Multivariate generalized Laplace distribution and related random fields

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Multivariate Laplace distribution is an important stochastic model that accounts for asymmetry and heavier than Gaussian tails, while still ensuring the existence of the second moments. A Lévy process based on this multivariate infinitely divisible distribution is known as Laplace motion, and its marginal distributions are multivariate generalized Laplace laws. We review their basic properties and discuss a construction of a class of moving average vector processes driven by multivariate Laplace motion. These stochastic models extend to vector fields, which are multivariate both in the argument and the value. They provide an attractive alternative to those based on Gaussianity, in presence of asymmetry and heavy tails in empirical data. An example from engineering shows modeling potential of this construction.

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1. Introduction

The classical Laplace distribution has been introduced in [19] as an alternative to the Gaussian distribution. Being neglected for many years, this distribution has been recently revived and extended to skew as well as multivariate settings, and is gaining popularity as an attractive alternative to Gaussian models (see [17] and references therein). While the term multivariate Laplace law is still a bit ambiguous, it applies most often to the class of symmetric, elliptically contoured distributions, which is given by setting \( \mu = 0 \) in the following asymmetric generalization known as multivariate asymmetric Laplace (AL) distribution (see [17])

\[
\phi(t) = \frac{1}{1 + \frac{1}{2} t' \Sigma t - i \mu' t}, \quad t \in \mathbb{R}^d,
\]

where \( \mu \in \mathbb{R}^d \) is the mean of the distribution and \( \Sigma \) is a \( d \times d \) non-negative definite matrix. The significance of AL distributions, denoted by \( AL_d(\Sigma, \mu) \), is partially due to fact that these arise rather naturally as the only distributional limits for (appropriately normalized) random sums

\[
X^{(1)} + \cdots + X^{(N_p)}
\]

of independent and identically distributed (IID) random vectors \( X^{(i)} \) with finite second moments as \( p \) converges to zero, where the integer-valued random variable \( N_p \) is independent of the \( \{X^{(i)}\} \) and has a geometric distribution with mean \( 1/p \).
Since the sums such as (2) frequently appear in many applied problems in biology, economics, insurance mathematics, reliability, and other fields (see examples in [15] and references therein), AL distributions have a wide variety of applications; see [17]. The AL distributions play an analogous role among the heavy tailed geometric stable laws approximating sums (2) without the restriction of finite second moment (see [16]), as the Gaussian distributions do among the stable laws—they have finite moments of all orders, and their theory is elegant and straightforward. However, in spite of finiteness of moments, their tails are substantially longer than those of the Gaussian laws. This, coupled with the fact that they allow for asymmetry, renders them more flexible and attractive for modeling data featuring heavy tails and asymmetries.

The AL distributions are infinitely divisible, allowing for natural extension to more general random processes and fields. In this paper, we discuss properties of some constructions in the multivariate setting, based on the ideas from [1], where one-dimensional case was discussed, as well as from [29], where basic elements of extensions to multivariate settings were first laid down. Namely, we consider random moving average fields driven by Laplace motion, which are multivariate in both the argument and the value. The extensions beyond the Gaussian setup give more flexibility in modeling of asymmetric features not only in the distributional sense but also in the geometrical properties of the observed records, [29]. This is due to the fact that they are no longer uniquely defined by the covariance functions and a shape of the possibly asymmetric kernel function is becoming responsible for the model properties.

Multivariate generalizations of the Laplace motion constitute building blocks and play a crucial role in our construction of multivariate random fields. The Laplace motion on the positive half-line is a Lévy process built upon AL distribution (1). The increments of the process are independent and homogeneous, and the ChF is a power of AL ChF (1) so that the marginal distributions belong to multivariate generalized asymmetric Laplace (GAL) laws, or Bessel function distributions (the latter name relates to the fact that their PDFs involve Bessel special functions). We shall first briefly review their properties in Section 2, where also several new results are presented. Section 3 contains the main construction of random fields, their basic properties, and a discussion of model fitting and estimation. Finally, an example of application for modeling two-dimensional process in time variable constitutes Section 4. The example shows the modeling potentials and illustrates the several distinctive features of the bivariate processes, in particular, underlining the role of a proper choice of the asymmetric kernel.

