About the piercing number of a family of intervals

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1 Introduction

Given a universe (a set) $\mathcal{U}$ and a property $\mathcal{P}$, (closed under inclusions, for subsets of $\mathcal{U}$). Results of the type “if every subset of cardinality $\mu$ of a finite family $\mathcal{F} \subset \mathcal{U}$ has property $\mathcal{P}$, then the entire family $\mathcal{F}$ has property $\mathcal{P}$’” are called Helly type theorems. The minimum number $\mu$ for which the result is true is called the Helly number of the Helly-type theorem $(\mathcal{U}, \mathcal{P}, \mu)$. In the case of Helly’s classical theorem, $\mathcal{U}$ is the family of convex sets in the euclidean space $\mathbb{R}^d$, the property $\mathcal{P}$ is to have a point in common and $\mu = d + 1$.

If the universe $\mathcal{U}$ is a family of sets, $\mathcal{F} \subset \mathcal{U}$ is called $n$–pierceable if there exists a set of $n$ points such that each member of the family contains at least one of the points. The minimum number $n$ for which the family is $n$–pierceable is called the piercing number and we will denote it by $\pi(\mathcal{F})$. Results of the type “if every

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subset of cardinality \( \mu \) of a finite family \( \mathcal{F} \subset \mathcal{U} \) is \( n \)-pierceable, then the entire family \( \mathcal{F} \) is \( n \)-pierceable\(^*\) are called Helly-Gallai type theorems. This type of theorems have been widely studied for different settings (see for instance surveys such as \([?], [?] \) ). It is well known, that finding such theorems is usually hard to find, in fact, Danzer and Grünbaum \([?]\) show that in general such theorems does not exist, even for the case of families of Boxes in \( \mathbb{R}^d \).

We are mainly interested in finding a Helly-Gallai type theorem for the universe of closed intervals in the \( d \)-dimensional euclidean space \( \mathbb{R}^d \). In this paper we prove that for every integer \( k \), there is an integer \( \alpha(k) \leq o(k^5) \) (only depending on \( k \) and not in the dimension \( d \)) such that if \( \mathcal{F} \) is a family of closed intervals in \( \mathbb{R}^d \), \( \pi(\mathcal{F}) \leq k \) if and only if for every subfamily \( \mathcal{F}' \subset \mathcal{F} \), with \( | \mathcal{F}' | \leq \alpha(k) \), \( \pi(\mathcal{F}') \leq k \).

Our strategy will be to obtain first the same results for a family of lines in \( \mathbb{R}^d \) and use this to obtain the similar results for families of closed intervals. Some of this results for lines are already known, see for instance \([?], [?] \), however we do not use this results directly.

On this paper we are interested also in the classical Hadwiger-Debrunner \((p,q)\)- problem \([?]\) which can be consider as another generalization of the Helly's theorem.

Given \( p \) and \( q \) integers with \( p \geq q \geq 2 \), \( \mathcal{F} \) is said to have the \((p,q)\)-property if \( \mathcal{F} \) contains at least \( p \) members and among every \( p \) members of \( \mathcal{F} \), some \( q \) have a common point. In general the \((p,q)\)-problem, for a family \( \mathcal{U} \) of sets, consists in determining or bounding the piercing number \( \pi(\mathcal{F}) \), in terms of \( p \) and \( q \), for every subfamily \( \mathcal{F} \subset \mathcal{U} \) with the \((p,q)\)-property. In the original Hadwiger-Debrunner theorem \([?]\), \( p \geq q \geq d + 1 \) and \( (d - 1)p < d(q - 1) \) where \( \mathcal{U} \) is a family of convex sets in \( \mathbb{R}^d \), and the "piercing number" is \( p - q + 1 \) been this number the smallest possible. On of the most important theorems in discrete geometry is the result of N. Alon and D. Kleitman, proving that the piercing number of a family of convex sets in \( \mathbb{R}^d \) with the \((p,q)\)-property is bounded by a function of \( p, q \) and \( d \).

