Optimization of perishable asset revenue management problems that allow prices as decision variables

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Abstract: This paper addresses a new slant to a problem which is general to the service industries – perishable asset revenue management. Traditional approaches have assumed that prices are fixed and solved for the optimal allocation quantities. Our approach recognizes that prices affect demand and should therefore be included as decision variables to be optimized. We solve three different types of problems: a) up to n price classes, distinct asset control mechanism, and no diversion, b) up to three price classes, nested asset control mechanism, and no diversion, c) up to three price classes, nested asset control mechanism, and diversion. Analytical results are provided in most cases and examples illustrate the results as well as the time required to solve these complex problems. Finally we look at the trade-offs involved between computational time and expected contribution when using heuristic decisions obtained from less realistic assumptions relative to the true optimal decisions. On average, the suboptimality ranged from 0.84% to 8.3% with a corresponding decrease in computing time required on the order of several minutes. Some trends are presented to help determine a priori which type of problems would tend to benefit most from the more accurate formulation. This helps managers decide when it is worth the extra computing time to come up with the true optimal solution.

Keywords: Pricing, yield management; perishable asset revenue management.

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Biographical notes: Lawrence R. Weatherford is an Assistant Professor in the College of Business at the University of Wyoming. He received his BA from Brigham Young University in 1982, and his MBA and PhD from the Darden Graduate School of Business at the University of Virginia in 1991. He received the Outstanding Teaching Award for the College of Business in his first year as a professor. In the ensuing years he has also earned the ‘Outstanding Faculty Member’ award by Alpha Kappa Psi, and the Outstanding Junior Research Award for the College of Business. He has published several scholarly articles in such journals as Operations Research, Transportation Science, Naval Research Logistics and Omega.

On the practitioner side, he was featured in the 'Questions and Answers' section of Scorecard (the Revenue Management Quarterly) in the second quarter of 1994. He also wrote the technical brief section of Scorecard for the same issue on the joint optimization of prices with allocation decisions. Larry has made presentations of his research to the Yield Management study group of Agifors for the last three years and the IATA International Revenue

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1 Introduction

This paper addresses a problem (Perishable Asset Revenue Management or PARM) which is general to the service industries. Initially, the methodology of yield management was applied to the airline industry [1]. Several other authors have recognized the similarity of the airline problem to numerous other service industries like hotels, rental cars, and cruise lines [2,3]. In fact the common threads of the problem that appeared throughout businesses in the service sector were identified [3] as:

1. a perishable good or service,
2. a fixed number of units, and
3. the possibility of segmenting price-sensitive customers.

It was recognized that fixed capacity was not necessary in order to practice PARM tools.

To date, however, the literature on this important subject has not addressed the fact that prices are truly decision variables and not fixed quantities. Pricing is not a separable issue, although pricing is often done by a separate department in the company. Several researchers recognized this fact in the inventory management literature as they addressed not only the optimal inventory policy, but also the fact that the price charged would influence demand and therefore should be a part of the overall optimal solution or policy [4-10]. Others have addressed different factors that affect demand like sales effort [11]. Because of the importance of pricing and segmentation in general, we also reviewed a couple of general articles on the subject [12-15]. Two more recent articles have looked at problems more closely related to the one we address [16,17]. In [16] optimal dynamic pricing for one price class over a finite horizon is provided using inventory control theory. In [17] an optimal pricing policy for cruise liners is provided where demand is a function of price, but demand is deterministic for a given price. In [18] the solution to the joint pricing and allocation news vendor problem (one price class) is presented as well as some preliminary results for the two price class problem with the lower price assumed as fixed.

The drawback of the above papers is that either they are dealing with problems with goods that are storable as opposed to the situation we address where they are perishable; or if they do deal with perishable goods, they only allow one price class, e.g., for an aeroplane they only address a full fare category and do not allow for discount price classes. Our analysis allows multiple classes. The converse problem (fix the allocations and solve for the optimal prices) which is a problem of interest to the TV and cable advertising industry has been addressed [19]. The issue of finding optimal prices for an airline given nonoptimally set allocation decisions has also been addressed [20].

Our paper treats the problem with perishable goods, more than one price class, and random demand. In all of the models we look at, there are two underlying assumptions which we explain and defend presently. First, we assume elasticity models where the mean of demand is assumed to be linear in price. This is both plausible and has support in previous literature [4,5]. The second common assumption is that demand is assumed to
be normally distributed. Finally, the demand distributions are assumed to be independent for the different price classes which is an approximation of reality. The problems are classified according to the standard taxonomy [3], see Appendix 1 for easy reference. It is assumed throughout that we have a continuous resource, fixed capacity, prices that are set jointly with the allocation decision, buildup willingness to pay, random and independent reservation demand, certain show-up of discount reservation, certain show-up of full price reservation, no group reservations, and no displacement.

As far as notation throughout the paper, let:

R_i represent the contribution (Revenue minus variable cost) from those who buy at the ith price class (decision variable),

X_i represent the random demand for price class i which is normally distributed with mean, μ_i = a_i*R_i + b_i, and standard deviation σ_i (random variable),

q_i represent the maximum number of units that can be sold to the ith price class (decision variable),

a_i is the slope parameter for the elasticity of μ_i, and

b_i is the intercept parameter for the elasticity of μ_i.

The remainder of the paper is organized as follows: Sections 2, 3, and 4 look at three different types of joint pricing and allocation problems that have been solved analytically and examples are presented. Section 5 presents a sensitivity analysis involving the trade-off involved between computational time and expected contribution when using the optimal decisions compared to heuristic decisions (much faster, yet suboptimal expected contribution) and Section 6 presents our conclusions and directions for further research.

2 Distinct asset control mechanism, no diversion parm problems

In this section, we look at the problem in one of its simplest forms. We assume that people who are willing to pay the full price do not divert and buy at the lower price(s) if they are available. We also use the simpler asset control mechanism in that we divide up the capacity between the price classes with no overlapping or nesting. For example, if we have capacity of 100 and we decide to set aside 40 for the discount price customers and 60 for the full price customers, that is it – when a 61st full price customer comes along, we turn him/her away even if we have not used all 40 of the discount price allocation. The following subsections develop the rules for different numbers of discount price classes.

2.1 Two price classes

Taxonomy = A2-B1-C3-D1-E1-F3-G1-H1-I1-J1-K1-L1-M1-N1 (see Appendix 1)

Recall that we have a situation with two price classes (discount and full price), each exhibiting random demand [X = N(μ, σ)] and X = N(μ, σ)] where the mean is a function of price or contribution (μ = a*R + b, where a < 0). The contribution from each is R and R respectively. Let q represent the maximum number of discount units we will
sell and \( q_0 \) represent the maximum number of full price units to sell. Because we have fixed capacity, there are really only three decision variables (\( R_1 \), \( R_0 \), and either \( q_i \) or \( q_0 \)) and two random variables (\( X_i \), \( X_0 \)). An expression for the contribution towards fixed costs (a surrogate for profit) can be expressed as follows:

\[
\text{Contribution} = \begin{cases} 
X_1 R_1 + X_0 R_0 & \text{if } X_1 \leq q_i \text{ and } X_0 \leq q_0 \\
X_1 R_i + q_0 R_0 & \text{if } X_1 \leq q_i \text{ and } X_0 > q_0 \\
q_1 R_1 + X_0 R_0 & \text{if } X_1 > q_i \text{ and } X_0 \leq q_0 \\
q_1 R_1 + q_0 R_0 & \text{if } X_1 > q_i \text{ and } X_0 > q_0 
\end{cases}
\]

(1)

