



THE STRUCTURE OF INTEGRAL REPRESENTATIONS IN TOPOLOGICAL VECTOR SPACES

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Abstract

The subject of the present work deals with integral representations of bounded operators, acting on linear spaces of vector valued functions.

First we will consider operators on the space $C_0(S, X)$ of all continuous functions $f : S \rightarrow X$ vanishing at infinity, endowed with the uniform topology, S being a locally compact space and X a Banach space. We will give a complete characterization of operators $T : C_0(S, X) \rightarrow X$, which enjoy an integral form with respect to a scalar measure μ on S .

Next we consider integral representations for operators on L_1 type spaces, with values in a Banach space or a locally convex space. The main setting is the Bochner integration process with respect to finite abstract measure. The integral representations obtained may be considered as generalizations of the classical Riesz Theorem.

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0. Introduction

In the first part, S will be a locally compact space and X a Banach space. We consider the function space $C_0(S, X)$ of all continuous functions $f : S \rightarrow X$ vanishing at infinity, endowed with the uniform topology. We shall be concerned with integral representations of linear bounded operators $T : C_0(S, X) \rightarrow X$. The main result is a complete characterization of those operators which enjoy an integral form with respect to a scalar measure μ on S . Furthermore, we show that such operators also have an integral representation with respect to an operator valued measure G on S with values in $\mathcal{L}(X, X)$, the space of bounded operators on X . Finally, relationships between the different measures are established and this allows to characterize the operators under consideration by their representing measures.

In the second part, we consider a finite measure space (S, \mathcal{F}, μ) , a Banach space X and form the Banach space $L_1(\mu, X)$ of all Bochner μ -integrable functions $f : S \rightarrow X$, with $L_1(\mu, X) = L_1(\mu)$ if $X = \mathbb{R}$. We introduce a class of linear bounded operators $T : L_1(\mu, X) \rightarrow X$, whose integral structure is much similar to that of bounded functionals on $L_1(\mu)$. We give two complete characterizations of this class. The first one, which may be considered as a Riesz type theorem, is obtained via integrals by functions in $L_\infty(\mu)$. Actually the identified class will be isometrically isomorphic to $L_\infty(\mu)$. The second characterization is more specific. It pertains to an operator valued measure, that will be attached to each operator of the class. This operator valued measure will be absolutely continuous with respect to μ and this property will be used to get another interesting characterization of the class under consideration.

In the third part, we assume that X is a locally convex space whose topology is defined by a family $\{p_\alpha\}$ of continuous seminorms. We assume that $\{p_\alpha\}$ is separating, this means that for each nonzero $x \in X$ there is a p_α such that $p_\alpha(x) \neq 0$. Moreover, we assume that X is sequentially

complete, that is, every Cauchy sequence in X is convergent. The construction of the Bochner integral we give in this context is, as far as we know, new (for other approaches see [1, 7, 17]). Finally, arrangements are made so that each part of this work is mostly self-contained and can be read independently.

Part 1

Integral Representations of Linear Bounded Operators on $C_0(S, X)$

1. Preliminaries

1.1. Let S be a locally compact space equipped with its Borel field \mathcal{B}_S and let X be a Banach space. A function $f : S \rightarrow X$ is said to vanish at infinity, if for every $\varepsilon > 0$ there exists a compact set $K = K_{\varepsilon, f} \subset S$ such that $\|f(x)\| < \varepsilon, \forall x \notin K$. We shall denote by $C_0(S, X)$ the vector space of all continuous functions $f : S \rightarrow X$ vanishing at infinity. If $X = \mathbb{R}$, then we put $C_0(S, X) = C_0(S)$.

For $f \in C_0(S, X)$, we put:

$$\|f\| = \sup_{s \in S} \|f(s)\|. \quad (1.2)$$

Then it is well known that:

Proposition 1.3. *Formula (1.2) defines a norm on $C_0(S, X)$, for which $C_0(S, X)$ is a Banach space.*

For a Banach space E , E^* will be the topological dual of E and E^{**} the topological dual of E^* . It is assumed that all measures considered here are defined on \mathcal{B}_S . If μ is a set function on \mathcal{B}_S with values in the Banach space E we refer the reader to Chapter IV of [5] and Chapter 1 of [2] for the definition of the variation $v(\mu)$ and the semivariation $\|\mu\|$ of μ .

1.4. We shall deal occasionally with additive set functions $G : \mathcal{B}_S \rightarrow \mathcal{L}(X, E)$, where $\mathcal{L}(X, E)$ is the space of linear bounded operators of the Banach space X into the Banach space E .

If G is such a function, then we define the semivariation of G by the set function:

$$B \in \mathcal{B}_S, \quad \tilde{G}(B) = \text{Sup} \left\| \sum_i G(A_i) \cdot x_i \right\|$$

the supremum being over all finite partitions $\{A_i\}$ of B in \mathcal{B}_S and all finite systems of vectors $\{x_i\}$ in X , with $\|x_i\| \leq 1 \forall i$.

The function G is said to be of *finite semivariation* if $\tilde{G}(B)$ is finite for all $B \in \mathcal{B}_S$.

The objective is to characterize bounded linear operators $T : C_0(S, X) \rightarrow X$ via integral representation either by a scalar measure or by a vector measure. In Section 2, we start with the identification of those $T : C_0(S, X) \rightarrow X$ that do have an integral form with respect to a scalar measure. This will extend the results of [10] to the setting of a locally compact space S . In Section 3, we will consider integral representation by operator valued measures for a simple integration process. Finally relationships between the various representations are considered.

2. The Class \mathcal{C}_{XX}

In this section and in Section 3 below, we assume that X is a fixed Banach space and S is a locally compact space.

Let μ be a bounded signed measure on S and let us consider the Bochner integral

$$f \in C_0(S, X), \quad T_\mu f = \int_S f d\mu. \quad (2.1)$$

For all properties of the Bochner integral we refer the reader to [8].

Proposition 2.2. (a) *Formula (2.1) defines a linear bounded operator from $C_0(S, X)$ into X .*

(b) *For every bounded operator $U : X \rightarrow E$ from X into the Banach space E we have $UT_\mu f = T_\mu Uf$ for all $f \in C_0(S, X)$, where Uf is the vector of $C_0(S, E)$ given by $Uf(t) = U(f(t))$, $t \in S$.*

(c) $\|T_\mu\| = v(\mu)$ (the variation of μ).

Now we want to identify those operators $T : C_0(S, X) \rightarrow X$, for which there is a signed measure μ on S such that $T = T_\mu$. To this end we shall extend the strategy used in [10] to the present setting. So we begin by introducing the family of bounded operators $U_{x^*} : C_0(S, X) \rightarrow C_0(S)$, $x^* \in X^*$ given by:

$$f \in C_0(S, X), \quad U_{x^*} f = x^* \circ f, \quad (2.3)$$

where $x^* \circ f(t) = x^*(f(t))$, $t \in S$.

We collect some facts about U_{x^*} for later use:

Proposition 2.4. U_{x^*} is onto for each $x^* \neq 0$.

Proof. If $x^* \neq 0$, then take $x \in X$ such that $x^*(x) = 1$. For $h \in C_0(S)$ put $f(s) = h(s) \cdot x$, $s \in S$. Then $f \in C_0(S, X)$ and we have $U_{x^*} f(s) = x^*(f(s)) = h(s) \cdot x^*(x) = h(s)$. \square

Definition 2.5. Let \mathcal{C}_{XX} be the class of linear bounded operators $T : C_0(S, X) \rightarrow X$ satisfying the following condition:

$$(C) \quad x^*, y^* \in X^*, f, g \in C_0(S, X) : U_{x^*} f = U_{y^*} g \Rightarrow x^* T f = y^* T g.$$

It is easy to check that \mathcal{C}_{XX} is a closed subspace of the space of all bounded operators $T : C_0(S, X) \rightarrow X$. Note also that every T_μ defined by (2.1) is in \mathcal{C}_{XX} , since by 2.2(b), applied successively with $U = x^*$ and $U = y^*$, we have $x^*(T_\mu f) = T_\mu(x^* \circ f)$ and $y^*(T_\mu g) = T_\mu(y^* \circ f)$. So if $x^* \circ f = y^* \circ g$, we get $x^*T_\mu f = y^*T_\mu g$.

The outstanding fact about \mathcal{C}_{XX} is:

Theorem 2.6. *Let T be an operator in \mathcal{C}_{XX} . Then there exists a unique bounded linear functional $\varphi : C_0(S) \rightarrow \mathbb{R}$ such that:*

$$\varphi \circ U_{x^*} = x^* \circ T, \text{ for every } x^* \in X^*. \quad (2.7)$$

Proof. Let $h \in C_0(S)$ and $x^* \in X^*$, $x^* \neq 0$; by Proposition 2.4 there is an $f \in C_0(S, X)$ such that $U_{x^*} f = h$. Then we put:

$$\varphi(h) = x^*Tf. \quad (2.8)$$

It is noteworthy that the functional φ does not depend on the choice of x^* but depends only on T . For if $U_{x^*} f = U_{y^*} g = h$, then $x^*Tf = y^*Tg$, by condition (C); so φ is well defined, and it is easy to see that φ is linear. We must show that φ is bounded. We may argue as follows: since U_{x^*} is bounded and onto, by the open mapping principle there exists a constant $K = K_{x^*} > 0$ such that for every $h \in C_0(S)$, there is a solution $f \in C_0(S, X)$ of $U_{x^*} f = h$, with $\|f\| \leq K \cdot \|h\|$. From (2.8) we deduce that $\|\varphi(h)\| \leq \|x^*\| \|T\| \|f\| \leq \|x^*\| \|T\| K \|h\|$, which proves that φ is bounded.