Many work has been done around this type of problems, see for instance \([?]\) for an excellent survey. It can be seen, that exact results are rare, and in most of the cases one can do little more than proving that the piercing number of the family is bounded, and in some cases the focus is on finding good approximations.

In this paper, we prove that there is an integer \( \theta(p) \) (depending only on \( p \) and not in the dimension \( d \)) such that for families \( \mathcal{F} \) of closed intervals in \( \mathbb{R}^d \) satisfying the \((p,3)\)-property, then the piercing number \( \pi(\mathcal{F}) \) is smaller or equal than \( \theta(p) \). We are only interested in the existence of \( \theta(p) \).
2 About the piercing number of a family of lines

In this section we will study piercing numbers of families of lines in the \(d\)-dimensional euclidean space. We begin with some natural bounds for families with property \((p,3)\).

**Proposition 2.1** Let \(F\) be a family of lines in \(\mathbb{RP}^d\) satisfying the \((p,3)\)–property. Then there exists a function \(\psi(p)\) such that \(\pi(F) \leq \psi(p)\).

**Proof.** A point \(x\), in a line \(l \in F\) is called a "\(k\)-point" is there are \(k\) lines concurrent in \(x\). Let \(F'\) be a subfamily of \(F\) created by removing the minimum number of lines of \(F\) in such a way \(F'\) does not contain any 3-points. Thus \(F'\) is the subfamily of \(F\) such that \(F' \cup l\) contains at least one 3-point, for any line \(l \in F\). Since \(F\) satisfy property \((p,3)\) then clearly \(F'\) has at most \(p - 1\) lines, furthermore since every two lines intersects each other, then maximum number of 2-points in \(F'\) is \(\binom{p-1}{2}\). Then the maximality of \(F'\) yields that

\[
\pi(F) \leq \psi(p) = \binom{p-1}{2}.
\]

\[\blacksquare\]

Some partial results are known for families with \((p,3)\)-property, for instance Dolnikov show that \((4,3)\) implies \(\pi(F) = 2\) and \((5,3)\) implies \(\pi(F) = 3\). Some other results following this spirit can be found in [?]

**Conjecture 2.1** Let \(F\) be a family of lines in \(\mathbb{RP}^d\) satisfying the \((p,3)\) property. Then the piercing number of \(F\)

\[
\pi(F) = \left\lfloor \frac{p}{2} \right\rfloor.
\]

**Definition 2.1** Let \(F\) be a family of sets. We say that \(F\) is \(k\)-critical if \(\pi(F) = k\) but \(\pi(F \setminus l) < k\) for every \(l \in F\).

As we mention before, a Helly-Gallai type theorem should have the following general set up.

(1) There exists a function \(\varphi(n)\) such that if \(\pi(F') \leq n\) for every \(F' \subset F\) with \(|F'| \leq \varphi(n)\) then \(\pi(F) \leq n\).
The following proposition proves that in order to obtain a Helly-type theorem
or Helly-Gallai type theorem such as (1) for $\mathcal{F} \subset \mathcal{U}$ (where $\mathcal{U}$
is a family of sets) it is enough to show that the size of $k$-critical subfamilies of $\mathcal{U}$
in question, is bounded by a function of $k$.

**Proposition 2.2** Let $\mathcal{U}$ be a family of sets. If there exist $\psi(k)$ such that every
$k$-critical family $\mathcal{F} \subset \mathcal{U}$ satisfy $|\mathcal{F}| \leq \psi(k)$, then a Helly-type theorem (1) is
obtained for $\mathcal{F} \subset \mathcal{U}$ with $k = n + 1$.