2. Generalized Laplace distributions

Here we review basic properties of the generalized asymmetric Laplace distributions, which play a crucial role in constructing Laplace random fields. Although some of these results are known (and taken from [17]), others are new and presented here for the first time. We start with a formal definition of these laws.

Definition 1 (Multivariate Generalized Laplace Law). A random vector in \( \mathbb{R}^d \) is said to have a multivariate generalized asymmetric Laplace distribution (GAL) if its ChF is given by

\[
\phi(t) = \left( \frac{1}{1 + \frac{1}{2} t^\top \Sigma t - i \mu \cdot t} \right)^s, \quad t \in \mathbb{R}^d, \tag{3}
\]

where \( s > 0, \mu \in \mathbb{R}^d, \) and \( \Sigma \) is a \( d \times d \) non-negative definite symmetric matrix. This distribution is denoted by \( \text{GAL}_d(\Sigma, \mu, s) \).

Remark 1. If \( d = 1 \), we obtain a one-dimensional \( \text{GAL}(\sigma, \mu, s) = \text{GAL}_1((\sigma^2), \mu, s) \) distribution studied in [17].

If the matrix \( \Sigma \) is positive-definite, the distribution is truly \( d \)-dimensional and has a PDF of the form (see [17])

\[
p(x) = \frac{2 \exp(\mu^\top \Sigma^{-1} x)}{(2\pi)^{d/2} \Gamma(s)|\Sigma|^{1/2}} \left( \frac{Q(x)}{C(\Sigma, \mu)} \right)^{s-d/2} K_{s-d/2}(Q(x)C(\Sigma, \mu)), \tag{4}
\]

where \( K_{s}(\cdot) \) is the modified Bessel function with index \( \lambda \) (see [22]) and

\[
Q(x) = \sqrt{x^\top \Sigma^{-1} x}, \quad C(\Sigma, \mu) = \sqrt{2 + \mu^\top \Sigma^{-1} \mu}. \tag{5}
\]

This follows from the interpretation of a GAL random vector \( Y \sim \text{GAL}_d(\Sigma, \mu, s) \) as a subordinated Gaussian process,

\[
Y \overset{d}{=} X(Z), \tag{6}
\]

where \( Z \) has a standard gamma distribution with shape parameter \( s \), while \( X \) is a \( d \)-dimensional Gaussian process with independent increments, \( X(0) = 0, \) and \( X(1) \sim N_d(\mu, \Sigma) \) (\( d \)-dimensional normal distribution with mean vector \( \mu \) and covariance matrix \( \Sigma \)). Thus, we have

\[
Y \overset{d}{=} \mu Z + Z^{1/2} X, \tag{7}
\]

where \( Z \) is as above and \( X \sim N_d(0, \Sigma) \), showing that GAL distributions are location-scale mixtures of normal distributions. Stochastic representation (7) leads to many further properties of GAL random vectors, including moments, marginal and conditional distributions, and linear transformations.
2.1. Infinite divisibility

All GAL distributions are infinitely divisible, and their Lévy measure, presented below for the first time, can be obtained from representation (6) as a subordinated Brownian motion and Lemma 7, VI.2 of Bertoin [3].

**Proposition 1.** Let \( Y \) have a truly \( d \)-dimensional GAL\(_d\) (\( \Sigma, \mu, s \)) law. Then, the ChF of \( Y \) is of the form

\[
\Psi(t) = \exp \left( \int_{\mathbb{R}^d} (e^{i\langle t, x \rangle} - 1) \Lambda(dx) \right)
\]

with

\[
\frac{d \Lambda}{dx}(x) = \frac{2s \exp(\mu^\top \Sigma^{-1}x)}{(2\pi)^{d/2} |\Sigma|^{1/2}} \left( \frac{Q(x)}{C(\Sigma, \mu)} \right)^{-d/2} K_{d/2}(Q(x)C(\Sigma, \mu)),
\]

where \( Q(x) \) and \( C(\Sigma, \mu) \) are given by (5).

2.2. The mean vector and the covariance matrix

The relation between the mean vector, the covariance matrix, and the parameters \( \mu \) and \( \Sigma \) of \( Y \sim \text{GAL}_d(\Sigma, \mu, s) \) can be obtained from representation (7) in a straightforward way, leading to \( \mathbb{E}(Y) = ms \) and \( \text{Cov}(Y) = s(\Sigma + \mu\mu^\top) \). In particular, the covariance of an AL\(_d\)(\( \Sigma, \mu) \) distribution is \( \Sigma + \mu\mu^\top \), and reduces to \( \Sigma \) in the symmetric case \( \mu = 0 \).