It remains to prove (2.7). For $f \in C_0(S, X)$ and $x^* \in X^*$, we have $h = U_{x^*} f \in C_0(S)$, and (2.8) gives $\varphi(h) = \varphi(U_{x^*} f) = x^*Tf$. Since f and x^* are arbitrary, (2.7) follows. \square

By a straightforward extension of Theorem 2.1 in [10], to the present setting, we get:

Theorem 2.9. *There is an isometric isomorphism between the Banach space \mathcal{C}_{XX} and the topological dual $C_0^*(S)$ of $C_0(S)$, for each non trivial Banach space X .*

Now we turn to the representation of operators T in the class \mathcal{C}_{XX} , via Bochner integrals.

Theorem 2.10. *Every operator T in \mathcal{C}_{XX} is of the form T_μ , for a unique bounded signed measure μ on S . In other words, to each T in the class \mathcal{C}_{XX} there corresponds a unique bounded signed measure μ on S such that:*

$$\forall f \in C_0(S, X), T(f) = \int_S f d\mu$$

$$\|T\| = v(\mu). \tag{2.11}$$

Proof. Let φ be the functional corresponding to the operator T according to Theorem 2.6, and let μ be the measure related to φ by the Riesz representation theorem ([15] Theorem 6.19). Then for every $h \in C_0(S)$, we have $\varphi(h) = \int_S h d\mu$; so, if $h = U_{x^*} f$ with $f \in C_0(S, X)$, we get, by Proposition 2.2(b): $\varphi(U_{x^*} f) = x^* T_\mu f$. But by (2.7) $\varphi(U_{x^*} f) = x^* T f$, therefore $x^* T f = x^* T_\mu f$; and since f and x^* are arbitrary, we deduce that $T = T_\mu$. □

For integral representations by weak integrals, see the approach in [12, 13].

3. Representations by Operator Valued Measures

In this section, we show that a linear bounded operator $T : C_0(S, X) \rightarrow X$ in the class \mathcal{C}_{XX} enjoy an integral representation by an operator

valued measure whose special form will be used to characterize the operator T as being a member of the class \mathcal{C}_{XX} . In what follows we first make precise the integration process which will be used in our representation.

3.1. A simple measurable function s on S with values in the Banach space X is a function of the form $s(\bullet) = \sum_i 1_{A_i}(\bullet) \cdot x_i$, where $\{A_i\}$ is a finite partition of S in \mathcal{B}_S , and $\{x_i\}$ is a finite system of vectors in the Banach space X . The symbol 1_{A_i} means the characteristic function of the set A_i . Let \mathfrak{S} be the set of all X -valued simple functions on S . A function $f : S \rightarrow X$ is said to be *measurable* if there is a sequence s_n in \mathfrak{S} converging uniformly to f on S . Let $\mathcal{M}(S, X)$ be the set of all measurable functions. Then \mathfrak{S} and $\mathcal{M}(S, X)$ are in an obvious way vector subspaces of the space \mathfrak{F} of all bounded functions $f : S \rightarrow X$. Actually the uniform convergence alluded to above is the convergence with respect to the supremum norm $\|f\| = \sup_{s \in S} \|f(s)\|$, for $f \in \mathfrak{F}$. It is the same to say that $\mathcal{M}(S, X)$ is the closure of \mathfrak{S} for this norm.

Now let $G : \mathcal{B}_S \rightarrow \mathcal{L}(X, E)$ be an additive set function on \mathcal{B}_S with values in $\mathcal{L}(X, E)$, the space of linear bounded operators from the Banach space X into the Banach space E . Assume that G has finite semivariation (see 1.4). We define the integral of the simple function $s(\bullet) = \sum_i 1_{A_i}(\bullet) \cdot x_i$, over the set $B \in \mathcal{B}_S$, with respect to G by:

$$\int_B s dG = \sum_i G(A_i \cap B) \cdot x_i. \quad (3.2)$$

It is easy to check that the integral is well defined and satisfies:

$$\left\| \int_B s dG \right\| \leq \|s\| \cdot \tilde{G}(B). \quad (3.3)$$

Let us observe that estimation (3.3) implies that the linear operator $U_B : \mathfrak{S} \rightarrow E$, with $U_B(s) = \int_B s dG$ is bounded. So we can extend it in a unique manner to a bounded operator on the closure $\mathcal{M}(S, X)$ of \mathfrak{S} . This extension will be our integration process on the space $\mathcal{M}(S, X)$ of measurable functions. We shall denote it also by U_B with $U_B = U$ if $B = S$. Note that if $f \in \mathcal{M}(S, X)$ and if s_n is a sequence in \mathfrak{S} such that $\|f - s_n\| \rightarrow 0$, then the integral of f is given by:

$$U_B(f) = \int_B f dG = \lim_n \int_B s_n dG. \quad (3.4)$$

By (3.3), the integral (3.4) does not depend on the sequence s_n chosen to converge to the function f . This simple integration process will be sufficient for our purpose. The outstanding facts are summarized in the following:

Theorem 3.5. *Let G be an additive $\mathcal{L}(X, E)$ -valued set function with finite semivariation on \mathcal{B}_S . Then:*

(a) *The integral $\int_B f dG$ is linear in $f \in \mathcal{M}(S, X)$ and satisfies:*

$$\tilde{G}(B) = \text{Sup} \left\{ \left\| \int_B f dG \right\|, \|f\| \leq 1, f \in \mathcal{M}(S, X) \right\} \quad (3.6)$$

*in other words the operator $U_B : \mathcal{M}(S, X) \rightarrow E$ given by $U_B(f) = \int_B f dG$ is bounded with norm $\|U_B\| = \tilde{G}(B)$, for each $B \in \mathcal{B}_S$.
Conversely:*

(b) *Let $U : \mathcal{M}(S, X) \rightarrow E$ be a bounded operator. Then there is a unique additive set function $G : \mathcal{B}_S \rightarrow \mathcal{L}(X, E)$, with finite semivariation such that:*

$$\forall f \in \mathcal{M}(S, X), \forall B \in \mathcal{B}_S, \quad U(f \cdot 1_B) = \int_B f dG. \quad (3.7)$$

(c) Let $\Lambda : E \rightarrow Y$ be a bounded operator from E into the Banach space Y . Let us define $\Lambda G : \mathcal{B}_S \rightarrow \mathcal{L}(X, Y)$ by $(\Lambda G)(B)x = \Lambda(G(B)x)$, $B \in \mathcal{B}_S$, $x \in X$. Then ΛG is an additive $\mathcal{L}(X, Y)$ -valued set function with finite semivariation and we have:

$$\forall f \in \mathcal{M}(S, X), \quad \int_S f d\Lambda G = \Lambda \left(\int_S f dG \right). \quad (3.8)$$

Proof. (a) To prove (3.6) start with f simple and use (3.2) and the definition of $\tilde{G}(B)$. For general f , use (3.4).

(b) Define $G : \mathcal{B}_S \rightarrow \mathcal{L}(X, E)$ by $G(B) \cdot x = U(1_B \cdot x)$, for $B \in \mathcal{B}_S$, and $x \in X$. Then G is additive since U is linear and G is $\mathcal{L}(X, E)$ -valued because U is bounded. Now (3.7) is easily checked by (3.2) and (3.4).

(c) To prove (3.8) start with f simple and use the definition of ΛG , then apply (3.4), (recall that the operator Λ is bounded). \square

Actually, part (b) of this theorem is an integral representation of a bounded operator U on the space $\mathcal{M}(S, X)$ by means of an $\mathcal{L}(X, E)$ -valued set function G on \mathcal{B}_S . In our context, we need representation for operators on the space of continuous functions $C_0(S, X)$. This is a less trivial problem which will be solved presently for operators in the class \mathcal{C}_{XX} . First let us observe:

Proposition 3.9. *We have $C_0(S, X) \subset \mathcal{M}(S, X)$, that is $C_0(S, X)$ is a subspace of $\mathcal{M}(S, X)$.*

Proof. Let us consider the set of all functions $f \in C_0(S, X)$ of the form $f(\bullet) = g(\bullet) \cdot x$, with $g \in C_0(S)$ and x fixed in X . We denote by $C_0(S) \otimes X$ the subspace spanned by this set of functions. Then it is known that $C_0(S) \otimes X$ is dense in $C_0(S, X)$ (see Proposition 1 Section 19 in [3]).