**Proof.** Let $\varphi(n) = \psi(n + 1)$, suppose that (1) is false, then there exist a family
of convex sets that conform a counterexample. Consider $\mathcal{F}$ to be the minimum
counterexample possible, clearly $|\mathcal{F}| > \varphi(n)$ and $\pi(\mathcal{F}') \leq n$ for every $\mathcal{F}' \subset \mathcal{F}$
with $|\mathcal{F}'| \leq \varphi(n) = \psi(n + 1)$, but such that $\pi(\mathcal{F}) > n$. Then by the minimality
of $\mathcal{F}$, $\pi(\mathcal{F} \setminus A) \leq n$ thus $\pi(\mathcal{F}) = n + 1$ which implies that $\mathcal{F}$ is $(n + 1)$-critical,
then $|\mathcal{F}| \leq \psi(n + 1) = \varphi(n)$ which yields a contradiction.  

Let $k \geq 1$ be an integer and let $\mathcal{F}$ be a family of lines in $\mathbb{R}^d$. Then the
following proposition is true.

**Proposition 2.3** The cardinality of any $k$-critical family $\mathcal{F}$ of lines in $\mathbb{R}^d$ is
smaller than $k^2 - k + 1$.

**Proof.** Let $\mathcal{F}$ be a family of lines in $\mathbb{R}^d$, $k$-critical and let $\{x_1, ..., x_k\}$ be the set
of points that pierce $\mathcal{F}$, by the criticality of $\mathcal{F}$ it is possible to pierce $\mathcal{F} \setminus l$ with
$k - 1$ points for every $l \subset \mathcal{F}$. Then, observe that throughout every vertex $x_i$
there is a bundle say $\mathcal{L}_i := \{L^1_i, L^2_i, \ldots, L^k_i\}$ with at most $m \leq k$ different lines;
otherwise if throughout some $x_i$ are more than $k$ different lines, say $L^1_i, L^2_i \ldots L^m_i$
with $m > k$, then $L^1_i, L^2_i, \ldots L^{m-1}_i$ can not be pierced by $k - 1$ points since $x_i$ is
not a point piercing $\mathcal{F} \setminus l_m$ otherwise $\pi(\mathcal{F}) < k$.

Suppose that every bundle contains exactly $k$ different lines, then observe that
given any two bundles say, $\mathcal{L}_i$ and $\mathcal{L}_j$ there is one line $L^m_i \in \mathcal{L}_j$ such that $L^m_i = L^s_j$
for some $1 \leq s \leq k$. Suppose the contrary, then $\mathcal{L}_i$ and $\mathcal{L}_j$ do not share any line.
Then $\pi(\mathcal{F} \setminus L^k_i) = k - 1$, let $y_1, y_2, \ldots y_{k-1}$ be the set of points piercing $\mathcal{F} \setminus L^k_i$
then, clearly $x_i$ is not one of this points. Furthermore since all lines in $\mathcal{L}_i$ already
intersect at $x_i$ then each one of the $y_i, i = \{1, \ldots, k - 1\}$ intersect one and only one
of the lines in $\mathcal{L} \setminus L^k_i$ and since no line in $\mathcal{L}_i$ meet $x_j$ then $x_j$ is not in a piercing
points either. Thus $\mathcal{L}_j$ needs $k$ piercing points, contradicting the fact that $\mathcal{F}$ is
critical. Then $|\mathcal{F}| \leq k^2 - (k - 1) = k^2 - k + 1$. If the bundles have less than $k$
lines each then this number still valid.
This bound on the number of critical families of lines, will be very useful to find a Helly-Gallai type theorems, for the piercing number of families of closed intervals.

The following theorem, have been done by Grisha Chelnokov and Vladimir Dol’nikov in a more general set up for quasialgebraic families of sets see ?? and by Subramanya Bharadwaj B. V. et al [?] for families of (Pseudo)lines. However, in order to obtain a Helly-Gallai type theorem for piercing number of families of closed intervals we needed to obtain a bound for critical families of lines, which allow us to get this same theorem, as a corollary of Proposition ??.

