A new adaptive control scheme for uncertain nonlinear systems with quantized input signal☆

Lantao Xing\textsuperscript{a}, Changyun Wen\textsuperscript{b}, Hongye Su\textsuperscript{a,}\textsuperscript{*}, Jianping Cai\textsuperscript{c}, Lei Wang\textsuperscript{a}

\textsuperscript{a}State Key Laboratory of Industrial Control Technology, Institute of Cyber-Systems and Control, Zhejiang University, 310027 Hangzhou, PR China
\textsuperscript{b}School of Electrical and Electronic Engineering, Nanyang Technological University, Nanyang Avenue, Singapore 639798, Singapore
\textsuperscript{c}Zhejiang University of Water Resources and Electric Power, 310027 Hangzhou, PR China

Received 1 January 2015; received in revised form 21 May 2015; accepted 26 June 2015
Available online 3 July 2015

Abstract

In this paper, we propose a new simple yet effective adaptive tracking control scheme for uncertain strict-feedback nonlinear systems whose input is quantized by a class of sector-bounded quantizers including the well-known logarithmic quantizer and the extended hysteresis quantizer. To deal with the error caused by quantization, a key technique is to utilize the sector bound property of the quantizers directly instead of decomposing it into a linear part and a nonlinear part. Compared with the existing results in adaptive control, the proposed scheme relaxes the global Lipschitz conditions of nonlinearities and the boundedness of their partial derivatives. It is shown that the designed adaptive controller ensures global boundedness of all the signals in the closed-loop system and enables the tracking error to exponentially converge towards a compact set which is adjustable.

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☆This work is partially supported by the 111 Project (B07031); National Natural Science Foundation of China (NSFC: 61134007, 61320106009, 61573322) and Zhejiang Provincial Natural Science Foundation of China (grant no. LY12F03019).

*Corresponding author.
E-mail address: hysu@iipc.zju.edu.cn (H. Su).
1. Introduction

Quantization can be seen as a map from continuous signals to discrete finite sets. Recently, a great deal of attention has been paid to the study of quantization problems because of its theoretical and practical importance in modern engineering such as hybrid systems, discrete-event systems, digital control systems and control with information constrains, see for example [1–4]. For systems with information quantization, a continuous control input or state is quantized by a quantizer which results in an inevitable quantization error. Thus the effects of the quantization error to the performances of the closed system, especially system stability, need to be carefully and clearly studied.

In [5–8], stability of linear and nonlinear systems with input quantization is studied. However, all these systems are assumed to be known completely. For uncertain nonlinear systems, which is more common in practice, quantized control is mainly studied with robust approaches, such as [9–12]. Besides robust control, adaptive control is another important approach to deal with system uncertainties as it can provide on-line estimation of unknown parameters. In [13,14], adaptive control with quantized input signals for linear systems is reported. In [15], adaptive quantized control for nonlinear systems is considered. However, the established stability conditions depend on the control signals, which are hardly checkable in advance. In [16], a backstepping-based adaptive control scheme is presented for a class of strict-feedback uncertain systems with input quantization. Although the proposed design method can make the stabilization error arbitrarily small and provide a guideline to choose the quantization parameters, it requires the nonlinearities in the system to satisfy global Lipschitz conditions with known Lipschitz constants and their partial derivatives to be bounded. This clearly restricts the class of systems to be controlled.

In this paper, we propose a new simple adaptive backstepping approach for a more general class of uncertain nonlinear systems with its input signal \( u \) quantized by a sector-bounded quantizer \( q(u) \). By using the sector bound property of the quantizers, the quantization error between the quantizer input \( u \) and output \( q(u) \) is effectively compensated for so that the conditions required for nonlinearities in [16] are relaxed. It is shown that the closed-loop system is globally bounded and the tracking error can exponentially converge towards a compact set which is adjustable by choosing suitable design parameters. Simulation results illustrate the effectiveness of the proposed design scheme.

The remaining part of this paper is organized as follows. Section 2 describes the problem formation and introduces the sector-bounded quantizer, mainly the hysteresis quantizer and the logarithmic quantizer. Section 3 presents adaptive controller design based on backstepping technique and analyzes the stability and tracking performance of the closed-loop system. Two simulation examples are given in Section 4. Finally Section 5 concludes this paper.