So it is enough to show that $C_0(S) \otimes X \subset \mathcal{M}(S, X)$. Let $f \in C_0(S) \otimes X$ of the form $f = g \cdot x$ with $g \in C_0(S)$. Since g is scalar measurable,

bounded on S , there exists a sequence t_n of simple scalar functions on S converging to g uniformly on S . But then the functions $s_n = t_n \cdot x$ are simple X -valued and converge uniformly to f on S . Thus $f \in \mathcal{M}(S, X)$. \square

Now we are in a position to give the main theorem of this section:

Theorem 3.10. *To each operator $T : C_0(S, X) \rightarrow X$ in the class \mathcal{C}_{XX} there corresponds a unique operator valued set function G on \mathcal{B}_S with values in $\mathcal{L}(X, X)$ such that*

$$\forall f \in C_0(S, X), \quad Tf = \int_S fdG. \quad (3.11)$$

Moreover, the function G is σ -additive in the uniform operator topology and takes its values in a compact set of $\mathcal{L}(X, X)$.

Proof. Let μ be the bounded signed measure attached to the operator T by Theorem 2.10.

Define the set function $G : \mathcal{B}_S \rightarrow \mathcal{L}(X, X)$ by:

$$B \in \mathcal{B}_S, \quad G(B) = \mu(B) \cdot I, \quad (3.12)$$

where I is the identity operator of X . From (3.12), we deduce that the semivariation of G is equal to the variation of μ and so it is finite. It is a simple routine job to check that the integration process (3.2)-(3.4) with this G is exactly the Bochner integration process with the measure μ . This means that for every $f \in \mathcal{M}(S, X)$ we have $\int_S fdG = \int_S fd\mu$. Note that the Bochner integral in the RHS is well defined since the functions f and μ are both bounded. From (2.11) we deduce the validity of (3.11). On the other hand formula (3.12) shows that G is σ -additive in the uniform operator topology of $\mathcal{L}(X, X)$ and takes values in the one dimensional subspace of $\mathcal{L}(X, X)$ spanned by I . But the range of G is bounded by the variation $v(\mu)$ of μ , so it is conditionally compact. \square

Let us point out the converse:

Theorem 3.13. *Let $T : C_0(S, X) \rightarrow X$ be a bounded operator such that (3.11) holds with G given by (3.12), where μ is a bounded scalar measure. Then $T \in \mathcal{C}_{XX}$, that is $T = T_\mu$.*

Proof. For $f \in C_0(S, X)$, we have $Tf = \int_S fdG$ and, by (3.8) with $\Lambda = x^*$, we have $x^*Tf = \int_S fdx^*G$, for each $x^* \in X^*$. From (3.12), we deduce that $x^*G(\bullet) = \mu(\bullet) \cdot x^*$, and then $x^*Tf = \int_S fd\mu \cdot x^* = \int_S x^* \circ fd\mu$, by standard tools of integration. The last integral is equal to $x^*T_\mu f$ by Proposition 2.2, (b). Thus, $x^*Tf = x^*T_\mu f$; since f and x^* are arbitrary, we deduce that $T = T_\mu$. \square

For the sake of completeness, we give the following integral representation valid for operators $T \in \mathcal{C}_{XX}$ in the case S compact. For the proof, the reader is referred to Theorem 5.3 in [6].

Theorem 3.14. *Suppose S compact. Then a linear bounded operator $T : C_0(S, X) \rightarrow X$ is in the class \mathcal{C}_{XX} if and only if T admits an integral representation by the set function $G : \mathcal{B}_S \rightarrow \mathcal{L}(X, X^{**})$, such that:*

$$B \in \mathcal{B}_S, \quad G(B) = \mu(B) \cdot \gamma, \quad (3.15)$$

where μ is a bounded scalar measure and $\gamma : X \rightarrow X^{**}$ the canonical isomorphism.

Proof. The necessity comes from Theorem 5.3 of [10]. The proof of the sufficiency is a straightforward adaptation of the proof of Theorem 3.13. \square

Theorem 3.16. *If S is compact, then for each $T \in \mathcal{C}_{XX}$, the representing measure G in (3.15) is σ -additive in the uniform topology of $\mathcal{L}(X, X^{**})$ and takes its values in a compact set of $\mathcal{L}(X, X^{**})$.*

Proof. We have: $G(\bullet) = \mu(\bullet) \cdot \gamma$, where μ is a bounded scalar measure and γ is the canonical isomorphism of X into X^{**} .

It is clear, that G is σ -additive in the uniform topology of $\mathcal{L}(X, X^{**})$. On the other hand, the values $G(A) = \mu(A) \cdot \gamma$, $A \in \mathcal{B}_S$, of the measure G , form a subset B of the space K_γ spanned by γ in $\mathcal{L}(X, X^{**})$. Since K_γ is one dimensional and since B is bounded by $\nu(\mu)$, we deduce that B is conditionally compact in K_γ and also in $\mathcal{L}(X, X^{**})$. \square

Part 2

The Integral Structure of Some Bounded Operators on $L_1(\mu, X)$

1. Operators on the Space of Bochner Integrable Functions

Let (S, \mathcal{F}, μ) be a finite measure space and let X be a Banach space. We denote by $L_1(\mu, X)$ the Banach space of all Bochner μ -integrable functions $f : S \rightarrow X$, with $L_1(\mu, X) = L_1(\mu)$ if $X = \mathbb{R}$. For all properties of the Bochner integral, we refer the reader to [5].

For $f \in L_1(\mu, X)$, we put:

$$\|f\|_1 = \int_S \|f(s)\| d\mu(s). \quad (1.2)$$

Then it is well known that:

Proposition 1.3. *Formula (1.2) defines a norm on $L_1(\mu, X)$, for which $L_1(\mu, X)$ is a Banach space. Moreover, the measurable simple functions $s : S \rightarrow X$ form a dense subspace of $L_1(\mu, X)$. This means that for each $f \in L_1(\mu, X)$ there is a sequence s_n of simple functions such that $\|f - s_n\|_1 \rightarrow 0$.*

The starting point that has motivated the present work is contained in the following simple observation:

Theorem 1.4. Fix a function g in $L_\infty(\mu)$ (the space of all μ -essentially bounded real functions on S) and consider the operator $T_g : L_1(\mu, X) \rightarrow X$ defined by:

$$f \in L_1(\mu, X), \quad T_g(f) = \int_S fg d\mu; \quad (1.5)$$

Then T_g is linear bounded and satisfies $\|T_g\| = \|g\|_\infty$.

Proof. Since $\|f(s)g(s)\| \leq \|f(s)\| \|g\|_\infty$ μ -a.e. we deduce from (1.5), $\|T_g(f)\| \leq \|g\|_\infty \cdot \int_S \|f(s)\| d\mu(s) = \|g\|_\infty \cdot \|f\|_1$. So the operator T_g is bounded and $\|T_g\| \leq \|g\|_\infty$. To prove the reverse inequality, apply T_g to a function $f \in L_1(\mu, X)$ of the form $f = \varphi \cdot x$, where $\varphi \in L_1(\mu)$, such that $\|\varphi\|_1 = 1$ and x fixed in X with $\|x\| = 1$. We get $\|f\|_1 = \|\varphi\|_1 = 1$ and $T_g(f) = \int_S \varphi g x d\mu = \left(\int_S \varphi g \cdot d\mu \right) \cdot x$, by standard integration tools.

So we deduce $\|T_g(f)\| = \left| \int_S \varphi g \cdot d\mu \right| \leq \|T_g\|$ and then

$$\text{Sup} \left\{ \left| \int_S \varphi g \cdot d\mu \right|, \varphi \in L_1(\mu), \|\varphi\|_1 = 1 \right\} \leq \|T_g\|.$$

But the LHS of the preceding inequality is equal to $\|g\|_\infty$ by the Riesz duality theorem for $L_1(\mu)$. So we get $\|g\|_\infty \leq \|T_g\|$ and then $\|T_g\| = \|g\|_\infty$. \square

Remark 1.6. Another way to put the conclusion of Theorem 1.4 is the following:

The map $\Phi : g \rightarrow T_g$ from $L_\infty(\mu)$ into $\mathcal{L}(L_1(\mu, X), X)$, the space of bounded operators $T : L_1(\mu, X) \rightarrow X$, is a linear isometry.

We can wonder whether Φ is onto. This is certainly true if $X = \mathbb{R}$ by the Riesz duality theorem for $L_1(\mu)$. But if dimension of X is greater than one, the following example shows that not all operators in $\mathcal{L}(L_1(\mu, X), X)$ can be written in the form (1.5) for some g in $L_\infty(\mu)$.

