given $k_e \ge 1$, $k_d, k_p > 0$ and the initial conditions are such that $|\tilde{E}| < c$ and $V(0) \le \frac{c^2}{2}$ for the choice of Lyapunov function candidate (7), then the control law,

$$\tau = \frac{\left(-\dot{q}_0 + k_e\tilde{E}\beta\dot{q}_0 - k_d f_{12} - k_p q_0\right)}{(k_e\tilde{E} + k_d f_{11})} \tag{13}$$

will drive the SPIR system (1) to the invariant set given by $\mathrm{Q} = \left\{(q_0, \theta, \dot{q}_0, \dot{\theta}) : q_0 \equiv 0, \dot{q}_0 \equiv 0, \alpha\dot{\theta}^2 = \frac{gL}{2}(1 - \cos\theta)\right\}$ $\bigcup\left\{(q_0, \theta, \dot{q}_0, \dot{\theta}) : (q_0, \theta, \dot{q}_0, \dot{\theta}) = (\epsilon, 0, \epsilon, 0), |\epsilon| < \epsilon^*\right\}$ where $\alpha = \frac{m(D^2+4L^2)}{24}$ and ϵ^* is arbitrarily small.

Proof: We will use LaSalle's Invariance Principle to show that the controller is stable. Since $\dot{V} = -\dot{q}_0^2 \le 0$, V is non-increasing. Therefore $q_0, \dot{q}_0, \theta, \dot{\theta}$ are bounded. Further, $\tilde{E}$ is bounded. Let the set Ω be the compact set where every solution for the soft robot system (1) remains in the same for all future time. Let $\Gamma \in \Omega$ such that $\Gamma = \{(q_0, \theta, \dot{q}_0, \dot{\theta}) : \dot{V} \equiv 0\}$. Let $\mathrm{Q} \in \Gamma$ be the largest invariant set in Γ. LaSalle's Invariance Principle ensures that every solution starting in Ω approaches Q as $t \to \infty$. In the following, we will compute the largest invariant set Q using the steps in [12].

Considering Γ, $\dot{V} \equiv 0$ which implies $\dot{q}_0 \equiv 0$. This implies that $q_0 =$ constant and $\ddot{q}_0 \equiv 0$. Also, V is constant. Therefore from (7) $\tilde{E}$ is constant. This implies that either $\tilde{E} = 0$ or $\tilde{E} \neq 0$. From control design (8) we obtain that,

$$0 = k_e\tilde{E}\tau + k_p q_0. \tag{14}$$

Now, from (14), if $\tilde{E} = 0$, then $q_0 = 0$ which means that since we considered $\dot{q}_0 = 0$, the conditions $c1) - c3)$ in (5) are satisfied and the trajectory belongs to the homoclinic orbit (6).

Next, considering $\tilde{E}$ constant and $\tilde{E} \neq 0$, the condition (14) implies that τ is constant. Since V is nonincreasing,

$$\begin{aligned}V = \frac{k_e\tilde{E}^2}{2} + \frac{k_p q_0^2}{2} &\le V(0) \\ \frac{k_p q_0^2}{2} &\le V \le V(0) \\ \sqrt{k_p}|q_0| &\le \sqrt{2V(0)}\end{aligned}$$

From (14),

$$\begin{aligned}k_e\tilde{E}\tau &= -\sqrt{k_p}\sqrt{k_p}q_0 \\ k_e|\tilde{E}\tau| = \sqrt{k_p}\sqrt{k_p}|q_0| &\le \sqrt{k_p}\sqrt{2V(0)} \\ k_e|\tilde{E}\tau| &\le \sqrt{k_p}c \\ |\tilde{E}\tau| &\le \frac{\sqrt{k_p}}{k_e}c.\end{aligned}$$

Therefore, we can select k_p and k_e appropriately such that, $|\tilde{E}\tau|$ will be small implying that τ is small since $|\tilde{E}|$ is bounded.

Now consider that, since $\dot{q}_0 \equiv 0$, q_0 is constant. Considering the base rotation, it should be either constant (i.e, $\dot{\theta} = 0$) or rotating (i.e, $\dot{\theta} \neq 0$). If $\dot{\theta} \neq 0$ (rotating) then it will impose a gravity induced torque to the soft robot body that will change the degree of curvature q_0, which contradicts the fact that the curvature q_0 = constant. Therefore, we conclude that $\dot{\theta} \equiv 0$ and hence θ is a constant.