2. Problem statement

2.1. System model

The following class of nonlinear systems with unknown parameters, similar to [16–18], are considered:

\[
\begin{align*}
\dot{x}_1 &= x_2 + f_1(x_1) \\
\dot{x}_2 &= x_3 + f_2(x_1,x_2)
\end{align*}
\]
\[ \begin{align*}
\dot{x}_{n-1} &= x_n + f_{n-1}(x_1, \ldots, x_{n-1}) \\
\dot{x}_n &= q(u(t)) + f_n(x) + q^T(x)\theta \\
y &= x_1(t)
\end{align*} \]

where \( x(t) = [x_1(t), \ldots, x_n(t)]^T \in \mathbb{R}^n \) and \( q(u(t)) : \mathbb{R}^1 \rightarrow \mathbb{R}^1 \) are the states and input of the system, respectively; \( \bar{x}_i(t) = [x_1(t), \ldots, x_i(t)]^T \in \mathbb{R}^i \), vector \( \theta \in \mathbb{R}^r \) contains all the unknown system parameters, \( q(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^r \) and \( f_i(\cdot) : \mathbb{R}^i \rightarrow \mathbb{R}^1 \) \( (i = 1, \ldots, n) \) are known nonlinear functions. The input \( q(u(t)) \) takes quantized values due to a sector-bounded quantizer, where \( u(t) \in \mathbb{R}^1 \) is the control input signal to be designed. Similar to the existing literature such as [16], we assume that the existence and uniqueness of solution are satisfied for this kind of nonlinear systems.

### 2.2. Sector-bounded quantizer

A sector-bounded quantizer is a quantizer with its quantization error satisfying a sector bounding condition. Based on [19,20], most practical quantizers belong to such a class with hysteresis quantizer and logarithmic quantizer being two typical examples.

#### 2.2.1. Hysteresis quantizer

The hysteresis quantizer was firstly introduced in [21]. In this paper, the hysteresis quantizer employed is described in the following form, similar to [16]:

\[ q_h(u(t)) = \begin{cases} 
    u_i \operatorname{sgn}(u), & \frac{u_i}{1 + \delta} < |u| \leq u_i, \dot{u} < 0, \text{or} \\
    u_i < |u| \leq \frac{u_i}{1 - \delta}, \dot{u} > 0 \\
    u_i(1 + \delta) \operatorname{sgn}(u), & u_i < |u| \leq \frac{u_i}{1 - \delta}, \dot{u} < 0, \text{or} \\
    \frac{u_i}{1 - \delta} < |u| \leq \frac{u_i(1 + \delta)}{1 - \delta}, \dot{u} > 0 \\
    0, & 0 \leq |u| < d, \dot{u} < 0, \text{or} \\
    \frac{d}{1 + \delta} \leq |u| \leq d, \dot{u} > 0 \\
    q_h(u(t^-)), & \dot{u} = 0
\end{cases} \]

where \( i = 1, 2 \ldots \). The parameters \( d > 0 \) and \( 0 < \rho < 1 \) determine the quantization density of \( q(u(t)) \) and also \( d \) determines the dead-zone of the quantizer around \( 0, \delta = \frac{1 - \rho}{1 + \rho}, u_i = \rho^{(1 - i)}d \). \( q_h(u) \) is in the set \( U = [0, \pm u_i, \pm u_i(1 + \delta)] \) and \( q_h(u(t^-)) \) denotes the status prior to \( q_h(u(t)) \). For this kind of quantizer, some detailed discussions can be found in [11,16].
2.2.2. Logarithmic quantizer

In this paper, the logarithmic quantizer described in [9] is considered. It is modelled as

\[
q_l(u(t)) = \begin{cases} 
    u_i, & \frac{u_i}{1+\delta} \leq u \leq \frac{u_i}{1-\delta} \\
    0, & 0 \leq u \leq \frac{d}{1+\delta} \\
    -q_l(-u(t)), & u < 0
\end{cases}
\]  

(3)

where \( u_i = \rho^{(1-i)}d \) with \( i = 1, 2, \ldots \). The parameters \( d > 0 \) and \( 0 < \rho < 1, \delta = \frac{1-\rho}{1+\rho} \) determine the quantization density of \( q_l(u) \). \( q_l(u) \) is in the set \( U = \{0, \pm u_i\} \). \( u_{\text{min}} = \frac{d}{1+\rho} \) determines the size of the dead-zone around 0 for \( q_l(u) \). For this quantizer, some detailed remarks are given in [11].

A very important property was provided in [19] to bound the magnitude of the quantization error of sector-bounded quantizers, including the above two quantizers \( q_h(u) \) and \( q_l(u) \), as below

\[
|q(u) - u| \leq \delta|u| + (1 - \delta)d.
\]  

(4)

The objective of this paper is to propose an adaptive control design scheme which can ensure global stability and make the output \( y = x_1(t) \) track a \( C^\alpha \) reference signal \( r(t) \) with the input quantized by a sector-bounded quantizer.