Integrating over all the possible combinations of \( X_0 \) and \( X_1 \) gives us the following expression for the expected contribution:

\[
\text{Exp. Contrib.} = \int_{X_1=0}^{q_1} \int_{X_0=0}^{q_0} (X_1 R_1 + X_0 R_0) f(X_1) f(X_0) \, dX_0 \, dX_1 \\
+ \int_{X_1=q_1}^{\infty} \int_{X_0=q_0}^{\infty} (q_1 R_1 + X_0 R_0) f(X_1) f(X_0) \, dX_0 \, dX_1 \\
+ \int_{X_1=0}^{q_1} \int_{X_0=q_0}^{\infty} (q_1 R_1 + q_0 R_0) f(X_1) f(X_0) \, dX_0 \, dX_1 \\
+ \int_{X_1=0}^{q_0} \int_{X_0=q_0}^{\infty} (X_1 R_1 + q_0 R_0) f(X_1) f(X_0) \, dX_0 \, dX_1.
\]

(2)

Applying the normal distribution assumption mentioned earlier for \( X_i \) and \( X_p \), we can evaluate the double integrals, simplify, and obtain a closed-form result for the expected contribution (see equation A-1 of Appendix 2). There are a couple of different approaches that can be used to solve for the values of \( q_1 \), \( R_0 \), and \( R_1 \) that maximize expected contribution (\( q_0 \) is not a true decision variable since it is determined once \( q_i \) is set, as equal to \( \text{CAPACITY} - q_i \)). The simplest way in this case is to use a spreadsheet-based nonlinear optimizer (like the one available in EXCEL or an add-in to Lotus 1-2-3). This is possible because all of the terms in equation (A-1) can be evaluated in a spreadsheet and it is simply a matter of identifying the objective function (MAXIMIZE Expected Contribution), the decision variables (\( q_i, R_0, R_1 \)), and the following constraints:

\begin{align*}
1) \ R_1 & \leq R_0 \\
2) \ R_0 & \leq \frac{3 \sigma_0 - b_0}{\omega} \\
3) \ R_1 & \leq \frac{3 \sigma_1 - b_1}{\alpha_i}
\end{align*}

(3)
The last two constraints are added due to the normality assumption of demand and require that the mean demand be at least three times the corresponding standard deviation to minimize the possibility of negative demand and effectively providing a truncated normal distribution. For this type of problem, the spreadsheet-based solution only takes on the order of seconds to be identified.

Another way to solve for the optimal solution (which becomes the only solution approach later) is to code a program (e.g., Turbo Pascal) that uses standard constrained nonlinear optimization techniques (e.g., Powell, Davidon-Fletcher-Powell, etc.). Four different possible methods are described in [21]. We tried all of these and found that Fletcher-Reeves-Polak-Ribiere (FRPR) was the best (i.e., gave the ‘best’ solutions in generally the shortest time). Both approaches (spreadsheet-based or Pascal code) obviously give the same solution. In this case, it is certainly much easier to solve the problems in the spreadsheet.

An example will serve to illustrate the solution. Assume we have the parameters given in Table 1. In just a matter of seconds, the spreadsheet solver tells us that \( q_1^* = 22.7 \), \( R_1^* = $279.12 \), and \( R_0^* = $495.07 \) which gives an expected contribution of $17,938.57. These values for the contributions (or prices) give mean values of demand, \( \mu_0 = 25.37 \) and \( \mu_1 = 24.18 \).

<table>
<thead>
<tr>
<th>Capacity = 50</th>
<th>( a_0 = -0.075 )</th>
<th>( a_1 = -0.2 )</th>
<th>( b_0 = 62.5 )</th>
<th>( b_1 = 80 )</th>
<th>( c = 5 )</th>
<th>( \sigma_i = 5 )</th>
</tr>
</thead>
</table>

Checking the optimality conditions, we find that they all equal zero and thus are satisfied.

An interesting comparison is made between this problem where the price and allocation decisions are jointly optimized and the traditional problem where the price is assumed fixed and the allocation decision is optimized (see [22] for a description of Belobaba’s EMSR decision rule which we use for our basis of comparison). We use the same parameters as Table 1 with the exception that \( R_0 \) and \( R_1 \) are fixed by the pricing department in a nonoptimal way. The table below (Table 2) summarizes three different possible scenarios for prices, the resulting optimal allocation decision and expected contribution. In each case, an indication is given of the percentage difference from the optimal value. For the prices given (varied from 10% below optimal to 25% above), we found the expected contribution to be suboptimal on the order of 1% to 12%.

| Table 2 Comparison of joint optimal with traditional optimization examples |
|-----------------|-----------------|-----------------|
| EXAMPLE #       | 1               | 2               | 3               |
| \( R_0 \)       | $500            | $450            | $550            |
| \% from optimal | 1%              | -9%             | 11%             |
| \( R_1 \)       | $350            | $300            | $250            |
| \% from optimal | 25%             | 7%              | -10%            |
| \( q_1^* \)     | 17              | 19.5            | 26.7            |
| \% from optimal | -24.9%          | -14%            | 18%             |
| Exp. Contrib.   | $15,877.74      | $17,702.82      | $17,547.21      |
| % from optimal  | -11.5%          | -13%            | -22%            |
2.2 Three price classes


Now we have a situation with three price classes (two discounts and full price), each exhibiting random demand \( X_2 \sim N(\mu_2, \sigma_2), X_1 \sim N(\mu_1, \sigma_1) \) and \( X_0 \sim N(\mu_0, \sigma_0) \) with the same conditions as before. Because we have fixed capacity, there are really only five decision variables \( (R_2, R_1, R_0, q_2, q_1) \) \( q_0 \) is not a true decision variable since it is determined once \( q_1 \) and \( q_2 \) are set, as equal to \( \text{CAPACITY} - q_0 \) and three random variables \( (X_2, X_1, X_0) \). An expression for the contribution towards fixed costs can be expressed as follows:

\[
\text{Contribution} = \begin{cases} 
X_2 R_2 + X_1 R_1 + X_0 R_0 & \text{if } X_2 \leq q_2, X_1 \leq q_1, X_0 \leq q_0 \\
X_2 R_2 + X_1 R_1 + q_0 R_0 & \text{if } X_2 \leq q_2, X_1 \leq q_1, X_0 > q_0 \\
X_2 R_2 + q_1 R_1 + X_0 R_0 & \text{if } X_2 \leq q_2, X_1 > q_1, X_0 \leq q_0 \\
X_2 R_2 + q_1 R_1 + q_0 R_0 & \text{if } X_2 \leq q_2, X_1 > q_1, X_0 > q_0 \\
q_2 R_2 + X_1 R_1 + X_0 R_0 & \text{if } X_2 > q_2, X_1 \leq q_1, X_0 \leq q_0 \\
q_2 R_2 + X_1 R_1 + q_0 R_0 & \text{if } X_2 > q_2, X_1 \leq q_1, X_0 > q_0 \\
q_2 R_2 + q_1 R_1 + X_0 R_0 & \text{if } X_2 > q_2, X_1 > q_1, X_0 \leq q_0 \\
q_2 R_2 + q_1 R_1 + q_0 R_0 & \text{if } X_2 > q_2, X_1 > q_1, X_0 > q_0.
\end{cases}
\]

\[(4)\]

Integrating over all the possible combinations of \( X_0, X_1, \) and \( X_2, \) and applying the normal distribution assumption, we can evaluate the triple integrals, simplify, and obtain the closed-form result for the expected contribution as shown in equation (A-6) in Appendix 2.

Again, we impose the following constraints:

1) \( R_2 \leq R_1 \)
2) \( R_1 \leq R_0 \)
3) \( R_0 \leq \frac{3\sigma_0 - b_0}{a_0} \)
\[(5)\]
4) \( R_1 \leq \frac{3\sigma_1 - b_1}{a_1} \)
5) \( R_2 \leq \frac{3\sigma_2 - b_2}{a_2} \)

If we require the solver to select integer values for \( q_1 \) and \( q_2 \), it takes about twice as long to solve.
2.3 General, n discount price classes


At this point, we can deduce a general pattern (it can also be proven without much difficulty) from equations (A-1) and (A-6) for the expected contribution given n price classes as:

\[ R_i \left[ E_{\mu_1}(X_i) + q_i(1 - F_{\mu_1}(q_i)) \right] \]

Again, the terms are all closed-form, so any problem with n price classes can be solved with a spreadsheet-based solver in a relatively quick amount of time. As we shall see, we are only so fortunate with this type of problem (distinct, no diversion). In the next two sections (no diversion with nested asset control and diversion with nested asset control) we are no longer able to obtain closed-form solutions.