Example 1.7. Let $X = \mathbb{R}^2$, equipped with the norm: $z = (z_1, z_2)$, $\|z\| = |z_1| + |z_2|$. If $f = (f_1, f_2) : S \rightarrow \mathbb{R}^2$ is Bochner μ -integrable with the Borel σ -field on \mathbb{R}^2 , then $f_1, f_2 : S \rightarrow \mathbb{R}$ are μ -integrable and $\int_S f d\mu = \left(\int_S f_1 d\mu, \int_S f_2 d\mu \right)$. Note also that $\|f(s)\| = |f_1(s)| + |f_2(s)|$, so that $\|f\|_1 = \int_S |f_1| d\mu + \int_S |f_2| d\mu$. Now define the operator $T : L_1(\mu, \mathbb{R}^2) \rightarrow \mathbb{R}^2$, by $Tf = T(f_1, f_2) = \left(\int_S f_1 d\mu, \alpha \int_S f_2 d\mu \right)$, where $0 < \alpha < 1$ is a fixed constant. It is clear that T is linear and we have $\|Tf\| = \left| \int_S f_1 d\mu \right| + \alpha \left| \int_S f_2 d\mu \right| \leq \|f\|_1$, so that T is bounded. If there were $g \in L_\infty(\mu)$ such that $T(f) = \int_S fg d\mu$, we would have $\int_S f_1 d\mu = \int_S f_1 g \cdot d\mu$ and $\alpha \int_S f_2 d\mu = \int_S f_2 \cdot g \cdot d\mu$, for all μ -integrable functions f_1, f_2 . Taking f_1, f_2 both characteristic functions of sets in \mathcal{F} , this would imply $g = 1$, μ -a.e and $g = \alpha$, μ -a.e. This is impossible by the choice of α . Consequently the operator T cannot be written in the form (1.5).

The aim is to characterize those bounded operators $T : L_1(\mu, X) \rightarrow X$ that have integral form (1.5) with a function $g \in L_\infty(\mu)$. This amounts to describe the range of the operator Φ in Remark 1.6. In Section 2, we give the ingredients of this characterization which allows a representation of operators on the space $L_1(\mu, X)$, much simpler than those given in [6, 14]. In Section 3, we prove integral representations by operator valued measures, for

operators introduced in Section 2. This leads to a rather precise description of such operators.

2. A Characterizing Class

In this section, we want to identify those operators $T \in \mathcal{L}(L_1(\mu, X), X)$, for which there is $g \in L_\infty(\mu)$ such that $T = T_g$. Let X^* be the topological dual of X . The technicalities involved here are much similar to that used in Section 2 of Part 1, mutatis mutandis. For each $x^* \in X^*$ consider the operator $\varphi_{x^*} : L_1(\mu, X) \rightarrow L_1(\mu)$, given by:

$$f \in L_1(\mu, X), \quad \varphi_{x^*} f = x^* \circ f, \quad (2.1)$$

where $(x^* \circ f)(t) = x^*(f(t))$, $t \in S$.

We collect some facts about φ_{x^*} for later use:

Proposition 2.2. (a) φ_{x^*} is linear bounded and $\|\varphi_{x^*}\| = \|x^*\|$.

(b) φ_{x^*} is onto for each $x^* \neq 0$.

(c) There exists $y^* \in X^*$ such that for each $h \in L_1(\mu)$ there is $f \in L_1(\mu, X)$ with

$$\|f\|_1 = \|h\|_1 \quad \text{and} \quad \varphi_{y^*} f = h.$$

Proof. (a)

$$\|\varphi_{x^*} f\| = \int_S |x^* \circ f| d\mu \leq \|x^*\| \int_S \|f(s)\| d\mu(s) = \|x^*\| \|f\|_1.$$

So φ_{x^*} is bounded and $\|\varphi_{x^*}\| \leq \|x^*\|$. To see the reverse inequality apply φ_{x^*} to a function $f \in L_1(\mu, X)$ of the form $f(\bullet) = g(\bullet) \cdot x$, with

$g \in L_1(\mu)$ such that $\|g\|_1 = 1$ and x fixed in X with $\|x\| = 1$. We get $\|f\|_1 = 1$ and $\|\varphi_{x^*} f\| = \int_S |x^* \circ f| d\mu = |x^*(x)|$. Thus, $|x^*(x)| \leq \|\varphi_{x^*}\|$ for every $x \in X$ with $\|x\| = 1$. Consequently

$$\|x^*\| = \text{Sup}\{|x^*(x)|, x \in X, \|x\|_1 = 1\} \leq \|\varphi_{x^*}\|.$$

(b) Let $x^* \neq 0$ and choose $x \in X$ such that $x^*(x) = 1$. Now if $h \in L_1(\mu)$ put $f = h \cdot x$, then clearly we have $\varphi_{x^*} f = h$.

(c) Choose $x \in X$ with $\|x\| = 1$, then choose $y^* \in X^*$ such that $y^*(x) = \|x\| = 1, \|y^*\| = 1$, this is possible by Hahn-Banach theorem. If $h \in L_1(\mu)$, the function $f = h \cdot x$ is in $L_1(\mu, X)$ and fits the conclusion. \square

The following class of operators will play an essential role for the characterization we need:

Definition 2.3. Let \mathfrak{D} be the class of linear bounded operators $T \in \mathcal{L}(L_1(\mu, X), X)$ satisfying the following condition:

$$x^*, y^* \in X^*, f, g \in L_1(\mu, X) : \varphi_{x^*} f = \varphi_{y^*} g \Rightarrow x^* T f = y^* T g. \quad (2.4)$$

It is easy to check that \mathfrak{D} is a closed subspace of $\mathcal{L}(L_1(\mu, X), X)$. Note also that every T_g as defined by (1.5) is in \mathfrak{D} .

The important fact about \mathfrak{D} is:

Theorem 2.5. *Let T be an operator in \mathfrak{D} . Then there exists a unique bounded linear functional $V : L_1(\mu) \rightarrow \mathbb{R}$ such that:*

$$V \circ \varphi_{x^*} = x^* \circ T \quad (2.6)$$

for every $x^* \in X^*$.

Proof. Let $h \in L_1(\mu)$ and $x^* \in X^*$, $x^* \neq 0$; by Proposition 2.2(b) there is an $f \in L_1(\mu, X)$ such that $\varphi_{x^*} f = h$. Then we put:

$$V(h) = x^* T f. \quad (2.7)$$

The functional V does not depend on the choice of x^* but depends only on T . For if V_{x^*} and V_{y^*} are defined as in (2.7), with $x^*, y^* \neq 0$, then $V_{x^*}(h) = x^* T f$ if $h = \varphi_{x^*} f$ and $V_{y^*}(h) = y^* T g$ if $h = \varphi_{y^*} g$; but condition (2.4) on T implies that $V_{x^*}(h) = V_{y^*}(h)$. It is easy to check that it is linear. We must show that V is bounded. Since φ_{x^*} is bounded and onto, by the open mapping principle there exists a constant $K = K_{x^*} > 0$ such that for every $h \in L_1(\mu)$, there is a solution $f \in L_1(\mu, X)$ of $\varphi_{x^*} f = h$, with $\|f\| \leq K \cdot \|h\|$. From (2.7) we deduce that $\|V(h)\| \leq \|x^*\| \|T\| \|f\| \leq \|x^*\| \|T\| K \|h\|$, which proves that V is bounded.

It remains to prove (2.6). For $f \in L_1(\mu, X)$ and $x^* \in X^*$, we have $h = \varphi_{x^*} f \in L_1(\mu)$, and (2.7) gives $V(h) = V(\varphi_{x^*} f) = x^* T f$. Since f and x^* are arbitrary, (2.6) follows. Uniqueness is clear from (2.6) since φ_{x^*} is onto. \square

As a consequence of the preceding theorem let us note:

Theorem 2.8. *There is an isometric isomorphism between the Banach space \mathcal{D} and the topological dual $L_1^*(\mu)$ of $L_1(\mu)$, for each non trivial Banach space X .*

Proof. Define the operator $\Psi : \mathcal{D} \rightarrow L_1^*(\mu)$ by: $T \in \mathcal{D}$, $\Psi(T) = V$, where V is the unique bounded functional on $L_1(\mu)$ attached to T by Theorem 2.5. It is not difficult to see that Ψ is linear. We have to show that

Ψ is an isometry, that is, $\|V\| = \|T\|$ if $\Psi(T) = V$. First we prove the estimation

$$\|V\| = \text{Sup}\{\|V \circ \varphi_{x^*}\| : x^* \in X^*, \|x^*\| \leq 1\}. \quad (2.9)$$

We have $\|V \circ \varphi_{x^*}\| \leq \|V\| \|\varphi_{x^*}\| = \|V\| \|x^*\|$, since $\|\varphi_{x^*}\| = \|x^*\|$ by 2.2(a). So we deduce $\|V \circ \varphi_{x^*}\| \leq \|V\|$, for all $x^* \in X^*$, with $\|x^*\| \leq 1$. Hence $\text{Sup}\{\|V \circ \varphi_{x^*}\| : x^* \in X^*, \|x^*\| \leq 1\} \leq \|V\|$. But $V \in L_1^*(\mu)$, consequently for each $\varepsilon > 0$ there is $h \in L_1(\mu)$ such that $\|h\|_1 \leq 1$ and $\|V\| - \varepsilon < |V(h)| \leq \|V\|$. Now let $y^* \in X^*$ as in 2.2(c) and choose $f \in L_1(\mu, X)$, such that $\|f\|_1 = \|h\|_1$ and $\varphi_{y^*} f = h$. Then $\|f\|_1 \leq 1$ and $|V \circ \varphi_{y^*}(f)| = |V(h)| \leq \|V \circ \varphi_{y^*}\| \|f\|_1$. Thus $|V(h)| \leq \|V \circ \varphi_{y^*}\| \leq \text{Sup}\{\|V \circ \varphi_{x^*}\| : x^* \in X^*, \|x^*\| \leq 1\}$. From the choice of h , we get $\|V\| - \varepsilon \leq \text{Sup}\{\|V \circ \varphi_{x^*}\| : x^* \in X^*, \|x^*\| \leq 1\}$. Letting $\varepsilon \downarrow 0$, we obtain $\|V\| \leq \text{Sup}\{\|V \circ \varphi_{x^*}\| : x^* \in X^*, \|x^*\| \leq 1\}$. So (2.9) is proved. To finish the norm equality $\|V\| = \|T\|$, we appeal to formula (2.6) and conclude:

$$\begin{aligned} \|V\| &= \text{Sup}\{\|V \circ \varphi_{x^*}\| : x^* \in X^*, \|x^*\| \leq 1\} \\ &= \text{Sup}\{\|x^* \circ T\| : x^* \in X^*, \|x^*\| \leq 1\} = \|T\|. \end{aligned}$$