Remark 1. The hysteresis quantizer can be seen as an extension of the logarithmic quantizer as it can be seen as a combination of two logarithmic quantizers with the same coarseness but different quantized values, so a control law which adopts a logarithmic quantizer can be also applied to a hysteresis quantizer, and vice versa. However, with the additional quantized values, a hysteresis quantizer can lessen chattering phenomenon. Some detailed discussions can be found in [15].

Remark 2. The difficulty to achieve the control objective is how to handle the quantization error. In [16], a hysteresis quantizer is considered in quantizing the input \( u \). Instead of using the sector bound property (4) directly, [16] decomposes the quantization error into a linear part and a nonlinear part. To compensate for the nonlinear part, the nonlinearities in the system are required to be globally Lipschitz continuous with known Lipschitz constants. The proposed scheme in this paper utilizes the sector bound property of the quantizers directly and it will be shown that the global Lipschitz condition of the nonlinearities is no longer needed.

3. Adaptive control scheme

3.1. Adaptive controller design

In this section, we will design an adaptive controller based on backstepping technique, which involves \( n \) recursive steps.

First, the change of coordinates is introduced

\[
z_1 = x_1 - r(t)
\]  

(5)

\[
z_i = x_i - \alpha_{i-1}, \quad i = 2, \ldots, n
\]  

(6)

where \( \alpha_{i-1} (i = 2, 3, \ldots, n) \) are virtual control laws and \( z_i (i = 1, 2, \ldots, n) \) are error variables. Following the standard procedures, the first \( n-1 \) steps are summarized as follows:
Step 1:

$$\alpha_1 = - \left( c_1 + \frac{1}{2} \right) z_1 - f_1(x_1) + \dot{r}$$  \hspace{1cm} (7)

Step $i$ ($i = 2, \ldots, n-1$):

$$\alpha_i = -(c_i + 1) z_i - f_i(\bar{x}_i) + \dot{\alpha}_{i-1}$$  \hspace{1cm} (8)

where $c_i$ ($i = 1, \ldots, n-1$) is a positive constant.

Step $n$: This is the last design step of determining the actual control signal $u$ and the parameter estimation law for $\theta$.

$$u = \frac{- \left( c_n + \frac{1}{2} \right) z_n - f_n(x) - \varphi(x) \dot{\theta} + \dot{\alpha}_{n-1} - (1 - \delta) \sigma \tanh \left( \frac{z_n}{\epsilon} \right)}{1 + \delta \tanh \left( \frac{z_n H}{\epsilon} \right)}$$  \hspace{1cm} (9)

$$\dot{\theta} = \Gamma \varphi(x) z_n - \Gamma \sigma \hat{\theta}$$  \hspace{1cm} (10)

where $\epsilon$, $\sigma$ and $c_n$ are all positive design parameters, $\Gamma$ is a positive definite matrix. A block diagram of the resulting closed loop feedback system is illustrated in Fig. 1. The control law (9) and the adaptive law (10) are obtained based on Lyapunov approach which will be made clear in the proof of Theorem 1 in the next subsection.
3.2. Closed-loop analysis

We now analyze the designed controller and establish the global boundedness of all the signals in the closed-loop system and its tracking performance, as stated in the following theorem:

**Theorem 1.** Consider the closed-loop system consisting of uncertain system (1) with the input signal $u$ quantized by a sector-bounded quantizer including the hysteresis quantizer (2) and the logarithmic quantizer (3), the control law (5)--(9) and parameter updating law (10). All the signals of the closed-loop system are ultimately bounded and the tracking error $y(t) - r(t)$ will exponentially converge towards a set which is adjustable by choosing suitable parameters.

**Proof.** Choose the Lyapunov function

$$V = \frac{1}{2} \sum_{i=1}^{n} z_i^2 + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}$$

where $\tilde{\theta} = \theta - \hat{\theta}$.

Then

$$\dot{V} = \sum_{i=1}^{n} z_i \dot{z}_i - \tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}}$$

$$\leq z_1 \left[ z_2 - \left( c_1 + \frac{1}{2} \right) z_1 \right] + \sum_{i=2}^{n-1} z_i [z_{i+1} - (c_i + 1) z_i]$$

$$+ z_n \left[ q(u) + f_n(x) + \phi(x)^T \theta - \dot{\alpha}_{n-1} \right] - \tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}} \tag{12}$$