2.3. Linear combinations and marginal distributions

The results presented below are new, and parallel to those connected with AL distributions discussed in [17]. First, we show that all linear combinations of the components of \( Y \sim \text{GAL}_d(\Sigma, \mu, s) \) are jointly GAL.

**Proposition 2.** Let \( Y = (Y_1, \ldots, Y_d) \sim \text{GAL}_d(\Sigma, \mu, s) \) and let \( A \) be an \( l \times d \) real matrix. Then, the random vector \( AY \) is \( \text{GAL}(\Sigma_A, \mu_A, s) \), where \( \mu_A = A\mu \) and \( \Sigma_A = A\Sigma A^\top \).

**Proof.** The assertion follows from the general relation

\[
\phi_{AY}(t) = \mathbb{E}e^{i\langle AY, t \rangle} = \mathbb{E}e^{i\langle YA^\top t \rangle} = \phi_Y(A^\top t)
\]

and the fact that the matrix \( A\Sigma A^\top \) is non-negative definite whenever \( \Sigma \) is. \( \square \)

In particular, it follows that all univariate and multivariate marginals as well as linear combinations of the components of a multivariate GAL vector are GAL.

**Corollary 1.** Let \( Y = (Y_1, \ldots, Y_d) \sim \text{GAL}_d(\Sigma, \mu, s) \), where \( \Sigma = [\sigma_{ij}]_{i,j=1}^d \). Then,

(i) for all \( n \leq d, (Y_1, \ldots, Y_n) \sim \text{GAL}_n(\tilde{\Sigma}, \tilde{\mu}, s) \), where \( \tilde{\mu} = (\mu_1, \ldots, \mu_n) \) and \( \tilde{\Sigma} \) is a \( n \times n \) matrix with \( \tilde{\sigma}_{ij} = \sigma_{ij} \) for \( i, j = 1, \ldots, n; \)

(ii) for any \( b = (b_1, \ldots, b_d) \in \mathbb{R}^d \), the random variable \( Y_b = \sum_{k=1}^d b_k Y_k \) is univariate GAL(\( \sigma, \mu, s \)) with \( \sigma = \sqrt{b^\top \Sigma b} \) and \( \mu = \mu^\top b \). Furthermore, if \( Y \) is symmetric (elliptically contoured) GAL, then \( Y_b \) is symmetric;

(iii) for all \( i \leq d, Y_i \sim \text{GAL}(\sigma, \mu, s) \) with \( \sigma = \sqrt{\sigma_i} \) and \( \mu = \mu_i \).

**Proof.** For part (i), apply Proposition 2 with \( n \times d \) matrix \( A = (a_{ij}) \) such that \( a_{ii} = 1 \) and \( a_{ij} = 0 \) for \( i \neq j \). For part (ii), apply Proposition 2 with \( l = 1 \) and compare the resulting ChF with (3). For part (iii) apply part (ii) with standard base vectors in \( \mathbb{R}^d \). \( \square \)

2.4. Polar representation

All GAL distributions with mean zero (\( \mu = 0 \)) are elliptically contoured, as their ChF depends on \( t \) only through the quadratic form \( t^\top \Sigma t \). With a non-singular \( \Sigma \), they are also elliptically symmetric, and admit a polar representation given below. This new result generalizes similar representation of asymmetric Laplace distributions discussed in [17].

**Proposition 3.** Let \( Y \sim \text{GAL}_d(\Sigma, 0, s) \), where \( |\Sigma| > 0 \). Then, \( Y \) admits the representation

\[
Y \sim \mathcal{R}\mathcal{H}U(d),
\]
where $H$ is a $d \times d$ matrix such that $HH' = \Sigma$, $U^{(d)}$ is a random vector uniformly distributed on the unit sphere $S_d$ of $\mathbb{R}^d$, and $R$ is a positive random variable, independent of $U^{(d)}$, with the density

$$f_R(x) = \frac{2x^{d/2+s-1}K_{d/2-s}((\sqrt{2})x)}{(\sqrt{2})^{d/2} \Gamma(s) \Gamma(d/2)}, \quad x > 0,$$

(9)

where $K_s$ is the modified Bessel function of the third kind.