3 Nested, no diversion parm problems

The important difference in this section is that we now ‘nest’ our buckets of assets. For example, if the total capacity is 100 and we decide to limit discount unit sales to 35, then \( q_1 = 35 \), but \( q_0 = 100 \) (i.e., not 65 as in §2). This means we will never turn down a request for a full price unit as long as we have any capacity remaining. This obviously makes a lot more sense in terms of how a particular service industry would want to operate. Figure 1 illustrates this for three price classes.

Figure 1 Nested price classes

3.1 Two price classes


We have a situation with two price classes (discount and full price), each exhibiting random demand \( [X_i \sim N(\mu_i, \sigma_i) \text{ and } X_0 \sim N(\mu_0, \sigma_0)] \) where the mean is again a linear function of price. The contribution from each is \( R_1 \) and \( R_0 \) respectively. Let \( q_1 \) represent the maximum number of discount units we will sell and \( q_0 \) represent the capacity (i.e., maximum number of full price units to sell). Because we have fixed capacity, there are three decision variables \( (R_1, R_0, \text{ and } q_1) \) and two random variables \( (X_i, X_0) \). An expression for the contribution towards fixed costs can be expressed as follows:
\[ \text{Contribution} = \begin{cases} 
X_1 R_1 + X_0 R_0 & \text{if } X_1 \leq q_1 \text{ and } X_0 \leq q_0 - X_1 \\
X_1 R_1 + (q_0 - X_1) R_0 & \text{if } X_1 \leq q_1 \text{ and } X_0 > q_0 - X_1 \\
q_1 R_1 + X_0 R_0 & \text{if } X_1 > q_1 \text{ and } X_0 \leq q_0 - q_1 \\
q_1 R_1 + (q_0 - q_1) R_0 & \text{if } X_1 > q_1 \text{ and } X_0 > q_0 - q_1 .
\end{cases} \]

Integrating over all the possible combinations of \(X_0\) and \(X_1\), and applying the normal distribution assumption, we can evaluate the double integrals, simplify, and obtain the following result for the expected contribution as shown in equation (A-8) of Appendix 3.

We now have three terms which are not closed-form. Because of the lack of a closed-form expression for expected contribution and the optimality conditions, our only choice for a solution approach (to solve for the values of \(q_0\), \(R_0\), and \(R_1\) that maximize expected contribution) is to code up a program (e.g., Tarbo Pascal) using the FRPR optimization technique described in §2.1. Again we add the following constraints:

1) \(q_1 \leq q_0\)
2) \(R_1 \leq R_0\)
3) \(R_0 \leq \frac{3\sigma_0 - b_0}{a_0}\)
4) \(R_1 \leq \frac{3\sigma_1 - b_1}{a_1}\). 

The first constraint is added due to the nested feature which now exists. For this type of problem, the solution takes on the order of minutes to be identified. NOTE: All computation times are measured on a standalone 486 personal computer, with a clock speed of 33 MHz.

An example will serve to illustrate the solution. Assume we have the same parameters as given in the previous example (Table 1). In 22 seconds, the Pascal program tells us that \(q_1^* = 40.6\), \(R_1^* = 386.27\), and \(R_0^* = 497.13\) which gives an expected contribution of \$18,630.81\). These values for the contributions (or prices) give mean values of demand, \(\mu_0 = 25.22\) and \(\mu_1 = 22.75\). In this example, the values for the prices are similar to the distinct, no diversion example, but \(q_1\) is quite different. If we were to use the optimal values from the distinct, no diversion example under the assumptions of this model (nested, no diversion), they would yield a suboptimal expected contribution of \$18,365.94\). Thus, the optimal nested decision variables provide a 1.42% improvement in expected contribution for a large increase in computing time.

Section 5 will look in more detail at the trade-off involved between computational time and expected contribution when using the optimal decisions obtained from assuming distinct asset control (obviously much faster) relative to the true optimal decisions under the nested asset control mechanism. This sensitivity analysis should help managers decide when it is worth the extra computing time to come up with the true optimal solution.
3.2 Three price classes

Taxonomy = A2-B1-C3-D1-E2-F3-G1-H1-I1-J1-K1-L1-M2-N1

Consider the situation with three price classes (two discounts and full price), each exhibiting random demand $[X_2 \sim N(\mu_2, \sigma_2), X_1 \sim N(\mu_1, \sigma_1)$ and $X_0 \sim N(\mu_0, \sigma_0)]$ with the same conditions as before. Because we have fixed capacity, there are five decision variables $(R_5, R_4, R_3, q_2, q_1)$ and three random variables $(X_2, X_1, X_0)$. An expression for the contribution towards fixed costs can be expressed as follows:

$$
\begin{align*}
\text{Contribution} = \\
X_2 R_2 &+ X_1 R_1 + X_0 R_0 & \text{if } X_2 \leq q_2, X_1 \leq q_1 - X_2, X_0 \leq q_0 - X_1 - X_2 \\
X_2 R_2 &+ X_1 R_1 + (q_0 - X_1 - X_2) R_0 & \text{if } X_2 \leq q_2, X_1 \leq q_1 - X_2, X_0 > q_0 - X_1 - X_2 \\
X_2 R_2 &+ (q_1 - X_2) R_1 + X_0 R_0 & \text{if } X_2 \leq q_2, X_1 > q_1 - X_2, X_0 \leq q_0 - q_1 \\
X_2 R_2 &+ (q_1 - X_2) R_1 + (q_0 - q_1) R_0 & \text{if } X_2 > q_2, X_1 > q_1 - X_2, X_0 > q_0 - q_1 \\
q_2 R_2 &+ X_1 R_1 + X_0 R_0 & \text{if } X_2 > q_2, X_1 \leq q_1 - q_2, X_0 \leq q_0 - X_1 - q_2 \\
q_2 R_2 &+ X_1 R_1 + (q_0 - X_1 - q_2) R_0 & \text{if } X_2 > q_2, X_1 \leq q_1 - q_2, X_0 > q_0 - X_1 - q_2 \\
q_2 R_2 &+ (q_1 - q_2) R_1 + X_0 R_0 & \text{if } X_2 > q_2, X_1 > q_1 - q_2, X_0 \leq q_0 - q_1 \\
q_2 R_2 &+ (q_1 - q_2) R_1 + (q_0 - q_1) R_0 & \text{if } X_2 > q_2, X_1 > q_1 - q_2, X_0 > q_0 - q_1.
\end{align*}
$$

(9)

Integrating over all the possible combinations of $X_0, X_1$, and $X_2$, and applying the normal distribution assumption, we can evaluate the triple integrals, simplify, and obtain the following result for the expected contribution as shown in equation (A-14) of Appendix 3.

In order to solve this we code up a computer program (e.g., Pascal) using an optimization technique like Powell's that does not use gradients. Using the same input parameters as the two price class example (see Table 1), with the addition of $a_2 = 0.3$, $b_2 = 110$, and $c_2 = 5$, the Pascal program using Powell's optimization technique finds the optimal decisions to be $q_2^* = 28$, $q_1^* = 28$, $R_5^* = \$310.20$, $R_4^* = \$310.59$, $R_3^* = \$356.68$ for an expected contribution of $\$20,067.2$. It makes sense that the expected contribution has increased (relative to the two price class example) since we have now further segmented the market. The computing time required has increased to 3.4 hours. If we use the optimal decision variables from the three price class, distinct asset control mechanism, and no diversion PARM problem from §2.2 ($q_2^* = 14.5$, $q_1^* = 27.1$, $R_5^* = \$301.53$, $R_4^* = \$315.1$, $R_3^* = \$531.54$), it yields an expected contribution of $\$19,831.12$ which is 1.2% less than the optimal value.