To achieve the proof it remains to prove that Ψ is onto. If $V \in L_1^*(\mu)$, then by the Riesz duality theorem, there is a unique $g \in L_\infty(\mu)$ such that $V(h) = \int_S h g d\mu$, for all $h \in L_1(\mu)$. Consider the operator T_g on $L_1(\mu, X)$ given by formula (1.5). We have $T_g \in \mathfrak{D}$ and it is straightforward that V and T_g are linked by equation (2.6). So from the definition of the operator Ψ we deduce that $\Psi(T_g) = V$. \square

Since $L_1^*(\mu)$ is isometrically isomorphic to $L_\infty(\mu)$, we deduce the following corollary:

Corollary. *The class \mathfrak{D} is isometrically isomorphic to $L_\infty(\mu)$. In other words, a bounded operator $T : L_1(\mu, X) \rightarrow X$ is in \mathfrak{D} iff there is a unique $g \in L_\infty(\mu)$ such that $T = T_g$ and in this case $\|T\| = \|g\|_\infty$.*

Now we turn to another description of the class \mathfrak{D} , namely by a space of measures. This will be achieved via integrals with respect to operator valued measures.

3. Operator Valued Measures Representing the Class \mathfrak{D}

3.1. The integration process we shall deal with in this section is performed by an operator valued additive set function $G : \mathcal{F} \rightarrow \mathcal{L}(X, E)$, where $\mathcal{L}(X, E)$ is the space of linear bounded operators of the Banach space X into the Banach space E . The integral will be defined for measurable functions $f : S \rightarrow X$, under the assumption that G is additive and with finite semivariation. Let us recall that semivariation means the set function

\tilde{G} on \mathcal{F} given by $\tilde{G}(B) = \text{Sup} \left\| \sum_i G(A_i) \cdot x_i \right\|$, where $B \in \mathcal{F}$, and the

supremum taken over all finite partitions $\{A_i\}$ of B in \mathcal{F} and all finite systems of vectors $\{x_i\}$ in X , with $\|x_i\| \leq 1 \forall i$. The function G is said to be

of *finite semivariation* if $\tilde{G}(B)$ is finite for all $B \in \mathcal{F}$. A simple measurable function s on S with values in the Banach space X is a function of the form $s(\bullet) = \sum_i 1_{A_i}(\bullet) \cdot x_i$, where $\{A_i\}$ is a finite partition of S in \mathcal{F} , and $\{x_i\}$ is a

finite system of vectors in X . The symbol 1_{A_i} means the characteristic function of the set A_i . A function $f : S \rightarrow X$ is said to be *measurable* if there is a sequence s_n of measurable simple functions converging uniformly to f on S . If we denote by \mathcal{I} and \mathcal{M} the sets of simple functions and

measurable functions, respectively then \mathcal{I} and \mathcal{M} are subspaces of the Banach space of all bounded functions $f : S \rightarrow X$, with supremum norm. Moreover, \mathcal{I} is dense in \mathcal{M} .

We define the integral of the simple function $s(\bullet) = \sum_i 1_{A_i}(\bullet) \cdot x_i$, over the set $B \in \mathcal{F}$, with respect to G by:

$$\int_B s dG = \sum_i G(A_i \cap B) \cdot x_i. \quad (3.2)$$

It is easy to check that the integral is well defined and satisfies:

$$\left\| \int_B s dG \right\| \leq \|s\| \cdot \tilde{G}(B) \quad (3.3)$$

($\|s\| =$ supremum norm).

Let us observe that estimation (3.3) implies that the linear operator $U_B : \mathcal{I} \rightarrow E$, with $U_B(s) = \int_B s dG$ is bounded. So we can extend it in a unique manner to a bounded operator on the closure \mathcal{M} of \mathcal{I} . This extension will be our integration process on the space \mathcal{M} of measurable functions. We shall denote it also by U_B with $U_B = U$ if $B = S$. Note that if $f \in \mathcal{M}$ and if s_n is a sequence in \mathcal{I} such that $\|f - s_n\| \rightarrow 0$, then the integral of f is given by:

$$U_B(f) = \int_B f dG = \lim_n \int_B s_n dG. \quad (3.4)$$

By (3.3) the integral (3.4) does not depend on the sequence s_n chosen converging to the function f . This simple integration process will be sufficient for our purpose. The outstanding facts are summarized in the following:

Theorem 3.5. *Let G be an additive $\mathcal{L}(X, E)$ -valued set function with finite semivariation on \mathcal{F} . Then:*

(a) The integral $\int_B fdG$ is linear in $f \in \mathcal{M}$ and satisfies:

$$\tilde{G}(B) = \text{Sup} \left\{ \left\| \int_B fdG \right\|, \|f\| \leq 1, f \in \mathcal{M} \right\} \quad (3.6)$$

in other words the operator $U_B : \mathcal{M} \rightarrow E$ given by $U_B(f) = \int_B fdG$ is bounded with norm $\|U_B\| = \tilde{G}(B)$, for each $B \in \mathcal{F}$. Conversely:

(b) Let $U : \mathcal{M} \rightarrow E$ be a bounded operator. Then there is a unique additive set function $G : \mathcal{F} \rightarrow \mathcal{L}(X, E)$, with finite semivariation such that:

$$\forall f \in \mathcal{M}, \forall B \in \mathcal{F}, U(f \cdot 1_B) = \int_B fdG. \quad (3.7)$$

(c) Let $\Lambda : E \rightarrow Y$ be a bounded operator from E into the Banach space Y . Let us define $\Lambda G : \mathcal{F} \rightarrow \mathcal{L}(X, Y)$ by $(\Lambda G)(B)x = \Lambda(G(B)x)$, $B \in \mathcal{F}$, $x \in X$. Then ΛG is an additive $\mathcal{L}(X, Y)$ -valued set function with finite semivariation and we have:

$$\forall f \in \mathcal{M}, \int_S fd\Lambda G = \Lambda \left(\int_S fdG \right). \quad (3.8)$$

Proof. (a) To prove (3.6) start with f simple and use (3.2) and the definition of $\tilde{G}(B)$. For general f use (3.4).

(b) Define $G : \mathcal{F} \rightarrow \mathcal{L}(X, E)$ by $G(B) \cdot x = U(1_B \cdot x)$, for $B \in \mathcal{F}$, and $x \in X$. Then G is additive since U is linear and G is $\mathcal{L}(X, E)$ -valued because U is bounded. Now (3.7) is easily checked by (3.2) and (3.4).

(c) To prove (3.8) start with f simple and use the definition of ΛG , then apply (3.4), (recall that the operator Λ is bounded). \square

Actually, part (b) of this theorem is an integral representation of a bounded operator U on the space \mathcal{M} by means of an $\mathcal{L}(X, E)$ -valued set function G on \mathcal{F} .

The next step is to extend the preceding integration process from \mathcal{M} to the space $L_1(\mu, X)$. The reader should observe that the space \mathcal{M} is

contained in $L_1(\mu, X)$, because functions in \mathcal{M} are bounded and μ is a finite measure. The extension of the integral (3.4) from \mathcal{M} to $L_1(\mu, X)$ will be achieved under the additional assumption that $\|G(A)\| \leq k \cdot \mu(A)$ for some constant $k > 0$ and all $A \in \mathcal{F}$.