**Theorem 2.2** If for every subfamily \( F' \subset F \) with \( |F'| \leq k^2+k+1 \) then \( \pi(F') \leq k \) then \( \pi(F) \leq k \).

### 3 About the piercing number of a family of closed intervals

**Theorem 3.1** Let \( F \) be a finite family of closed intervals in \( \mathbb{R}^d \) satisfying the \((p,3)\) property. Then the piercing number

\[
\pi(F) \leq \frac{p(p-1)(p-2)}{2}.
\]

**Proof.** Suppose first, that there exist open, pairwise disjoint intervals \( I_1, \ldots, I_m \), \( 1 \leq m < \infty \), such that every interval \( I \in F \), \( I \subset \bigcup_1 I_i \). Hence, in this situation, \( \pi(F) \leq p - 1 \). This is so, because in the line the piercing number of a family of closed intervals is the maximum number of pairwise disjoint intervals in the family. Take now a point \( x_0 \) and the subfamily \( F_{x_0} \) of all intervals in \( F \) that lies in a line through \( x_0 \), then \( \pi(F_{x_0}) \leq p \). This is so because we can pierce all the intervals containing \( x_0 \) with one point and all the other intervals of \( F_{x_0} \) with \( p - 1 \) points, due to the fact that all this intervals lie in a finite collection of open, pairwise disjoint intervals.

For every interval \( I \in F \), choose a line \( L_I \) containing \( I \), and let \( F_I = \{ L_I \mid I \in F \} \). Let \( F' \) be subfamily of \( F_I \) without triple points (points in three different lines of \( F' \)) and a maximal number of lines. Note that \( F' \) has at most \( p - 1 \) lines. Let \( X \) be the collection of points in two different lines of \( F' \). Note that \( X \) has at most \( \frac{(p-1)(p-2)}{2} \) points. For every \( x \in X \), consider the subfamily \( F_x \) of all intervals in \( F \).
that lies in a line through \( x \), then \( \pi(F_x) \leq p \). Moreover, note that \( \bigcup_{x \in X} F_x = F \), otherwise we contradict the maximality of \( F \). Therefore, we can pierce \( F \) with \( \frac{p(p-1)(p-2)}{2} \) points. 

**Conjecture 3.2** Let \( F \) be a family of closed intervals in \( \mathbb{R}^d \) satisfying the \( (p,3) \) property. Then the piercing number of \( F \), \( \pi(F) \leq p - 1 \).

**Theorem 3.3** For every integer \( k \geq 1 \), there is an integer \( a(k) \) with the property that if \( F \) is a family of closed intervals in \( \mathbb{R}^d \), then \( \pi(F') \leq k \) for every subfamily \( F' \subset F \mid \pi(F') \leq a(k) \). Then \( \pi(F') \leq k \).

**Proof.** We will start by analyzing a \( k \)-critical family \( F \) of closed intervals in \( \mathbb{R}^d \). Observe, that we may assume, that no element of \( F \) is a single point, because if a \( \{x\} \in F \) and \( F \) is \( k \)-critical, then \( F \setminus \{x\} \) is \( (k-1) \)-critical too. By Proposition ??, it will be enough to prove that a \( k \)-critical family of closed intervals in \( \mathbb{R}^d \) has at most \( \frac{1}{2}(k^2 - k + 1)(k^3 + 1) \) intervals.

For every non trivial closed interval \( I \in F \), let \( L_I \) be the line containing \( I \) and let \( F_I = \{L_I \mid I \in F\} \). As in the proof of Theorem ??, we know that there are at most \( k^2 - k + 1 \) of such lines. So, our next purpose, is to prove that if \( F \) is a \( k \)-critical collection of non trivial closed intervals in a line \( L \), then the number of elements of \( F \) containing \( L \) is smaller or equal than \( \frac{k^3 + 1}{2} \).