**Proof.** Write $\Sigma = HH'$, where $H$ is a $d \times d$ non-singular lower triangular matrix (see, e.g., [11, p. 566], for a recipe of obtaining such matrix from a given non-singular $\Sigma$). Then, the random vector $X \sim N_d(0, \Sigma)$ from (7) has the representation $X = HN$, where $N \sim N_d(0, I)$. Further, $N$, which is elliptically contoured, has the well known representation $N \stackrel{d}{=} R_N U^{(d)}$. Here, $R_N$ and $U^{(d)}$ are independent, $U^{(d)}$ is uniformly distributed on $S_d$, and $R_N$ is positive with the PDF

$$f_{R_N}(x) = \frac{d \cdot x^{d/2-1} \exp(-x^2/2)}{2^{d/2} \Gamma(d/2 + 1)}, \quad x > 0.$$  

(10)

Therefore, in view of representation (7) with $\mu = 0$, it is sufficient to show that $Z^{1/2}R_N$ has density (9). To see this, apply standard conditioning argument and write the PDF of $Z^{1/2}R_N$ as

$$f_{Z^{1/2}R_N}(y) = dy \int_0^\infty \frac{x^{d/2-2} \exp\left(-\frac{1}{2} (x + 2y^2/x)\right)}{2^{d/2} \Gamma(s) \Gamma(d/2 + 1)} \left(\frac{y^2}{x}\right)^{s-1} dx.$$  

Let

$$f_{\lambda, \chi, \psi}(x) = \frac{(\psi/\chi)^{1/2}}{2K_s(\sqrt{x/\psi})} x^{s-1} e^{-1/2(x/\chi + \psi)}, \quad x > 0,$$

(12)

be a generalized inverse Gaussian density with $\psi = 1$, $\chi = 2y^2$, and $\lambda = d/2 - s$. Then, the above relation becomes

$$f_{Z^{1/2}R_N}(y) = \frac{2d \cdot y^{2s-1} K_s(\sqrt{2y})}{2^{d/2} \Gamma(s) \Gamma(d/2 + 1)(\chi/\psi)^{s/2}} \int_0^\infty f_{\lambda, \chi, \psi}(x) dx,$$

(13)

which, after some algebra, yields (9) since the function $f_{\lambda, \chi, \psi}$ integrates to one. \qed

2.5. Limits of random sums

Recall that multivariate asymmetric Laplace distributions are the only possible (weak) limiting distributions of (normalized) geometric random sums (2) as $p \to 0$ (and $N_p \mathcal{P} \sim \infty$); see [17]. A similar result holds true for the GAL distributions under negative binomial (NB) random summation. Let $N_{p,s}$ be an NB random variable with parameters $p \in (0, 1)$ and $s > 0$, so that

$$P(N_{p,s} = k) = \frac{\Gamma(s + k)}{\Gamma(s)k!} p^k(1-p)^s, \quad k = 0, 1, 2, \ldots.$$  

(14)

The following new result is an extension of Theorem 6.10.1 concerning AL distributions from [17] to the GAL case.

**Theorem 1.** Let $X^{(i)}$ be IID random vectors in $\mathbb{R}^d$ with mean vector $\mu$ and covariance matrix $\Sigma$. For $p \in (0, 1)$, let $N_{p,s}$ be a NB random variable (14), independent of the sequence $(X^{(i)})$. Then, as $p \to 0$,

$$a_p \sum_{j=1}^{N_{p,s}} (X^{(i)} + b_p) \rightarrow Y \sim GAL_d(\Sigma, \mu, s),$$  

(15)

where $a_p = p^{1/2}$ and $b_p = \mu(p^{1/2} - 1)$.