Theorem 3.9. *Let G be an additive $\mathcal{L}(X, E)$ -valued set function with finite semivariation on \mathcal{F} . Assume that:*

$$\|G(A)\| \leq k \cdot \mu(A) \quad (3.10)$$

for some constant $k > 0$ and all $A \in \mathcal{F}$. Then we have:

(a) *The integral (3.4) is a linear operator from \mathcal{M} to E which is continuous with the $L_1(\mu, X)$ -topology on \mathcal{M} and satisfies:*

$$\forall f \in \mathcal{M}, \quad \left\| \int_S f dG \right\| \leq k \int_S \|f\| d\mu. \quad (3.11)$$

(b) *The integral $\int_S f dG$, $f \in \mathcal{M}$, admits a unique extension to $L_1(\mu, X)$, still denoted by $\int_S f dG$, such that:*

$$\forall f \in L_1(\mu, X), \quad \left\| \int_S f dG \right\| \leq k \int_S \|f\| d\mu. \quad (3.12)$$

(c) *The operator $f \rightarrow \int_S f dG$ is linear and bounded from $L_1(\mu, X)$ to E .*

Proof. (a) Let $s(\bullet) = \sum_i 1_{A_i}(\bullet) \cdot x_i$ be a simple measurable function with values in X . From (3.10), we deduce

$$\begin{aligned} \left\| \int_S s dG \right\| &= \left\| \sum_i G(A_i) \cdot x_i \right\| \leq \sum_i \|G(A_i)\| \cdot \|x_i\| \\ &\leq \sum_i k\mu(A_i) \cdot \|x_i\| = k \sum_i \mu(A_i) \cdot \|x_i\| = k \int_S \|s\| d\mu. \end{aligned}$$

So (3.11) is true for every $s \in \mathcal{I}$. Now if $f \in \mathcal{M}$, let $s_n \in \mathcal{I}$ be such that $s_n \rightarrow f$ uniformly on S . As μ is finite we deduce that $\int_S \|f - s_n\| d\mu \rightarrow 0$ and so $\int_S \|s_n\| d\mu \rightarrow \int_S \|f\| d\mu$. But $\left\| \int_S s_n dG \right\| \rightarrow \left\| \int_S f dG \right\|$ by (3.4). From the estimation above we know that $\left\| \int_S s_n dG \right\| \leq k \int_S \|s_n\| d\mu$, for all n . Letting $n \rightarrow \infty$ the validity of (3.11) follows. Hence, the continuity of the operator $f \rightarrow \int_S f dG$ with the $L_1(\mu, X)$ -topology on the space \mathcal{M} . Next to prove (b), we shall construct an E -valued integration process on $L_1(\mu, X)$ with the set function G , that coincides with the integral (3.4) on \mathcal{M} . This will be the desired extension. Recall that the integral $\int_S s dG$, for s simple, has been defined by formula (3.2). Now if $f \in L_1(\mu, X)$, there exists a sequence $s_n \in \mathcal{I}$ such that $\int_S \|f - s_n\| d\mu \rightarrow 0$. By (3.11), the sequence $\int_S s_n dG$ is fundamental in the Banach space E , so the limit $\lim_n \int_S s_n dG$ exists in E and it is easy to check that this limit is independent of the choice of the sequence s_n converging to f in $L_1(\mu, X)$. So we can define:

$$f \in L_1(\mu, X), \quad \int_S f dG = \lim_n \int_S s_n dG, \quad (3.13)$$

where s_n is any sequence in \mathcal{I} converging to f in the $L_1(\mu, X)$ sense.

Now if f is a function in \mathcal{M} , every sequence $s_n \in \mathcal{I}$ which converges uniformly to f , converges also in the $L_1(\mu, X)$ sense. So the integrals (3.4) and (3.13) are the same for such f and this proves that (3.13) is an extension of (3.4). To see the inequality (3.12), let $s_n \in \mathcal{I}$ converging in $L_1(\mu, X)$ to the function $f \in L_1(\mu, X)$. By (3.11), we have $\left\| \int_S s_n dG \right\| \leq k \int_S \|s_n\| d\mu$, for all n . Taking limits for both sides we get (3.12) from which uniqueness of the extension follows. Part (c) is clear. \square

As a converse let us point out the following:

Theorem 3.10. *Let $T : L_1(\mu, X) \rightarrow E$ be a bounded operator from $L_1(\mu, X)$ to E . Then there exists a unique set function $G : \mathcal{F} \rightarrow \mathcal{L}(X, E)$ with finite semivariation satisfying (3.10), with the constant $k = \|T\|$ and such that:*

$$f \in L_1(\mu, X), \quad Tf = \int_S fdG. \quad (3.14)$$

Moreover, G is σ -additive in the uniform topology of $\mathcal{L}(X, E)$.

Proof. Define G on \mathcal{F} by the formula:

$$A \in \mathcal{F}, x \in X, \quad G(A) \cdot x = T(1_A(\bullet) \cdot x). \quad (3.15)$$

It is clear that $G(A)$ is linear on X for each $A \in \mathcal{F}$ and we have $\|G(A) \cdot x\| = \|T(1_A(\bullet) \cdot x)\| \leq \|T\| \cdot \mu(A) \cdot \|x\|$. So we deduce that the function G sends \mathcal{F} to $\mathcal{L}(X, E)$ and satisfies $\|G(A)\| \leq \|T\| \cdot \mu(A)$, hence the validity of (3.10) with $k = \|T\|$. On the other hand, (3.14) is easily checked from (3.15) for simple functions by linearity, and then extended to arbitrary $f \in L_1(\mu, X)$, by the appropriate limiting process. Finally to get the σ -additivity of G , let A_n be a sequence in \mathcal{F} with $A_n \searrow \phi$. Then $\mu(A_n) \rightarrow 0$ and since $\|G(A_n)\| \leq \|T\| \cdot \mu(A_n)$ for all n , we obtain $G(A_n) \rightarrow 0$ in the uniform topology of $\mathcal{L}(X, E)$, whence the σ -additivity of G . \square

Now we consider operators T in the class \mathfrak{D} . We prove that the operator valued function G attached to an operator $T \in \mathfrak{D}$, according to (3.15), allows an interesting characterization of such operators.

Theorem 3.16. *Let $T : L_1(\mu, X) \rightarrow X$ be a bounded operator on $L_1(\mu, X)$ into X . Then T is in the class \mathfrak{D} if and only if the operator valued function attached to it according to (3.15) is of the form:*

$$A \in \mathcal{F}, x \in X, \quad G(A) \cdot (\bullet) = \lambda(A) \cdot I(\bullet), \quad (3.17)$$

where λ is a bounded measure absolutely continuous with respect to μ , and I is identity operator of X .

Proof. If T is in the class \mathcal{D} , then by the corollary of theorem (2.8) there is a unique $g \in L_\infty(\mu)$ such that $T = T_g$, that is for all $f \in L_1(\mu, X)$, $Tf = \int_S fg d\mu$. On the other hand, we have from (3.14), $Tf = \int_S fdG$ with G given by (3.15). So taking $f = 1_A(\bullet) \cdot x$, for $A \in \mathcal{F}$, $x \in X$, in the two preceding expressions of Tf , we get $G(A) \cdot x = \left(\int_A g \cdot x d\mu \right) = \left(\int_A g d\mu \right) \cdot x$. Hence, the validity of (3.17) with $\lambda(A) = \int_A g d\mu$. Since μ is finite, the function g is in $L_1(\mu)$ and it is clear that λ is a bounded measure absolutely continuous with respect to μ . Now suppose that the operator valued function attached to T according to (3.15) is of the form: $G(A) \cdot x = \lambda(A) \cdot x$, with λ a bounded measure absolutely continuous with respect to μ . So we can write $\lambda(A) = \int_A g d\mu$, $A \in \mathcal{F}$, for some unique $g \in L_1(\mu)$. Actually the function g belongs to $L_\infty(\mu)$. Indeed by (3.15), $G(A) \cdot x = T(1_A(\bullet) \cdot x)$ and we deduce that

$$\|G(A) \cdot x\| = \left\| \left(\int_A g d\mu \right) \right\| \cdot \|x\| \leq \|T\| \|\mu(A)\| \|x\|,$$

which implies $\left\| \left(\int_A g d\mu \right) \right\| \leq \|T\| \|\mu(A)\|$, for all $A \in \mathcal{F}$. Consequently $\|g\|_\infty \leq \|T\|$, that is $g \in L_\infty(\mu)$. Now let us write the formula $G(A) \cdot x = \lambda(A) \cdot x$ as $\int_S 1_A \cdot x dG = \int_S g \cdot 1_A \cdot x d\mu$, and extend it by linearity to $\int_S s dG = \int_S g \cdot s d\mu$, for s simple in $L_1(\mu, X)$. If $f \in L_1(\mu, X)$, let s_n be a sequence of simple functions converging to f in $L_1(\mu, X)$. Then $g \cdot s_n$ converges to $g \cdot f$ in $L_1(\mu, X)$, since $g \in L_\infty(\mu)$, so we deduce that $\int_S g \cdot s_n d\mu$ goes to

$\int_S g \cdot f d\mu$. But $\int_S s_n dG = \int_S g \cdot s_n d\mu$, for all n and by (3.13), $\int_S f dG = \lim_n \int_S s_n dG$, consequently $\int_S g \cdot f d\mu = \int_S f dG$, for all $f \in L_1(\mu, X)$. But from (3.14) we have, $Tf = \int_S f dG$ for $f \in L_1(\mu, X)$, thus $Tf = \int_S g \cdot f d\mu = T_g f$, that is $T \in \mathcal{D}$. \square

Part 3

Bochner Integral in Locally Convex Spaces

Let X be a locally convex Hausdorff space, whose topology is generated by a family $\{p_\alpha\}$ of continuous seminorms. We assume that $\{p_\alpha\}$ is separating, this means that for each nonzero $x \in X$ there is a p_α such that $p_\alpha(x) \neq 0$. Moreover, we assume that X is sequentially complete, that is, every Cauchy sequence in X is convergent. For all details on such spaces, the reader is referred to [17], especially the sections 1.25; 1.36; 1.37 there. The construction of the Bochner integral we give in this context is, as far as we know, new (for other approaches see [1, 7, 18]). On the space $L_1(\mu, X)$ of Bochner integrable functions we define a family of separating seminorms that make this space locally convex. Finally we introduce a special class of bounded operators from $L_1(\mu, X)$ into X whose structure is, in many respects, similar to some well known operators from $L_1(\mu)$ into \mathbb{R} .