Suppose that \( L = \mathbb{R} \). Note that since \( F \) is \( k \)-critical, then no interval is totally contained in another, that is for every \( I,J \in F \cup L \), such that \( I \cap J \neq \emptyset \) then if \( I = [a_1, b_1] \) and \( J = [a_2, b_2] \) then \( a_1 < a_2 \leq b_1 < b_2 \). Let \( I_1 \) be the first interval to the left of \( L = \mathbb{R} \) then the set of intervals that intersect \( I_1 \) conform a "ladder" configuration \( \Omega_1 = \{I_1, \ldots, I_{m_1}\} \) where \( I_{i_1} = I_1, I_{i_i} = [a_1_i, b_1_i] \) and \( a_{i_1} < a_{i_2} < \cdots < a_{i_m} \leq b_{i_1} < b_{i_2} < \cdots < b_{i_m} \).

Let \( \tau_1 \) be the number of intervals of \( F \) transversal to \( \mathbb{R} \) at some point of \( I_1 \), we observe that \( m - 1 \) is smaller or equal to \( \tau_1 \).

For every \( i \in \{2, \ldots, m\} \) consider \( F \setminus I_{i_1} \). By hypothesis, \( \pi(F \setminus I_{i_1}) = k - 1 \). Let \( \{y_1, \ldots, y_{k-1}\} \) be \( k - 1 \) points piercing \( F \setminus I_{i_1} \). Note that \( \{y_1, \ldots, y_{k-1}\} \cap I_{i_1} = \emptyset \), otherwise \( \pi(F) = k - 1 \). Without loss of generality, we may assume that \( y_i \in I_{i_{i-1}} \setminus I_{i_1} \), and consider now \( \{y_1, \ldots, y_{i-1}, a_1, y_{i+1}, \ldots, y_{k-1}\} \). Clearly by the criticality of \( F \), \( \{y_1, \ldots, y_{i-1}, a_1, y_{i+1}, \ldots, y_{k-1}\} \) does not pierce \( F \). So, there must be and interval \( I_i \) of \( F \), which is not pierced by \( \{y_1, \ldots, y_{i-1}, a_1, y_{i+1}, \ldots, y_{k-1}\} \), but it is pierced by \( \{y_1, y_2, \ldots, y_{k-1}\} \). This implies that \( y_i \in I_i \) but \( a_1 \notin I_i \). If
I_i \subset \mathbb{R}, then I_i must be in \Omega_1, because y_i \in I_{i-1} \cap I_i. But this contradicts the fact that a_1 \not\in I_i. Then I_i is transversal to \mathbb{R} at y_i. Let I_i \in F_i be the line containing I_i. Note now that I_i \neq I_j for i \neq j, 2 \leq i, j \leq m. This implies that \mid \Omega_1 \mid = m \leq \tau_1 + 1.

Let J_{2} = [a_{21}, b_{21}] be the first interval to the left of \mathcal{F} \setminus \Omega_1 in \mathbb{R}. Then the set of intervals of \mathcal{F} \setminus \Omega_1 in \mathbb{R} that intersect J_2 have a "ladder" configuration \Omega_2 = \{I_{21}, I_{22}, \ldots, I_{2m_2}\}, with I_2 = [a_{21}, b_{21}] and a_{21} < a_{22} < \cdots < a_{2m_2} \leq b_{21} < b_{22} < \cdots < b_{2m_2}.

Next we observe that if \tau_2 is the number of intervals of \mathcal{F} transversal to \mathbb{R} at some point of I_2, then m_2 \leq \mid \Omega_1 \mid + \tau_2 + 1.