Multiplying $|z_n|$ on both sides of Eq. (4), after some manipulation, the following inequality can be derived:

$$z_n q(u) \leq z_n u + \delta |z_n| u + (1 - \delta) d |z_n| \tag{13}$$

To deal with the error caused by quantization, we need to use the following property of hyperbolic tangent function $\tanh(\cdot)$ which is introduced in [22]:

$$0 \leq \phi(q) - q \tanh\left( \frac{q}{\epsilon} \right) \leq 0.2785 \epsilon \tag{14}$$

where $\epsilon > 0$ and $q \in \mathbb{R}$. Substituting Eq. (14) into Eq. (13), the following inequality can be obtained:

$$z_n q(u) \leq z_n u + \delta z_n u \tanh\left( \frac{z_n u}{\epsilon} \right) + (1 - \delta) d z_n \tanh\left( \frac{z_n}{\epsilon} \right)$$

$$+ 0.2785(1 - \delta) d \epsilon + 0.2785 \delta \epsilon \tag{15}$$

Substituting Eq. (15) into Eq. (12), we have

$$\dot{V} \leq z_1 \left[ z_2 - \left( c_1 + \frac{1}{2} \right) z_1 \right] + \sum_{i=2}^{n-1} z_i [z_{i+1} - (c_i + 1) z_i]$$

$$+ z_n \left[ \mu + f_n(x) + \phi(x)^T \theta - \dot{\alpha}_{n-1} + (1 - \delta) d \tanh\left( \frac{z_n}{\epsilon} \right) + \delta u \tanh\left( \frac{z_n u}{\epsilon} \right) \right]$$

$$- \tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}} + 0.2785(1 - \delta) d \epsilon + 0.2785 \delta \epsilon \tag{16}$$
Using the control law (9) and the adaptive law (10), we can get

\[
\dot{V} \leq - \sum_{i=1}^{n} c_i \xi_i^2 - \frac{1}{2} \xi_1^2 + \sum_{i=1}^{n-1} z_i \xi_{i+1} - \sum_{i=2}^{n-1} \xi_i^2 + \frac{1}{2} \xi_n^2 + \sigma \hat{\theta}^T \hat{\theta} \\
+ 0.2785 (1 - \delta) \, d\epsilon + 0.2785 \delta \epsilon \\
\leq - \sum_{i=1}^{n} c_i \xi_i^2 + \sigma \hat{\theta}^T \hat{\theta} + 0.2785 (1 - \delta) \, d\epsilon + 0.2785 \delta \epsilon
\]  

(17)

Note that \( \sigma \) can be chosen as a constant based on the \( \sigma \)-modification scheme proposed in [23]. Let \( \xi = \min \left\{ 2c_1, \ldots, 2c_n, \frac{\sigma}{\lambda_{\text{max}}(\Gamma^{-1})} \right\} \) where \( \lambda_{\text{max}}(\Gamma^{-1}) \) is the maximum eigenvalue of \( \Gamma^{-1} \). Then using Young’s inequalities \( \hat{\theta}^T \hat{\theta} \leq - \frac{1}{2} \hat{\theta}^T \hat{\theta} + \frac{1}{2} \theta^T \theta \), we can get

\[
\dot{V} \leq - \frac{\xi}{2} \sum_{i=1}^{n} \xi_i^2 - \frac{\xi}{2} \hat{\theta}^T \Gamma^{-1} \hat{\theta} + 0.2785 (1 - \delta) \, d\epsilon + 0.2785 \delta \epsilon + \frac{\sigma}{2} \theta^T \theta \\
\leq - \xi V + \Delta
\]

(18)

where \( \Delta = 0.2785 (1 - \delta) \, d\epsilon + 0.2785 \delta \epsilon + \frac{\sigma}{2} \theta^T \theta \).

From Eq. (18) we have

\[
\frac{1}{2} \| z(t) \|^2 = \frac{1}{2} \sum_{i=1}^{n} (\xi_i^2) \leq V(t) \leq e^{-\xi t} V(0) + (\Delta / \xi) \left( 1 - e^{-\xi t} \right)
\]

(19)

So \( \| z(t) \|^2 \) is bounded by a function that converges exponentially towards a compact set \( \Omega = \left\{ z \mid \| z \|^2 \leq 2 \Delta / \xi = 2 \times \frac{0.2785(1 - \delta) \, d\epsilon + 0.2785 \delta \epsilon + \frac{\sigma}{2} \theta^T \theta}{\xi} \right\} \) at a rate of \( \xi \). When the convergence
speed $\xi$ is set, the size of $\Omega$ can be reduced by decreasing $\lambda_{\text{max}}(\Gamma^{-1})$ because $\frac{\sigma}{\xi} \geq \lambda_{\text{max}}(\Gamma^{-1})$ or decreasing $\varepsilon$. □

4. Simulation results

In this section, two examples are given to illustrate the performances of the control scheme proposed in this paper.