Note that the curvature and the base rotation are stationary (i.e, q_0 = constant and θ = constant) only if τ is exactly compensating gravity and the stiffness induced forces. We earlier obtained that τ is chosen to be small. If q_0 is far from 0 then, τ will always be large to compensate for the stiffness. Therefore, q_0 has to be close to zero. For small τ, then $\theta = 0$ (upright) or $\theta = \pi$ (bottom-down). Since $\theta = \pi$ is excluded considering the initial conditions, we conclude that θ is close to zero. Thus, both $|q_0| < \epsilon^*$ and $|\theta| < \epsilon^*$ for $\epsilon*$ arbitrarily small. Moreover, if $\tau = 0, \dot{q}_0 = 0, \ddot{q}_0 = 0$ and $q_0 = 0$, using (2) we can easily show that $\theta \equiv 0$.

Therefore, we conclude that the largest invariant set Q is given by the set satisfying the homoclinic orbit (6) with $q_0 \equiv 0$ and $\dot{q}_0 \equiv 0$, and the interval $(q_0, \theta, \dot{q}_0, \dot{\theta}) = (\epsilon, 0, \epsilon, 0)$ where $|\epsilon| < \epsilon^*$ with ϵ^* arbitrarily small. This completes the proof. ∎

The main objective of the swing up controller is to bring the soft robotic system to a region of attraction of a stabilizing controller. In this work, we will use a linear quadratic regulator (LQR) as the stabilizing controller.

IV. LQR for upright stabilization

We will discuss the LQR to stabilize the inverted soft pendulum in the upright equilibrium.

Let the state vector, $\mathbf{x}(\mathbf{t}) = [q_0(t), \theta(t), \dot{q}_0(t), \dot{\theta}(t)]^\top$ and control input $u(t) = \tau(t)$. Then linearizing [24] around the upright equilibrium $(q_o, \theta, \dot{q}_0, \dot{\theta}) = (0, 0, 0, 0)$, $\tau = 0$ we get the linear time invariant (LTI) system,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(\mathbf{t}) + \mathbf{B}u(t)$$

where $\mathbf{A} \in \mathbb{R}^{4\times 4}$ is the system matrix and $\mathbf{B} \in \mathbb{R}^{4\times 1}$ is the input matrix. Using $\mathbf{A}$ and $\mathbf{B}$ matrices we can show that this system is controllable.

For stabilization at the vertical upright position, we can formulate the LQR problem,

$$\min_u \int_0^T \frac{1}{2}\left(\mathbf{x}^\top \mathbf{Q}\mathbf{x} + \mathbf{R}u^2\right) dt$$

to find the control $u(t)$ which is of the form,

$$u(t) = -(\mathbf{R}^{-1}\mathbf{B}^\top \mathbf{P})\mathbf{x}(t) = -\mathbf{K}\mathbf{x}(t). \quad (15)$$

Here $\mathbf{R} > 0$ and $\mathbf{P}$ is the solution to the algebraic Riccati equation,

$$\mathbf{A}^\top \mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top \mathbf{P} + \mathbf{Q} = 0$$

where $\mathbf{Q}$ is a positive semidefinite matrix. In this work, we used $\mathbf{Q} = \mathbb{I}_{4\times 4}$ and $\mathbf{R} = 1$ and find the gain $\mathbf{K}$ using MATLAB command `lqr(A,B,Q,R)`.

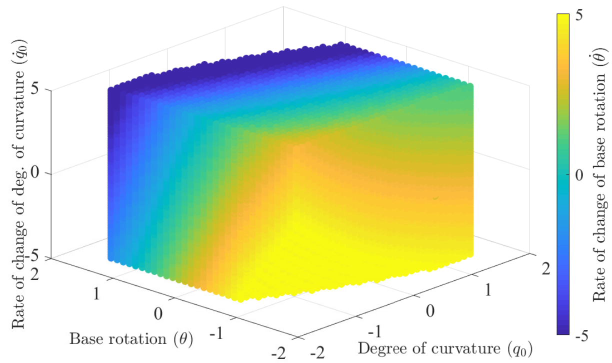


Fig. 3. Numerically computed region of attraction (ROA) for the LQR in the selected range of initial conditions. The deg. of curvature q_0 and base rotation θ are in radians and their rate of change are in rad/s.

V. Simulation results

We present the simulation results for the combined swing up control and upright stabilization in this section. We considered a soft robotic system with length $L = 1$ m, width $D = 0.1$ m, mass $m = 2$ kg, stiffness $k = 0.5$ Nm/rad and damping $\beta = 0.1$ Nms/rad. The simulations were performed in MATLAB 2019a using the ode45 function. For the swing up control, we used $k_p = 1.1$, $k_d = 1.25$ and $k_e = 3$.