For every i \in \{2, \ldots, m_2\}, consider \mathcal{F} \setminus J_2, by hypothesis, \pi(\mathcal{F} \setminus J_2) = k - 1. Let \{z_1, \ldots, z_{k-1}\} be (k-1) points piercing \mathcal{F} \setminus J_2. Note that \{z_1, \ldots, z_{k-1}\} \cap I_{2i} = \emptyset, otherwise \pi(\mathcal{F}) = k - 1. Without loss of generality, we may assume that z_i \in I_{2i-1} \setminus J_2. Consider now \{z_1, \ldots, z_{i-1}, a_2, z_{i+1}, \ldots, z_{k-1}\}. Clearly \{z_1, \ldots, z_{i-1}, a_2, z_{i+1}, \ldots, z_{k-1}\} does not pierce \mathcal{F}. So, there must be an interval J_i of F which is not pierced by \{z_1, \ldots, z_{i-1}, a_2, z_{i+1}, \ldots, z_{k-1}\}, but it is pierced by \{z_1, z_2, \ldots, z_{k-1}\}. This implies that z_i \in J_i, but a_2 \not\in J_i. If J_i \subset \mathbb{R}, then J_i must be in \Omega_1, because if not, the fact that z_i \in I_2 \cap J_i implies that J_i \in \Omega_2, contradicting the fact that a_2 \not\in J_i. This implies that either J_i is transversal to \mathbb{R} through z_i, or J_i \in \Omega_1 and J_i \cap (I_{2i-1} \setminus J_2) \neq \emptyset. Therefore J_i \neq J_j for i \neq j, 2 \leq i, j \leq m_2. Then \mid \Omega_2 \mid = m_2 \leq \mid \Omega_1 \mid + \tau_2 + 1 \leq \tau_1 + \tau_2 + 2.

By repeating inductively this argument we obtain that I_{\lambda_1} the first interval to the left of \mathcal{F} \setminus \bigcup_{i=1}^{\lambda - 1} \Omega_i in \mathbb{R} and \Omega_\lambda the set of all intervals of \mathcal{F} \setminus \bigcup_{i=1}^{\lambda - 1} \Omega_i in \mathbb{R} that intersect \Omega_{\lambda_1}, with the property that \mid \Omega_\lambda \mid \leq \tau_\lambda + \sum_{i=1}^{\lambda - 1} \mid \Omega_i \mid \leq \sum_{i=1}^{\lambda} \tau_i + \lambda. Note that \{I_{\lambda_1}, I_{\lambda_2}, \ldots, I_{\lambda_1}\} is a pairwise non intersecting collection of closed intervals of \mathcal{F} and consequently \lambda \leq \pi(\mathcal{F}) = k. Finally, note that we may repeat our argument until \mathcal{F} = \bigcup_{i=1}^{n} \Omega_i, where \mid \Omega_j \mid \leq \sum_{i=1}^{j} \tau_i + j and n \leq k.

This implies that if \mathcal{F} is a k-critical family of non trivial closed intervals then every line contains at most \sum_{j=1}^{k} \mid \Omega_j \mid \leq \sum_{j=1}^{k} \sum_{i=1}^{j} (\tau_i + j) \leq \sum_{i=1}^{k} \sum_{j=1}^{i} \tau_i + \sum_{j=1}^{k} j intervals of \mathcal{F}. Since \sum_{j=1}^{k} t_j \leq \mid \mathcal{F} \mid - 1 \leq k^2 - 1, hence \sum_{j=1}^{k} \sum_{i=1}^{j} \tau_i \leq \frac{(k-1)(k^2-1)}{2} and consequently \sum_{j=1}^{k} \mid \Omega_j \mid \leq \frac{(k-1)(k^2-1)}{2} + \frac{k(k+1)}{2} \leq \frac{k^3+1}{2}. So, \mathcal{F} has at most \frac{k^3+1}{2}
intervals containing in $\mathbb{R}$.

This, together with the fact that $\mathcal{F}_t$ has at most $k^2 - k + 1$ lines implies that $\mathcal{F}$ contains at most $\frac{1}{2}(k^2 - k + 1)(k^3 + 1)$ intervals.

Finally applying proposition ?? to this bound we obtain that $\alpha(k)$ is bounded function depending only on $k$ and $\alpha(k) < o(k^5)$

References