4 Nested, diversion parm problems

The important difference in this section is that we now allow the realistic behaviour of a higher price customer diverting to pay a lower price if it is available. For example, a customer may be willing to pay up to $\$500$ for a unit, but if when he makes the reservation one is available for $\$350$, of course, he will buy the cheaper priced unit. Airlines, hotels, and rental cars try to prevent this behaviour by putting up 'fences' [23].
The following subsections assume a strict arrival sequence (deepest discount class arrives first, followed by the next discount class, etc., and finally the full price customers arrive last) in order to implement the diversionary behaviour.

4.1 Two price classes

Taxonomy = A2-B1-C3-D1-E1-F3-G1-H1-I1-J2-K1-L1-M2-N1

We have a situation with two price classes (discount and full price), each exhibiting random demand \(X_t \sim N(\mu_t, \sigma_t)\) and \(X_0 \sim N(\mu_0, \sigma_0)\) where the mean is again a linear function of price. The contribution from each is \(R_t\) and \(R_0\) respectively. Let \(q_t\) represent the maximum number of discount units we will sell and \(q_0\) represent the capacity (i.e., maximum number of full price units to sell). Because we have fixed capacity, there are three decision variables \((R_t, R_0, \text{and } q_t)\) and two random variables \((X_t, X_0)\). An expression for the contribution towards fixed costs can be expressed as follows:

\[
\text{Contribution} = \begin{cases} 
(X_0 + X_t)R_t & \text{if } X_0 \leq q_t - X_t \\
q_tR_t + (X_0 + X_t - q_t)R_0 & \text{if } X_t \leq q_t \text{ and } X_0 \leq X_0 + X_t \leq q_0 \\
q_tR_t + X_0R_0 & \text{if } X_t > q_t \text{ and } X_0 \leq q_0 - q_t \\
q_tR_t + (q_0 - q_t)R_0 & \text{if } X_t \leq q_t \text{ and } X_0 + X_t > q_0 \\
\text{OR } X_t > q_t \text{ and } X_0 > q_0 - q_t & 
\end{cases}
\]

(10)

Integrating over all the possible combinations of \(X_0\) and \(X_t\), and applying the normal distribution assumption, we can evaluate the double integrals, simplify, and obtain the following result for the expected contribution as shown in (A-17) of Appendix 4.

We note that we now have six terms which are not closed-form. Because of the lack of a closed-form expression for expected contribution and the optimality conditions, our only choice for a solution approach (to solve for the values of \(q_t, R_0, \text{and } R_t\) that maximize expected contribution) is to code up a program (e.g., Turbo Pascal) using the FRPR optimization technique described in §2.1. Again we add the following reasonable constraints:

1) \(q_t \leq q_o\)
2) \(R_t \leq R_0\)
3) \(R_0 \leq \frac{3\sigma_0 - b_0}{a_0}\)
4) \(R_t \leq \frac{3\sigma_t - b_1}{a_t}\)

(11)

An example will serve to illustrate the solution. Assume we have the same parameters as given in the previous example (Table 1). In about ten minutes, the Turbo Pascal program tells us that \(q_t^* = 22, R_t^* = $274.37, \text{and } R_0^* = $484.19\) which gives an expected contribution of
$17,820.74. These values for the contributions (or prices) give mean values of demand, 
μ₀ = 26.19 and μ₁ = 25.13. If we were to use the optimal values from the nested, no 
diversion example under the assumptions of this model (nested, diversion), they would 
yield a suboptimal expected contribution of $14,443.4. Thus, the optimal decision 
variables under the more realistic assumptions (nested with diversion) provide a 19%
 improvement in expected contribution for an increase in computing time of 8–9 minutes.

Although it is not the main thrust of this paper, it is worth discussing some 
implementation issues. Obviously, the time required to solve some of these problems is 
very lengthy. In practice, the airlines have different fares for all possible fareclass and 
origin/destination combinations for each departure of a flight. These fares are changing 
all the time. How could a model like this be used to help a real airline? First of all, the 
sensitivity analysis in Section 5 that explores in more detail the trade-off involved 
between computational time and expected contribution when using the optimal decisions 
obtained from assuming nested, no diversion PARM problems (obviously much faster) 
relative to the true optimal decisions under the nested, diversion assumptions should help. 
In other words, this analysis should help managers decide when it is worth the extra 
computing time to come up with the true optimal solution. Therefore, not all flight 
departures need to use the more complicated models that take so long to solve. Also, 
experience tells that it is often necessary to take theoretically sound models and make 
modifications that allow them to be implemented in the real world. Finally, the speed of 
computers keeps increasing at an exponential pace. Already with Pentium chips available 
and soon the P6 chip will be out, the time to solve such problems is cut by a factor of 
four or more. This trend will continue into the foreseeable future.

4.2 Three price classes

Taxonomy = A2-B1-C3-D1-E2-F3-G1-H1-I1-J2-K1-L1-M2-N1

Consider the situation with three price classes (two discounts and full price), each 
exhibiting random demand [X₂ ~ N(μ₂, σ₂), X₁ ~ N(μ₁, σ₁) and X₀ ~ N(μ₀, σ₀)] with the same 
conditions as before. Because we have fixed capacity, there are five decision 
variables (R₂, R₁, R₀, q₂, q₁) and three random variables (X₂, X₁, X₀). An expression for 
the contribution towards fixed costs can be expressed as follows:

\[
\text{Contribution} = \begin{cases} 
(X₀ + X₁ + X₂)R₂ & \text{if Case 1} \\
q₂R₂ + (X₀ + X₁ + X₂ - q₂)R₁ & \text{if Cases 2 or 3} \\
q₁R₂ + (X₀ + X₁)R₁ & \text{if Case 4} \\
q₂R₂ + (q₁ - q₂)R₁ + (X₀ + X₁ + X₂ - q₁)R₁ & \text{if Cases 5 or 6} \\
q₁R₂ + (q₁ - q₂)R₁ + X₁R₀ & \text{if Cases 7 or 9} \\
q₁R₂ + (q₁ - q₂)R₁ + (X₀ + X₁ - q₁)R₀ & \text{if Case 8} \\
q₂R₂ + (q₁ - q₂)R₁ + (q₁ - q₂)R₀ & \text{if Cases 10, 11, 12, 13, or 14} 
\end{cases}
\]

(12)

where cases refer to situations diagrammed in Figure 2.
Integrating over all the possible combinations of $X_0$, $X_1$, and $X_2$, and applying the normal distribution assumption, we can evaluate the triple integrals, simplify, and obtain the following result for the expected contribution as shown in (A-24) of Appendix 4. In order to solve, we code up a computer program (e.g., \texttt{C/C++}) using an optimization technique like Powell's that does not use gradients.

Using the same input parameters as §3.2, the \texttt{pascal} program using Powell’s optimization technique finds the optimal decisions to be $q_1^*=31$, $q_2^*=31$, $R_2^* = $315, $R_3^* = $315, $R_6^* = $515 for an expected contribution of $19,423.00. It makes sense that the expected contribution has decreased (relative to the three price class example under nested, no diversion assumptions) because we now allow for the realistic feature that some of the customers who are willing to pay full price have diverted and obtained a lower price. The computing time required was two hours. If we use the optimal decision variables from the three price class, nested asset control mechanism, and no diversion PARM problem from §3.2 ($q_1^* = 28$, $q_2^* = 28$, $R_2^* = $310.2, $R_3^* = $310.99, $R_6^* = $536.68), it yields an expected contribution of $10,083.86 which is 48% less than the optimal value.

5 Sensitivity analysis.

The results from the previous three sections indicate that there is a trade-off between the more realistic assumptions and the time required to solve the resulting problem. It would be most helpful to managers to explore how far from the optimal expected contribution we are if we use decisions ($R_1$, $R_6$, $q_1$) obtained more quickly under less realistic assumptions (distinct or no diversion) as surrogates for decisions that would be obtained under more realistic assumptions (nested or diversion).