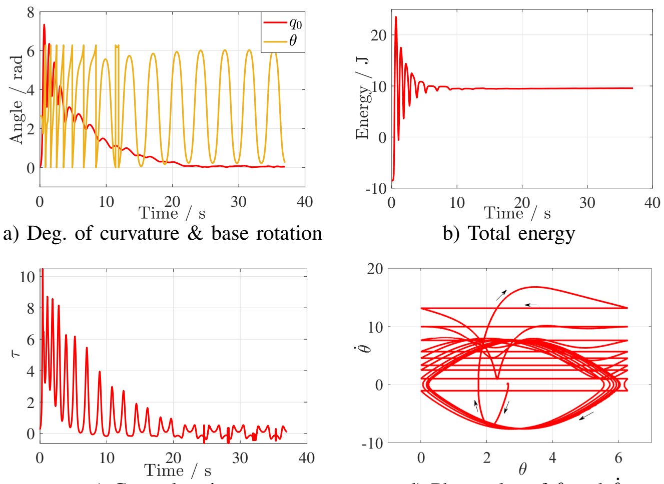
Considering the LQR, the computed gain was $\mathbf{K} = [-84.74, -251.19, -23.54, -65.46]$. The region of attraction (ROA) for the LQR was computed using numerical simulation applying the controller at different initial conditions on the following ranges: $q_0 \in [-1.5, 1.5]$, $\theta \in [-1.5, 1.5]$, $\dot{q}_0 \in [-5, 5]$ and $\dot{\theta} \in [-5, 5]$. Here q_0, θ are in radians and $\dot{q}_0$, $\dot{\theta}$ are in rad/s. The computed ROA is illustrated in Fig.3.

First, we illustrate the performance of the swing up control showing that as the curvature is getting closer to zero, the base rotation will remain swinging and it converges to a homoclinic orbit as in Fig.4. Here the initial conditions were $(q_o, \theta, \dot{q}_0, \dot{\theta}) = (\pi/72, \pi/1.2, 0, 0)$.

Next, we present the simulation results for the combined control of swing up control and stabilization at the upright position using LQR. Once the swing up control takes the system to the ROA, the control switched to the LQR and the soft robot is stabilized in the upright position. The results are shown in Fig.5. Here also the system was initialized $(q_o, \theta, \dot{q}_0, \dot{\theta}) = (\pi/72, \pi/1.2, 0, 0)$. Around 9.35s the control switches to LQR as indicated in the Fig.5.

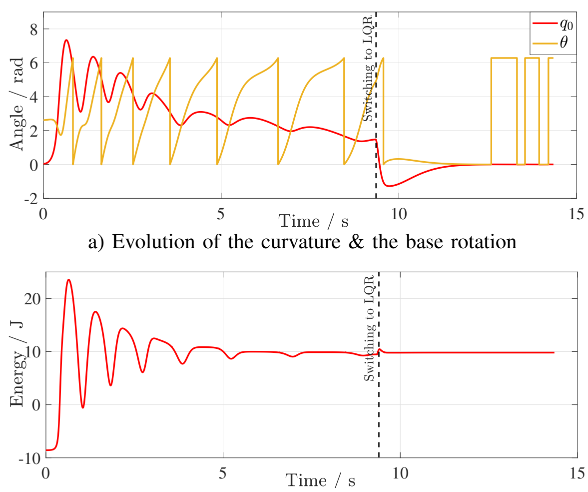
VI. Conclusion and future work

A novel underactuated soft robotic system- soft inverted pendulum with revolute base (SIPR) was introduced in this paper. The soft robot was considered to have a constant curvature which was actuated while the base was allowed to rotate freely. The dynamics of the system were derived and a switching based control was developed. For swing up, an energy based control method was developed and for stabilization of the soft system at the upright unstable



a) Deg. of curvature & base rotation
b) Total energy
c) Control action
d) Phase plot of θ and $\dot{\theta}$

Fig. 4. Illustration of the simulation results for swing up only starting from $(q_o, \theta, \dot{q}_0, \dot{\theta}) = (\pi/72, \pi/1.2, 0, 0)$. The phase plot shows that the trajectory is converging o a homoclinic orbit. The vertical lines on the base rotation plot and
the horizontal lines on the phase plot are due to the fact that $\theta \in \mathbb{S}^1$.



a) Evolution of the curvature & the base rotation
b) Evolution of the total energy of the system

Fig. 5. Simulation results for combined control of swing up and LQR are shown here. The control is switched from swing up to LQR around 9.35s. The vertical lines on the base rotation plot are due to the fact that $\theta \in \mathbb{S}^1$.

equilibrium an LQR was used. The stability of the proposed control approach was analyzed and simulation results were presented to illustrate the efficacy of the proposed control framework.

Future work include, extending the considered soft system for general non-constant curvature soft robots and validating the results in physical experiments.

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