For the needs of measurability and integration, we fix an abstract measure space (S, \mathcal{F}, μ) , where \mathcal{F} is a σ -field on the set S and μ is a finite positive measure on \mathcal{F} .

1. Measurability

Definition 1.1. A function $f : S \rightarrow X$ is called *elementary* if its range $f(S)$ is finite.

If we put $f(S) = \{x_1, x_2, \dots, x_n\}$ and $A_j = \{s : f(s) = x_j\}$, then the sets A_j form a partition of S and we can write f in the consolidated form

$$f(\bullet) = \sum_{j=1}^n x_j 1_{A_j}(\bullet), \text{ where } 1_{A_j} \text{ is the characteristic function of the set } A_j.$$

Definition 1.2. An elementary function $f(\bullet) = \sum_{j=1}^n x_j 1_{A_j}(\bullet)$ is measurable if we have $A_j \in \mathcal{F}$ for every j . We denote by $\mathcal{E}(X)$ the set of all elementary measurable functions $f : S \rightarrow X$. Then we have:

Proposition 1.3. $\mathcal{E}(X)$ is a vector space on \mathbb{R} .

Proof. Let f, g be in $\mathcal{E}(X)$ and $\lambda \in \mathbb{R}$. Put $f(\bullet) = \sum_n x_n 1_{A_n}(\bullet)$, $g(\bullet) = \sum_m y_m 1_{B_m}(\bullet)$, then $(f + g)(\bullet) = \sum_{n,m} (x_n + y_m) 1_{A_n \cap B_m}(\bullet)$ and $(\lambda f)(\bullet) = \sum_n \lambda x_n 1_{A_n}(\bullet)$. □

Remark 1.4. Let T be any mapping from X into Y .

If $f(\bullet) = \sum_n x_n 1_{A_n}(\bullet)$, then $(T \circ f)(\bullet) = \sum_n T(x_n) 1_{A_n}(\bullet)$.

Definition 1.5. A function $f : S \rightarrow X$ is measurable if there is a sequence (f_n) of elementary measurable functions such that:

$$\lim_n p_\alpha(f_n - f) = 0$$

for each p_α .

This means that for each $s \in S$, each $\varepsilon > 0$, and each p_α , there is $N = N_{s, \varepsilon, p_\alpha} \geq 1$ such that $\forall n \geq N$, $p_\alpha(f_n(s) - f(s)) < \varepsilon$.

Proposition 1.6. The set $M(X)$ of all measurable functions $f : S \rightarrow X$ is a vector space on \mathbb{R} .

Proof. Let f, g be in $M(X)$ and let f_n, g_n be sequences of elementary functions such that $p_\alpha(f_n - f) \rightarrow 0$ and $p_\alpha(g_n - g) \rightarrow 0$, for each p_α . Then we have $p_\alpha((f_n + g_n) - (f + g)) \leq p_\alpha(f_n - f) + p_\alpha(g_n - g)$, so the sequence of elementary functions $f_n + g_n$ gives the measurability of $f + g$. Likewise for $\lambda \in \mathbb{R}$, we have $p_\alpha(\lambda f_n - \lambda f) = |\lambda| p_\alpha(f_n - f) \rightarrow 0$, which gives $\lambda f \in M(X)$. \square

2. Bochner Integration

Definition 2.1. Let $f(\bullet) = \sum_{j=1}^n x_j 1_{A_j}(\bullet)$ be an elementary measurable

function. We define the integral of f by the vector $\int_S f d\mu \in X$:

$$\int_S f d\mu = \sum_{j=1}^n \mu(A_j) \cdot x_j.$$

Since μ is finite, this integral is well defined.

Proposition 2.2. (a) *The integral is linear from $\mathcal{E}(X)$ into X .*

(b) *For every $f \in \mathcal{E}(X)$ and every p_α we have*

$$p_\alpha\left(\int_S f d\mu\right) \leq \int_S p_\alpha(f) d\mu,$$

where $p_\alpha(f)$ is the positive elementary function given by

$$p_\alpha(f)(\bullet) = \sum_{j=1}^n p_\alpha(x_j) 1_{A_j}(\bullet)$$

whose integral is $\int_S p_\alpha(f) d\mu = \sum_{j=1}^n p_\alpha(x_j) \mu(A_j)$.

Proof. (a) Put $f(\bullet) = \sum_{j=1}^n x_j 1_{A_j}(\bullet)$, $g(\bullet) = \sum_{k=1}^m y_k 1_{B_k}(\bullet)$, then
 $(f + g)(\bullet) = \sum_{j,k} (x_j + y_k) 1_{A_j \cap B_k}(\bullet)$ and $(\lambda f)(\bullet) = \sum_j \lambda x_j 1_{A_j}(\bullet)$.

This yields

$$\begin{aligned} & \int_S (f + g) d\mu = \sum_{1 \leq k \leq m} \sum_{1 \leq j \leq n} (x_j + y_k) \mu(A_j \cap B_k) \\ & = \sum_{j=1}^n \mu(A_j) \cdot x_j + \sum_{k=1}^m \mu(B_k) \cdot y_k = \int_S f d\mu + \int_S g d\mu. \end{aligned}$$

Likewise we can prove that $\int_S \lambda \cdot f d\mu = \lambda \cdot \int_S f d\mu$ for $\lambda \in \mathbb{R}$.

(b) We have

$$\begin{aligned} p_\alpha\left(\int_S f d\mu\right) &= p_\alpha\left(\sum_{j=1}^n \mu(A_j) \cdot x_j\right) \\ &\leq \sum_{j=1}^n \mu(A_j) \cdot p_\alpha(x_j) = \int_S p_\alpha(f) d\mu. \quad \square \end{aligned}$$

Proposition 2.3. Let $T : X \rightarrow Y$ be a linear operator from X into a locally convex space Y .

Let $f \in \mathcal{E}(X)$. Then $T \circ f \in \mathcal{E}(Y)$ and we have:

$$T\left(\int_S f d\mu\right) = \int_S T \circ f d\mu.$$

Proof. Let $f(\bullet) = \sum_{j=1}^n x_j 1_{A_j}(\bullet)$, with $\int_S f d\mu = \sum_{j=1}^n \mu(A_j) \cdot x_j$. Then

$$(T \circ f)(\bullet) = \sum_{j=1}^n T(x_j) 1_{A_j}(\bullet) \text{ and}$$

$$\int_S T \circ f d\mu = \sum_{j=1}^n \mu(A_j) \cdot T(x_j) = T \left(\sum_{j=1}^n \mu(A_j) \cdot x_j \right),$$

by the linearity of T , so we deduce that $T \left(\int_S f d\mu \right) = \int_S T \circ f d\mu$. \square

Definition 2.4. A measurable function $f : S \rightarrow X$ is Bochner integrable if there is a sequence f_n of elementary measurable functions such that for each p_α , $\text{Lim}_n p_\alpha(f_n - f) = 0$ uniformly on S . Since the measure μ is assumed finite, this implies that $\text{Lim}_n \int_S p_\alpha(f_n - f) d\mu = 0$, for each p_α .

To define the Bochner integral of f let us observe that if f_n is such a sequence of elementary functions we have:

$$\int_S p_\alpha(f_n - f_m) d\mu \leq \int_S p_\alpha(f_n - f) d\mu + \int_S p_\alpha(f_m - f) d\mu.$$

So $\text{Lim}_{n,m} \int_S p_\alpha(f_n - f_m) d\mu = 0$. But

$$p_\alpha \int_S (f_n - f_m) d\mu \leq \int_S p_\alpha(f_n - f_m) d\mu$$

by Proposition 2.2(b), this implies that the sequence of integrals $\int_S f_n d\mu$ is Cauchy. As the space X is assumed sequentially complete, $\int_S f_n d\mu$ converges. This allows to define the Bochner integral of f by the vector:

$$\int_S f d\mu = \text{Lim}_n \int_S f_n d\mu.$$

If g_n is another sequence of elementary functions such that $p_\alpha(g_n - f) \rightarrow 0$ uniformly on S , it is easy to check, from the continuity of p_α that $\text{Lim}_n \int_S f_n d\mu = \text{Lim}_n \int_S g_n d\mu$, so the Bochner integral $\int_S f d\mu$ is well defined.

In the sequel, we will denote by $L_1(\mu, X)$ the set of all Bochner integrable functions $f : S \rightarrow X$, where as usual two integrable functions are considered as identical if they are equal μ -almost everywhere.

Proposition 2.5. $L_1(\mu, X)$ is a vector space on \mathbb{R} and we have:

(a) The integral as defined is linear from $L_1(\mu, X)$ into X .