We will make two studies of this effect. Both will assume two price classes. In the first study we will compare the nested asset control mechanism vs. the distinct asset control mechanism (under the assumption of no diversion). The second study will compare diversion vs. no diversion (under the assumption of a nested asset control mechanism). Our experimental design consists of six different combinations of elasticity parameters, three different values for capacity ($q_0$), and seven different $\sigma_j/\sigma_i$ combinations for a total of 126 total different combinations.

5.1 Nested vs distinct (no diversion) sensitivity analysis

Over the 126 cases, the average improvement was 0.84% while the average time increased from seconds to 2.49 minutes. Figure 3 demonstrates a probability distribution function for the % improvements over the 126 cases. As can be seen, nearly 80% of the cases showed no improvement from using the more realistic assumptions. Another 10% of the cases showed between 0-1% improvement. There were some extreme cases where the % improvement was as high as 19%.
We then tried to classify the cases to try and predict a priori from the input parameters which cases would tend to have larger % improvements. The following trends were noticed:

- As $a_0$ and $a_1$ ↓ (more elastic), % improvement ↑
- As $b_0$ ↑ and $b_1$ ↓, % improvement ↑
- As $\sigma_0$ ↑ and $\sigma_1$ ↓, % improvement ↑

5.2 Diversion vs. no diversion (nested) sensitivity analysis

Over the 126 cases, the average improvement was 8.3% while the average time increased from 5.83 minutes to 6.96 minutes. Figure 4 demonstrates a probability distribution function for the % improvements over the 126 cases. As can be seen, nearly 70% of the cases showed no improvement from using the more realistic assumptions. Another 10% of the cases showed between 0-10% improvement, but some 20% of the cases were at least 30% off. There were some extreme cases where the % improvement was as high as 150%.

We then tried to classify the cases to try and predict a priori from the input parameters which cases would tend to have larger % improvements. The following trends were noticed:

- As $a_0$ and $a_1$ ↓ (more elastic), % improvement ↑
- As $b_0$ ↑ and $b_1$ ↓, % improvement ↑
- As $\sigma_0$ ↓ and $\sigma_1$ ↓ (less uncertainty), % improvement ↑
- As $q_0$ ↑, % improvement ↑

6 Conclusions and future research directions

In summary, this paper has addressed a key issue for many service industries. For typical PARM problems, the prices are assumed to be fixed when in reality, they should be part of the optimization process. This paper shows how this can be done for several different types of PARM problems. As the assumptions become more realistic, obviously the solutions become less tractable. But even for the difficult problems, examples were given to show the optimal solutions can still be obtained via numerical methods.
Next we looked in more detail at the trade-off involved between computational time and expected contribution when using the optimal decisions obtained from less realistic assumptions (e.g., no diversion or distinct asset control mechanism) relative to the true optimal decisions under the more realistic assumptions (e.g., diversion or nested asset control mechanism). Obviously solutions can be obtained much faster under the less realistic assumptions, but they will yield suboptimal solutions. On average, the suboptimality ranged from 0.84% (in comparing the nested vs. the distinct asset control mechanism) to 8.3% (in comparing diversion vs. no diversion). Some trends were noticed in helping to determine a priori which type of problems would tend to benefit most from the more accurate formulation. This should help managers decide when it is worth the extra computing time to come up with the true optimal solution.

Future work needs to look at a similar sensitivity analysis for the case of three price classes and random variable distributions other than the normal where $X_i$ is only defined for non-negative values (e.g., Poisson).

References

Appendix 1  Taxonomy for perishable asset revenue management problems

<table>
<thead>
<tr>
<th>ELEMENTS</th>
<th>DESCRIPTORS</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
<tr>
<td>B-CAPACITY</td>
<td>Fixed/Nonfixed</td>
</tr>
<tr>
<td>C-PRICES</td>
<td>Predetermined/Sec optimally/ Set jointly</td>
</tr>
<tr>
<td>D-WILLINGNESS TO PAY</td>
<td>Buildup/Drawdown</td>
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<td>E-DISCOUNT PRICE CLASSES</td>
<td>1/2/3/.../1</td>
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<tr>
<td>F-RESERVATION DEMAND</td>
<td>Deterministic/Mixed/Random-independent/Random-correlated</td>
</tr>
<tr>
<td>G-SHOW-UP OF DISCOUNT RESERVATION</td>
<td>Certain/Uncertain without cancellation/Uncertain with cancellation</td>
</tr>
<tr>
<td>H-SHOW-UP OF FULL PRICE RESERVATION</td>
<td>Certain/Uncertain without cancellation/Uncertain with cancellation</td>
</tr>
<tr>
<td>I-GROUP RESERVATIONS</td>
<td>No/Yes</td>
</tr>
<tr>
<td>J-DIVERSION</td>
<td>No/Yes</td>
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<tr>
<td>K-DISPLACEMENT</td>
<td>No/Yes</td>
</tr>
<tr>
<td>L-BUMPING PROCEDURE</td>
<td>None/Full price/Discount/ FCFS/Auction</td>
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<td>M-ASSET CONTROL MECHANISM</td>
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<td>N-DECISION RULE</td>
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Appendix 2  Technical derivations for Section 2

2.1 Two price classes

Applying the normal distribution assumption mentioned earlier for $X_1$ and $X_0$, we can
evaluate the double integrals in equation (2), simplify, and obtain the closed-form result for the expected contribution as:

\[
\begin{align*}
&= R_0[E_0^g(X_0) + q_0(1 - F_{X_0}(q_0))] + R_1[E_1^g(X_1) + q_1(1 - F_{X_1}(q_1))] \\
&+ F_{X_1}(0)F_{X_0}(q_0)[q_0R_0] + F_{X_1}(0)[- q_0R_0] + E_0^g(X_1)F_{X_0}(0)[- R_1] \\
&+ E_0^g(X_0)F_{X_1}(0)[- R_0] + F_{X_0}(0)[- q_1R_1] + F_{X_0}(0)F_{X_1}(q_1)[q_1R_1].
\end{align*}
\] (A-1)

Because we assume that demand is distributed normally, we add constraints to ensure that the mean of demand is at least three times the corresponding standard deviation. This assumption causes the last six terms of equation (A-1) to be equal to zero. To maximize the expected contribution, we take the derivative of expected contribution with respect to each of the three decision variables, set them equal to zero, and obtain the following three optimality conditions:

\[
0 = R_1[1 - F_{X_1}(q_1)] - R_0[1 - F_{X_0}(q_0)]
\] (A-2)

\[
0 = F_{X_0}(q_0)[2a_0R_0 + b_0 - q_0] - \sigma_0^2f_{X_0}(q_0) + q_0
\] (A-3)

which we note is independent of \(R_1\), and

\[
0 = F_{X_1}(q_1)[2a_1R_1 + b_1 - q_1] - \sigma_1^2f_{X_1}(q_1) + q_1
\] (A-4)

which we note is independent of \(R_p\). The second order conditions show that this is indeed a maximum, as opposed to having found a minimum.