4.1. Example one

Consider the same nonlinear example in [16]

$$\ddot{x} + \theta \sin(\dot{x}) + \tanh(x) = q(u)$$

(20)

where $q(u)$ is the quantized input and $\theta$ is unknown. In simulation, we choose $\Gamma = 1$, $\sigma = 1$, and $\varepsilon = 1$. To make comparison, a hysteresis quantizer is used and the following parameters are set to be the same values as in [16], i.e., $d = 0.02$, $\delta = 0.2$, $\theta = 1$, $c_1 = c_2 = 1$, $x(0) = \dot{x}(0) = 0.5$, and $\dot{\theta}(0) = 0.82$. Fig. 2 shows the trajectories of $x$ and $\dot{x}$ with the scheme in this paper and the scheme in [16], respectively, while the continuous input $u$ and the quantized $q(u(t))$ are presented in Fig. 3. From these simulation results, it is observed that, by using the controller proposed in this paper, the states converge to zero with a faster speed and less oscillation. The magnitude of the control signal with our scheme is also smaller. These results illustrate that the proposed design method gives better performances.
Fig. 4. Tracking performance with a hysteresis quantizer.

Fig. 5. Control $u$ and $q(u)$ with a hysteresis quantizer.
Fig. 6. Tracking performance with a logarithmic quantizer.

Fig. 7. Control $u$ and $q(u)$ with a logarithmic quantizer.
4.2. Example two

Consider the following nonlinear system studied in pages 182 and 183 of [24] where the nonlinearities do not satisfy global Lipschitz condition:

\[
\dot{x} = \theta_1 \frac{1 - e^{\theta_1 t}}{1 + e^{\theta_1 t}} - \theta_2 (\dot{x}^2 + 2x) - 0.2\theta_3 \sin (3t) + q(u)
\]  

(21)

where \(q(u)\) is the quantized input and \(\theta_1, \theta_2, \theta_3\) are unknown constants. The objective is to make the output \(y = x_1(t)\) track a reference signal \(r(t) = 2.5 \sin(t)\). For simulation, we choose \(\theta_1 = \theta_2 = \theta_3 = 1, d = 0.02, \delta = 0.2, x_1(0) = 1, x_2(0) = 1.05, \) and \(\theta(0) = [1.5 1.2 0.5]\). We set the convergence rate \(\xi = 0.5, \sigma_1 = \sigma_2 = 1\), so the size of the ultimate tracking error bound \(\Lambda\) is mainly determined by \(\lambda_{\max}(\Gamma^{-1})\) and \(\varepsilon\). For comparison, we choose the following three sets of parameters: (1) \(\Gamma^{-1} = \text{diag}[10, 10, 10], \sigma = 5, \varepsilon = 100\); (2) \(\Gamma^{-1} = \text{diag}[1, 1, 1], \sigma = 0.5, \varepsilon = 1\); (3) \(\Gamma^{-1} = \text{diag}[0.1, 0.1, 0.1], \sigma = 0.05, \varepsilon = 0.1\).

4.2.1. Hysteresis quantizer

For the system with a hysteresis quantizer, Fig. 4 shows the tracking performance of \(y(t)\), Fig. 5 shows the continuous input \(u\) and the quantized \(q(u(t))\).

4.2.2. Logarithmic quantizer

For the system with a logarithmic quantizer, Fig. 6 shows the tracking performance of \(y(t)\), Fig. 7 shows the continuous input \(u\) and the quantized \(q(u(t))\).

From the above comparison, we can obtain that the ultimate tracking error bound can be reduced to a very small set around zero by decreasing \(\varepsilon\) or \(\Gamma^{-1}\), which is consistent with the proved theoretical results. It should be noted that by decreasing \(\varepsilon\) or \(\Gamma^{-1}\), the control signal becomes more chattering, which is observed in Figs. 5 and 7. So there exists a trade-off between the tracking error performance and the control cost.

**Remark 3.** Compared with logarithmic quantizer, the hysteresis quantizer can lessen the chattering phenomenon because whenever \(u\) hits a new value it will stay at this value for some time. This can be seen from the simulation results in Figs. 5 and 7. So under the same conditions, the hysteresis quantizer gives a better performance.

5. Conclusions

In this paper, we develop a simple adaptive tracking control scheme for a general class of uncertain strict-feedback nonlinear system with the input quantized by a class of sector-bounded quantizers. The proposed approach can deal with systems whose nonlinearities are not globally Lipschitz continuous and thus such a restrictive requirement in existing results is relaxed. By choosing design parameters, the tracking error can converge towards an adjustable set. Simulation results illustrate that the proposed scheme yields good control performances.
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