(b) For every $f \in L_1(\mu, X)$ and every p_α we have

$$p_\alpha\left(\int_S f d\mu\right) \leq \int_S p_\alpha(f) d\mu.$$

Proof. (a) Let f, g be in $L_1(\mu, X)$ and let f_n, g_n be in $\mathcal{E}(X)$ such that $p_\alpha(f_n - f) \rightarrow 0$ and $p_\alpha(g_n - g) \rightarrow 0$, uniformly on S . Since we have $p_\alpha((f + g) - (f_n + g_n)) \leq p_\alpha(f_n - f) + p_\alpha(g_n - g) \rightarrow 0$, it follows that $p_\alpha((f + g) - (f_n + g_n)) \rightarrow 0$ uniformly on S . This yields

$$\begin{aligned} \int_S (f + g) d\mu &= \lim_n \int_S (f_n + g_n) d\mu = \lim_n \int_S f_n d\mu + \lim_n \int_S g_n d\mu \\ &= \int_S f d\mu + \int_S g d\mu. \end{aligned}$$

Likewise we have $\int_S \lambda \cdot f d\mu = \lambda \cdot \int_S f d\mu$.

(b) Let f_n be in $\mathcal{E}(X)$ defining $\int_S f d\mu$. By Proposition 2.2(b)

$p_\alpha\left(\int_S f_n d\mu\right) \leq \int_S p_\alpha(f_n) d\mu$ for all n . This implies

$$\begin{aligned} p_\alpha\left(\int_S f d\mu\right) &= p_\alpha\left(\lim_n \int_S f_n d\mu\right) = \left(\lim_n p_\alpha\left(\int_S f_n d\mu\right)\right) \\ &\leq \liminf_n \int_S p_\alpha(f_n) d\mu \\ &\leq \liminf_n \left(\int_S p_\alpha(f_n - f) d\mu + \int_S p_\alpha(f) d\mu\right) \\ &= \int_S p_\alpha(f) d\mu. \end{aligned} \quad \square$$

Proposition 2.6. *Let $T : X \rightarrow Y$ be a linear continuous operator from X into a locally convex space Y .*

Let $f \in L_1(\mu, X)$. Then $T \circ f \in L_1(\mu, Y)$ and we have:

$$T\left(\int_S f d\mu\right) = \int_S T \circ f d\mu.$$

Proof. Let f_n be in $\mathcal{E}(X)$ defining $\int_S f d\mu$, i.e., $\lim_n p_\alpha(f_n - f) \rightarrow 0$ uniformly on S . By the continuity of T , if q is a seminorm on Y there is a seminorm p_α on X such that $q(Tx) \leq p_\alpha(x)$, for every $x \in X$. It follows that $q(Tf_n - Tf) = qT(f_n - f)d\mu \leq p_\alpha(f_n - f) \rightarrow 0$ uniformly on S . We deduce that $q(Tf_n - Tf) \rightarrow 0$ uniformly on S for each q . So the sequence Tf_n , which is in $\mathcal{E}(Y)$ by Proposition 2.3, is defining the integral of Tf by $\int_S Tf d\mu = \lim_n \int_S Tf_n d\mu$. By Proposition 2.3 once more we have $\int_S Tf_n d\mu = T \int_S f_n d\mu$ for all n .

Since $\lim_n \int_S f_n d\mu = \int_S f d\mu$, we get $\lim_n \int_S Tf_n d\mu = T\left(\int_S f d\mu\right)$, by the continuity of T . This gives $T\left(\int_S f d\mu\right) = \int_S T \circ f d\mu$. \square

3. Bounded Operators on $L_1(\mu, X)$

First we start by defining on $L_1(\mu, X)$ a family $\{\tilde{p}_\alpha\}$ of continuous seminorms which will make $L_1(\mu, X)$ a locally convex space.

Let us observe that for each p_α , we have $p_\alpha(f)$ bounded on S if $f \in L_1(\mu, X)$. To see this let f_n be in $\mathcal{E}(X)$ defining $\int_S f d\mu$, i.e., $\lim_n p_\alpha(f_n - f) = 0$ uniformly on S , (Definition 2.4), so if $\varepsilon > 0$, there is $N \geq 1$ such that $|p_\alpha(f) - p_\alpha(f_N)| \leq p_\alpha(f_N - f) < \varepsilon$ uniformly on S . We

deduce that $p_\alpha(f) < \varepsilon + p_\alpha(f_N)$ on S and $p_\alpha(f_N)$ is bounded on S since $f_N \in \mathcal{E}(X)$.

Now define \tilde{p}_α on $L_1(\mu, X)$ by:

$$f \in L_1(\mu, X) \quad \tilde{p}_\alpha(f) = \sup_{t \in S} p_\alpha(f(t)). \quad (3.1)$$

Then \tilde{p}_α is a seminorm on $L_1(\mu, X)$ and the family $\{\tilde{p}_\alpha\}$ is separating. To see this, let f be in $L_1(\mu, X)$ with $f \neq 0$, that is $f(t) \neq 0$ for some $t \in S$. Since the family $\{p_\alpha\}$ is assumed separating on X , there is a p_α such that $p_\alpha(f(t)) > 0$, so that $\tilde{p}_\alpha(f) > 0$.

Since the family of seminorms $\{\tilde{p}_\alpha\}$ is separating, it makes $L_1(\mu, X)$ a locally convex space such that each \tilde{p}_α is continuous ([16]; Section 1.37).

In what follows we define a special class of bounded operators from $L_1(\mu, X)$ into X which are, in many respects, similar to some well known operators from $L_1(\mu)$ into \mathbb{R} . First let us observe:

Lemma 3.2. *Let $g \in L_\infty(\mu)$. Then for every $f \in L_1(\mu, X)$, $g \cdot f \in L_1(\mu, X)$.*

Proof. Since $g \in L_\infty(\mu)$, there is a sequence (g_n) of simple measurable functions $g_n : S \rightarrow \mathbb{R}$ converging uniformly to g on S . Since $f \in L_1(\mu, X)$, there is a sequence f_n of elementary measurable functions such that for each p_α , $\lim_n p_\alpha(f_n - f) = 0$ uniformly on S . But $g_n \cdot f_n$ is elementary measurable, and we have:

$$\begin{aligned} p_\alpha(g_n \cdot f_n - g \cdot f) &= p_\alpha[(g_n \cdot f_n - g_n \cdot f) + (g_n \cdot f - g \cdot f)] \\ &\leq |g_n - g| \cdot p_\alpha(f) + |g_n| \cdot p_\alpha(f_n - f) \\ &\leq |g_n - g| \cdot \tilde{p}_\alpha(f) + |g_n| \cdot p_\alpha(f_n - f) \rightarrow 0, n \rightarrow \infty, \end{aligned}$$

uniformly on S .

Consequently we have $f \cdot g \in L_1(\mu, X)$. \square

Now we define a class $\{T_g, g \in L_\infty(\mu)\}$ of operators T_g , by the following recipe:

Definition 3.3. For each fixed $g \in L_\infty(\mu)$, T_g sends $L_1(\mu, X)$ into X by the formula:

$$f \in L_1(\mu, X), \quad T_g(f) = \int_S fg d\mu.$$

Theorem 3.4. Let $L_1(\mu, X)$ be endowed with the seminorms $\{\tilde{p}_\alpha\}$ given by (3.1), and let X be equipped with the seminorms $\{p_\alpha\}$. Then the operators T_g are linear and bounded.

Proof. The linearity is clear from 2.5(a). To see boundedness, let p_α be a seminorm on X , by 2.5(b) we have:

$$p_\alpha(T_g(f)) = p_\alpha\left(\int_S fg d\mu\right) \leq \int_S p_\alpha(fg) d\mu.$$

Since $p_\alpha(fg) = |g| p_\alpha(f)$, we deduce that $p_\alpha(T_g(f)) \leq \int_S |g| p_\alpha(f) d\mu \leq \|g\|_\infty \cdot \tilde{p}_\alpha(f) \cdot \mu(X)$, which proves that T_g is bounded. \square

In what follows, we quote some properties of the operators T_g , whose proof comes from facts about Bochner integral (2.5)-(2.6). We denote by E' the strong dual of the space E :

Proposition 3.5. (a) If $\theta \in X'$, then $\theta \circ T_g \in L'_1(\mu, X)$.

(b) If $\theta \in X'$, then $\theta \circ T_g(f) = \int_S g\theta f d\mu$, for every $f \in L_1(\mu, X)$.

(c) If θ, σ are in X' , and φ, ψ in $L_1(\mu, X)$, then:

$$\theta \circ \varphi = \sigma \circ \psi \Rightarrow \theta \circ T_g(\varphi) = \sigma \circ T_g(\psi).$$

These properties, especially property (c), lead to the following:

Open problem. Let $T : L_1(\mu, X) \rightarrow X$ be a linear bounded operator from $L_1(\mu, X)$ into X satisfying condition 3.5(c), that is:

If θ, σ are in X' , and φ, ψ in $L_1(\mu, X)$, then:

$$\theta \circ \varphi = \sigma \circ \psi \Rightarrow \theta \circ T(\varphi) = \sigma \circ T(\psi).$$

Does there exist a $g \in L_\infty(\mu)$ such that:

$$T(f) = \int_S fg d\mu, \text{ for all } f \in L_1(\mu, X).$$

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