2.2 Three price classes

Starting with expression (4), we integrate over all the possible combinations of \(X_0\), \(X_1\), and \(X_2\) which gives the following expression for the expected contribution (density functions – \(f(X_2), f(X_1), f(X_0)\) and differential notation – \(dX_2, dX_1, dX_0\) have been omitted to save space):
\[ \text{ExpContribution} = \int_{x_2=0}^{q_2} \int_{x_1=0}^{q_1} \int_{x_0=0}^{q_0} x_2 r_2 + x_1 r_1 + x_0 r_0 \\\+ \int_{x_2=0}^{q_2} \int_{x_1=0}^{q_1} \int_{x_0=q_0}^{q_0} x_2 r_2 + x_1 r_1 + q_0 r_0 \\\+ \int_{x_2=q_2}^{q_2} \int_{x_1=q_1}^{q_1} \int_{x_0=q_0}^{q_0} x_2 r_2 + q_1 r_1 + x_0 r_0 \\\+ \int_{x_2=q_2}^{q_2} \int_{x_1=q_1}^{q_1} \int_{x_0=q_0}^{q_0} x_2 r_2 + q_1 r_1 + q_0 r_0 \\\+ \int_{x_2=q_2}^{q_2} \int_{x_1=q_1}^{q_1} \int_{x_0=q_0}^{q_0} x_2 r_2 + q_1 r_1 + q_0 r_0 \\\+ \int_{x_2=q_2}^{q_2} \int_{x_1=q_1}^{q_1} \int_{x_0=q_0}^{q_0} x_2 r_2 + q_1 r_1 + q_0 r_0. \] 

(A-5)

Applying the normal distribution assumption mentioned earlier for \(x_2, x_1\) and \(x_0\), we can evaluate the triple integrals, simplify, and obtain the closed-form result for the expected contribution as:

\[ = R_0[\mathcal{E}^{q_2}(x_2) + q_0(1 - F_{X_0}(q_0))] + R_1[\mathcal{E}^{q_1}(x_1) + q_1(1 - F_{X_1}(q_1))] \\
+ R_2[\mathcal{E}^{q_2}(x_2) + q_2(1 - F_{X_2}(q_2))]. \] 

(A-6)

Because we assume that demand is distributed normally, constraints are added to ensure that the mean of demand is at least three times the corresponding standard deviation. This assumption causes us to ignore numerous terms in the derivation of equation (A-6) because they are equal to zero. In this case, we do not provide the five first-order optimality conditions (derivatives of expected contribution with respect to the five decision variables) because we know from §2.1 that we can solve any problem in a matter of seconds with the spreadsheet solver.
Appendix 3  Technical derivations for section 3

3.1 Two price classes

Integrating equation (7) over all the possible combinations of $X_0$ and $X_1$ gives us the following expression for the expected contribution (dX$_1$ and dX$_0$ omitted for brevity):

\[
\text{Exp. Contrib. } = \int_{X_1 = 0}^{q_1} \int_{X_0 = 0}^{q_0} (X_1 R_1 + X_0 R_0) f(X_1) f(X_0) + \int_{X_1 = 0}^{q_1} \int_{X_0 = q_0 - X_1}^{q_0} [X_1 R_1 + (q_0 - X_1) R_0] f(X_1) f(X_0) \\
+ \int_{X_1 = q_1}^{q_1} \int_{X_0 = 0}^{q_0 - q_1} (q_1 R_1 + X_0 R_0) f(X_0) f(X_1) + \int_{X_1 = q_1}^{q_1} \int_{X_0 = q_0 - q_1}^{q_0} [q_1 R_1 + (q_0 - q_1) R_0] f(X_1) f(X_0). \tag{A-7}
\]

Applying the normal distribution assumption mentioned earlier for $X_1$ and $X_0$, we can evaluate the double integrals, simplify, and obtain the following result for the expected contribution as:

\[
\text{Exp. Contrib. } = F_{X_1}(q_1)[-q_1 R_1 + q_1 R_0] + E_0^0(X_1)[-R_0 + R_1] \\
+ F_{X_0}(q_0 - q_1)[-q_0 R_0 + q_1 R_0] + F_{X_1}(q_0 - q_1) F_{X_1}(q_1) [+ q_0 R_0 - q_1 R_0] \\
+ E_0^0 - q_i (X_0) F_{X_1}(q_1)[-R_0] + E_0^0 - q_i (X_0) [R_0] + q_1 R_1 + q_0 R_0 - q_1 R_0 \\
+ [R_0] \int_{X_1 = 0}^{q_1} E_{X_0 - X_1}(X_0) f(X_1) dX_1 \\
+ \left[ \int_{X_1 = 0}^{q_1} q_0 R_0 \right] \int_{X_1 = 0}^{q_1} f(X_1) F_{X_0}(q_0 - X_1) dX_1 \\
+ [R_0] \int_{X_1 = 0}^{q_1} X_1 f(X_1) F_{X_0}(q_0 - X_1) dX_1. \tag{A-8}
\]

We note that we now have three terms which are not closed-form. Instead of repeating these integrals in future expressions, we will define them as NI1 (i.e., Numerical Integral #1), NI2, and NI3 respectively.
To maximize the expected contribution, we take the derivative of expected contribution with respect to each of the three decision variables, set them equal to zero, and obtain the following three optimality conditions:

\[ q_i: \quad 0 = F_{X_1}(q_i)[R_0 - R_1] + R_0 F_{X_0}(q_0 - q_1) \]
\[ - R_0 F_{X_1}(q_i) F_{X_0}(q_0 - q_1) + (R_1 - R_0) \]  \hspace{1cm} (A-9)

\[ R_1: \quad 0 = F_{X_1}(q_i)[- q_i + a_1 R_1 - a_1 R_0] + E^{q_i}(X_1) \]
\[ + f_{X_1}(q_i) F_{X_0}(q_0 - q_1)[- a_1 q_0 R_0 + a_1 q_1 R_0] \]
\[ + E^{q_0 - q_i}(X_0) f_{X_1}(q_i)[a_1 R_0] + q_1 + NI1 \left[ - \frac{R_0 a_1 \mu_1}{\sigma_1^2} \right] \]  \hspace{1cm} (A-10)

\[ + NI2 \left[ \frac{q_0 R_0 a_1 \mu_1}{\sigma_1^2} \right] + NI3 \left[ - \frac{q_0 R_0 a_1}{\sigma_1^2} - \frac{R_0 a_1 \mu_1}{\sigma_1^2} \right] \]
\[ + NI8 \left[ \frac{R_0 a_1}{\sigma_1^2} \right] + NI9 \left[ \frac{R_0 a_1}{\sigma_1^2} \right] \]

where:

\[ NI/8 = \int_{X_1 = 0}^{q_1} X_1 f/(X_1) E^{q_0 - X_1}(X_0) dX_1 \]  \hspace{1cm} (A-11)

\[ NI/9 = \int_{X_1 = 0}^{q_1} X_1^2 f/(X_1) F_{X_0}(q_0 - X_1) dX_1 . \]

\[ R_0: \quad 0 = q_i F_{X_1}(q_i) - E^{q_i}(X_1) + F_{X_0}(q_0 - q_1)[q_1 - q_0 + a_0 R_0] \]
\[ + F_{X_0}(q_0 - q_1) F_{X_1}(q_i)[q_0 - q_1 - a_0 R_0] - E^{q_0 - q_i}(X_0) F_{X_1}(q_i) \]
\[ + E^{q_0 - q_i}(X_0) + (q_0 - q_1) + NI1 + NI2(R_0 a_0 - q_0) + NI3 \]  \hspace{1cm} (A-12)

3.2 Three price classes

Integrating (9) over all the possible combinations of \( X_0, X_1, \) and \( X_2 \) gives us the following expression for the expected contribution (\( dX_2, dX_1, \) and \( dX_0 \) are omitted for brevity):
Exp Contrib = \int_{x_2 = 0}^{q_2} \int_{x_1 = 0}^{q_1 - X_2} \int_{x_0 = 0}^{q_0 - X_1 - X_2} [X_2 R_2 + X_1 R_1 + X_0 R_0] f(X_2) f(X_1) f(X_0) \, dx_2 \, dx_1 \, dx_0 \\
+ \int_{x_2 = 0}^{q_2} \int_{x_1 = 0}^{q_1 - X_2} \int_{x_0 = 0}^{q_0 - X_1 - X_2} [X_2 R_2 + X_1 R_1 + (q_0 - X_1 - X_2) R_0] f(X_2) f(X_1) f(X_0) \, dx_2 \, dx_1 \, dx_0 \\
+ \int_{x_2 = 0}^{q_2} \int_{x_1 = q_1 - X_2}^{q_0 - X_2} \int_{x_0 = 0}^{q_0 - X_1 - X_2} [X_2 R_2 + (q_1 - X_2) R_1 + X_0 R_0] f(X_2) f(X_1) f(X_0) \, dx_2 \, dx_1 \, dx_0 \\
+ \int_{x_2 = 0}^{q_2} \int_{x_1 = q_1 - X_2}^{q_0 - X_2} \int_{x_0 = q_0 - X_1 - X_2}^{q_0} [X_2 R_2 + (q_1 - X_2) R_1 + (q_0 - q_1) R_0] f(X_2) f(X_1) f(X_0) \, dx_2 \, dx_1 \, dx_0 \\
+ \int_{x_2 = q_2}^{q_2} \int_{x_1 = 0}^{q_1 - X_2} \int_{x_0 = 0}^{q_0 - X_1 - X_2} [q_2 R_2 + X_1 R_1 + X_0 R_0] f(X_2) f(X_1) f(X_0) \, dx_2 \, dx_1 \, dx_0 \\
+ \int_{x_2 = q_2}^{q_2} \int_{x_1 = q_1 - X_2}^{q_0 - X_2} \int_{x_0 = 0}^{q_0 - X_1 - X_2} [q_2 R_2 + (q_1 - q_2) R_1 + X_0 R_0] f(X_2) f(X_1) f(X_0) \, dx_2 \, dx_1 \, dx_0 \\
+ \int_{x_2 = q_2}^{q_2} \int_{x_1 = q_1 - q_2}^{q_0 - X_2} \int_{x_0 = q_0 - X_1 - X_2}^{q_0} [q_2 R_2 + (q_1 - q_2) R_1 + (q_0 - q_1) R_0] f(X_2) f(X_1) f(X_0) \, dx_2 \, dx_1 \, dx_0 \\
+ \int_{x_2 = q_2}^{q_2} \int_{x_1 = q_1 - q_2}^{q_0 - X_2} \int_{x_0 = q_0 - q_1}^{q_0} [q_2 R_2 + (q_1 - q_2) R_1 + (q_0 - q_1) R_0] f(X_2) f(X_1) f(X_0) \, dx_2 \, dx_1 \, dx_0 .

(A-13)

Applying the normal distribution assumption mentioned earlier for $X_2$, $X_1$, and $X_0$, we can evaluate the triple integrals, simplify, and obtain the following result for the expected contribution as:
Exp. Contrib. = NI12[N111]\{R_0\} + NI14[N111]{- q_0 R_0}
+ NI14[N113]\{R_0\} + NI14[N115]\{R_0\} + NI16[R_1 - R_0]
+ NI17[R_1 - R_0] + NI18[q_i R_0 - q_1 R_1 + q_0 R_0 F_{X0}(q_0 - q_i)]

- R_0 E_0^{q_i} q(X_0) - q_1 R_0 F_{X0}(q_0 - q_i)] + NI19[q_0 R_0 F_{X2}(q_2)
- q_1 R_0 + q_2 R_0 - q_2 R_0 F_{X2}(q_2)]
+ NI20[R_0 - R_0 F_{X2}(q_2)] + NI21[R_0 - R_0 F_{X2}(q_2)]

+ E_0^{q_2}(X_2)[R_2 - R_1] + E_0^{q_0} q_2(X_1)[R_1 - R_0]
+ E_0^{q_0} q_2(X_0)[R_0] + F_{X2}(q_2)[q_2 R_1 - q_2 R_2]
+ F_{X1}(q_1 - q_2)[q_2 R_1 - q_1 R_1 + q_1 R_0 - q_2 R_0]
+ F_{X0}(q_0 - q_1)[q_1 R_0 - q_0 R_0]
+ F_{X2}(q_2)F_{X1}(q_1 - q_2)[q_1 R_1 - q_2 R_1 + q_2 R_0 - q_1 R_0]
+ F_{X1}(q_1 - q_2)F_{X0}(q_0 - q_1)[q_0 R_0 - q_1 R_0]
+ E_0^{q_i} q_2(X_1)F_{X2}(q_2)[R_0 - R_1] + E_0^{q_0} q_2(X_0)F_{X1}(q_1 - q_2)[- R_0]
+ E_0^{q_0} q_2(X_0)F_{X2}(q_2)F_{X1}(q_1 - q_2)[R_0]
+ F_{X2}(q_2)F_{X1}(q_1 - q_2)F_{X0}(q_0 - q_1)[q_1 R_0 - q_0 R_0]
+ q_2 R_2 + q_1 R_1 - q_2 R_1 + q_0 R_0 - q_1 R_0

(A-14)

where:
Appendix 4  Technical derivations for section 4

4.1 Two price classes

Integrating (10) over all the possible combinations of $X_0$ and $X_1$ gives us the following expression for the expected contribution:
Optimization of perishable asset revenue management problems

\[ \text{Exp. Contrib.} = \int_{x_1 = 0}^{q_1} \int_{x_0 = 0}^{q_1 - x_1} [(X_0 + X_1) R_1] f(X_1) f(X_0) dX_1 dX_0 \]
\[ + \int_{x_1 = 0}^{q_1} \int_{x_0 = q_1 - x_1}^{q_0} [(q_1 R_1 + (X_0 + X_1 - q_1) R_0) f(X_0) f(X_1) dX_1 dX_0 \]
\[ + \int_{x_1 = q_1}^{q_0} \int_{x_0 = q_0 - x_1}^{\infty} [q_1 R_1 + X_0 R_0] f(X_1) f(X_0) dX_1 dX_0 \]
\[ + \int_{x_1 = 0}^{q_1} \int_{x_0 = q_0 - x_1}^{\infty} [q_1 R_1 + (q_0 - q_1) R_0] f(X_1) f(X_0) dX_1 dX_0 \].

(A-16)

Applying the normal distribution assumption mentioned earlier for \( X_1 \) and \( X_0 \), we can evaluate the double integrals, simplify, and obtain the following result for the expected contribution as:

\[ = N1(R_0) + N2(-q_0 R_0) + N3(R_0) + N4(q_1 R_0 - q_1 R_1) \]
\[ + N5(R_1 - R_0) + N6(q_0 R_0) + F_{X_0}(q_0 - q_1) F_{X_1}(q_1) [q_0 R_0 - q_1 R_0] \]
\[ + F_{X_0}(q_0 - q_1) [q_1 R_0 - q_0 R_0] + E^{q_0 - q_1}(X_0) [R_0] \]
\[ + E^{q_0 - q_1}(X_0) F_{X_1}(q_1) [- R_0] + q_1 R_1 + q_0 R_0 - q_1 R_0 \]

(A-17)

where the first three numerical integrals are the same as defined in equation (A-8) and the new ones are:

\[ N4 = \int_{x_1 = 0}^{q_1} f(X_1) F_{X_1}(q_1 - X_1) dX_1 \]
\[ N5 = \int_{x_1 = 0}^{q_1} f(X_1) E^{X_1}(X_0) dX_1 \]
\[ N6 = \int_{x_1 = 0}^{q_1} X_1 f(X_1) F_{X_1}(q_1 - X_1) dX_1 \].

(A-18)

We note that we now have six terms which are not closed-form.

To maximize the expected contribution, we take the derivative of expected contribution with respect to each of the three decision variables, set them equal to zero, and obtain the three following optimality conditions:
\[ q_i = 0 = NI4[R_0 - R_1] + F_{x0}(q_0 - q_i)F_{x1}(q_i)[- R_0] \]
\[ + F_{x0}(q_0 - q_i)R_0 + R_1 - R_0 \] \hspace{1cm} (A-19)

\[ R_1 : 0 = NI5 \left[ - \frac{R_1 \mu_1 a_1}{\sigma_1^2} + \frac{R_0 \mu_1 a_1}{\sigma_1^2} + 1 \right] + NI2 \left[ \frac{q_0 R_0 \mu_1 a_1}{\sigma_1^2} \right] \]
\[ + NI3 \left[ - \frac{q_0 R_0 a_1}{\sigma_1^2} - \frac{\mu_1 a_1 R_0}{\sigma_1^2} \right] + NI4 \left[ - \frac{q_1 R_0 \mu_1 a_1}{\sigma_1^2} + \frac{q_1 R_1 \mu_1 a_1}{\sigma_1^2} - q_i \right] \]
\[ + NI7 \left[ \frac{R_1 a_1}{\sigma_1^2} - \frac{R_0 a_1}{\sigma_1^2} \right] + NI6 \left[ \frac{q_1 R_0 a_1}{\sigma_1^2} - \frac{q_1 R_1 a_1}{\sigma_1^2} - \frac{\mu_1 R_1 a_1}{\sigma_1^2} + \frac{\mu_1 a_1 R_0}{\sigma_1^2} + 1 \right] \]
\[ + NI1 \left[ - \frac{R_0 \mu_1 a_1}{\sigma_1^2} \right] + NI8 \left[ \frac{R_0 a_1}{\sigma_1^2} \right] + NI9 \left[ \frac{R_0 a_1}{\sigma_1^2} \right] + NI10 \left[ \frac{R_1 a_1}{\sigma_1^2} - \frac{R_0 a_1}{\sigma_1^2} \right] \]
\[ + f_{x1}(q_i)F_{x0}(q_0 - q_i)[- a_1 q_0 R_0 + a_1 q_1 R_0] \]
\[ + f_{x1}(q_i)E_0^{u - \xi}(X_0)[a_1 R_0] + q_i \] \hspace{1cm} (A-20)

\[ R_0 : 0 = + NI1 + NI2(a_0 R_0 - q_o) \]
\[ + NI3 + NI4(q_1 + a_0 R_1 - a_0 R_0) \]
\[ - NI5 + NI6 + (q_0 - q_i) + E_0^{u - \xi}(X_0) - F_{x1}(q_i)E_0^{u - \xi}(X_0) \]
\[ + F_{x0}(q_0 - q_i)[q_1 - q_0 + a_0 R_0] \]
\[ + F_{x1}(q_i)F_{x0}(q_0 - q_i)[q_0 - q_1 - a_0 R_0] \] \hspace{1cm} (A-21)

where some of the numerical integrals have been previously defined in equations (A-8), (A-11) and (A-18) and the new ones are:

\[ NI/7 = \int_{X_1 = 0}^{q_i} X_1 f(X_1) E_0^{u - \xi}(X_0) \ dX_1 \]

\[ NI/8 = \int_{X_1 = 0}^{q_i} f(X_1) E_0^{u - \xi}(X_0) \ dX_1 \]

\[ NI/9 = \int_{X_1 = 0}^{q_i} X_1^2 f(X_1) F_{x0}(q_0 - X_1) \ dX_1 \]

\[ NI10 = \int_{X_1 = 0}^{q_i} X_1^2 f(X_1) F_{x0}(q_1 - X_1) \ dX_1 \]
4.2 Three price classes

Integrating (12) over all the possible combinations of $X_0$, $X_1$, and $X_2$ gives us the following expression for the expected contribution ($dX_0$, $dX_1$, and $dX_2$ are omitted for brevity):
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Applying the normal distribution assumption mentioned earlier for \( X_1, X_2, \) and \( X_0 \), we can evaluate the triple integrals, simplify, and obtain the following result for the expected contribution as:

\[
\text{Exp Contrib} = (R_2 - R_1)(N(I24[I22] + N[I24[I23]) + N[I22[I24]) + (R_1 - R_0)(N[I24[I25]) + N[I14[I26] + N[I14[I27] + N[I14a[I28] \\
+ N[I4a[I27b]) + N[I4[I24](- q_1 R_2 + q_2 R_1) \\
+ N[I4[I25](q_1 R_0 - q_1 R_i) + N[I4[I24a](- q_2 R_2 + q_2 R_1) \\
+ R_0(N[I12[I28] + N[I14[I29] + N[I14[I30] - (N[I18] E^{\phi - q_0}(X_0) \\
+ E^{\phi - q_0}(X_0) - E^{\phi - q_0}(X_0)F_X(q_1 - q_2) \\
+ E^{\phi - q_0}(X_0)F_X(q_1 - q_2)F_X(q_2) + N[I4a[I29b] \\
+ N[I4a[I30b] + N[I14a[I25b](q_2 R_1 + q_1 R_0 - q_1 R_i) \\
+ R_1(N[I14[I22a] - N[I14[I23a] \\
- N[I2[I24a]) - q_0 R_0(N[I14[I28]) \\
+ (q_0 R_0 - q_1 R_0)(N[I18] F_X(q_0 - q_1) - F_X(q_0 - q_1) \\
+ F_X(q_0 - q_1)F_X(q_1 - q_2) - F_X(q_0 - q_1)F_X(q_1 - q_2)F_X(q_2) \\
+ (q_2 R_2 + q_1 R_1 - q_2 R_1 - q_1 R_0)(N[I4a[I28b] \\
+ (q_2 R_2 + q_1 R_1 - q_2 R_1 + q_0 R_0 - q_1 R_0)F_X(q_2)(N[I28b]) \\
- N[I28b] + q_2 R_2 + q_1 R_1 + q_0 R_0 - q_2 R_1 - q_1 R_0
\]

where some numerical integrals were defined before in (A-15), and the new ones are:
\[ \int f(X_2) \text{[otherNI]} dX_2 \]

\[ \int f(X_1) E_0^{q_2, X_1} X_2(X_0) dX_1 \]

\[ \int X_1 \left( f(X_1) F_{X_0}(q_2 - X_1 - X_2) \right) dX_1 \]

\[ \int f(X_1) F_{X_0}(q_2 - X_1 - X_2) dX_1 \]

\[ \int f(X_1) F_{X_0}(q_1 - X_1 - X_2) dX_1 \]  \hspace{1cm} (A-25)

\[ \int f(X_1) E_0^{q_1, X_1} X_2(X_0) dX_1 \]

\[ \int X_1 f(X_1) F_{X_0}(q_1 - X_1 - X_2) dX_1 \]

\[ \int f(X_1) F_{X_0}(q_0 - X_1 - X_2) dX_1 \]

\[ \int f(X_1) E_0^{q_0, X_1} X_2(X_0) dX_1 \]

\[ \int X_1 f(X_1) F_{X_0}(q_0 - X_1 - X_2) dX_1 \]
\[ N_{22a}^{\mathcal{J}} = \int_{x_1 = q_2 - q_1}^{x_2} f(x_1) E_0^{q_2 - q_1}(X_0) dX_1 \]

\[ N_{23a}^{\mathcal{J}} = \int_{x_1 = q_2 - q_1}^{q_1 - q_2} X_1 f(x_1) F_{X_0}(q_2 - X_1 - X_2) dX_1 \]

\[ N_{24a}^{\mathcal{J}} = \int_{x_1 = q_2 - q_1}^{q_1 - q_2} f(x_1) F_{X_0}(q_2 - X_1 - X_2) dX_1 \]

\[ N_{25b}^{\mathcal{J}} = \int_{x_1 = q_2 - q_1}^{0} f(x_1) F_{X_0}(q_1 - X_1 - q_2) dX_1 \]

\[ N_{26b}^{\mathcal{J}} = \int_{x_1 = q_2 - q_1}^{0} f(x_1) E_0^{q_2 - X_1 - q_1}(X_0) dX_1 \]  \hspace{1cm} (A.26)

\[ N_{27b}^{\mathcal{J}} = \int_{x_1 = q_2 - q_1}^{q_1 - q_2} X_1 f(x_1) F_{X_0}(q_1 - X_1 - q_2) dX_1 \]

\[ N_{28b}^{\mathcal{J}} = \int_{x_1 = q_2 - q_1}^{0} f(x_1) F_{X_0}(q_0 - X_1 - q_2) dX_1 \]

\[ N_{29b}^{\mathcal{J}} = \int_{x_1 = q_2 - q_1}^{0} f(x_1) E_0^{q_1 - X_1 - q_2}(X_0) dX_1 \]

\[ N_{30b}^{\mathcal{J}} = \int_{x_1 = q_2 - q_1}^{0} X_1 f(x_1) F_{X_0}(q_0 - X_1 - q_2) dX_1 . \]