**Proof.** By the Cramér–Wald device, convergence (15) is equivalent to

$$c' a_p \sum_{j=1}^{N_{p,s}} (X^{(i)} + b_p) \rightarrow c' Y, \quad \text{as } p \to 0,$$

for all vectors $c$ in $\mathbb{R}^d$. Denoting $W_j = c'(X^{(i)} - \mu)$, $\mu = c'\mu$, $b_p = p^{1/2}\mu$, and $Y = c' Y$, we have

$$a_p \sum_{j=1}^{N_{p,s}} (W_j + b_p) \rightarrow Y \sim GAL(\sigma, \mu, s), \quad \text{as } p \to 0.$$  

(16)
Here, \((W_j)\) are IID with mean zero and variance \(\sigma^2 = c^2 \Sigma c\), and \(Y\) is a univariate GAL random variable. Writing (16) in terms of the ChFs we obtain
\[
\left( \frac{p}{1 - (1 - p)e^{\psi(t)p/2}} \right)^s \to \phi(t),
\]
where \(\psi\) is the ChF of the \((W_j)\). Note that (17) is equivalent to the convergence
\[
e^{-i\mu t} \frac{1 - (1 - p)\psi(p/2)}{p} = I + II \to 1 + \sigma^2 t^2/2 - i\mu t.
\]
It is easy to see that \(l \to -i\mu t\) as \(p \to 0\). To show the convergence
\[
I = \frac{1 - (1 - p)\psi(p/2)}{p} \to 1 + \sigma^2 t^2/2
\]
we use Theorem 8.44 from [9]: since \(W_j\) has the first two moments, its ChF can be written as 
\[
\psi(u) = 1 + iu\mathbb{E}W_j + (iu)^2(\mathbb{E}X_j^2 + \delta(u))/2,
\]
where \(\delta\) denotes a bounded function of \(u\) such that \(\lim_{u \to 0} \delta(u) = 0\). Since \(\mathbb{E}W_j = 0\) and \(\mathbb{E}W_j^2 = \sigma^2\), we apply the above with \(u = p^{1/2}/2\) to the lhs of (18) to obtain
\[
t^2/2(\sigma^2 + \delta(p^{1/2})t) + 1 - pt^2/2(\sigma^2 + \delta(p^{1/2})),
\]
which converges to \(1 + t^2\sigma^2/2\) as \(p \to 0\).

3. Moving average fields built upon generalized Laplace distributions

In this section, we give a short account of stochastic fields driven by Laplace motion. Since the main ideas are taken from [1], we focus mostly on these elements of the construction of Laplace moving averages (LMA) that differ from the one dimensional case; see also [29]. We discuss two constructions here, where the first one is based on one-dimensional models for each coordinate of the vector process, while the second one is intrinsically multivariate. The first model is used in Section 4, where it is applied to parallel road tracks records. The properties of these stochastic fields are discussed first, followed by an overview of available tools for model fitting and statistical inference.

3.1. Multivariate Laplace random measures

The main building block in our constructions is an independently scattered random measure, which has a generalized asymmetric multivariate Laplace distribution as the marginals. These measures, and their direct relation to multivariate Laplace motion, are discussed first. To build stochastic models for the generalized Laplace distribution it is convenient to use its infinite divisibility property. This property stands behind two important and related general concepts: an independently scattered random measure and Lévy motion.

**Definition 2 (Laplace Motion).** A vector valued Laplace motion \(\Lambda(t)\) in \(\mathbb{R}^d\) with parameters \((\Sigma, \mu, \nu)\) defined on a positive real line is a process with independent and homogeneous increments such that the increment over \(t\) and \(t + s\) has the multivariate GAL\(_d\)(\(\Sigma, \mu, s/\nu\)) distribution.

Such a process can be conveniently represented as a multivariate Brownian motion subordinated to a Gamma process. Namely, if \(A = \sqrt{\Sigma}\) (here a \(d \times d\) matrix \(\Sigma\) is assumed to be positive definite and the square root matrix \(\sqrt{\Sigma}\) is obtained by taking the square roots of the eigenvalues in the spectral representation of \(\Sigma\)), \(\mu \in \mathbb{R}^d\), \(B(t)\) is a \(d\)-dimensional Brownian motion, and \(\Gamma '(t)\) is a standard gamma process (so that \(\Gamma '(1)\) has the standard exponential distribution), then the process
\[
\Lambda(t) = AB(\Gamma '(t/\nu)) + \Gamma '(t/\nu)\mu
\]
is a Laplace motion, which follows directly from (6) and (7).

Every Lévy motion can be extended to the case of multidimensional argument through the concept of stochastic measure. In what follows, \(m(A)\) is the Lebesgue measure of a Borel-measurable set \(A \in \mathbb{R}^d\), and \(\mathcal{F}_d\) is the space of random vectors in \(\mathbb{R}^d\).

**Definition 3 (Stochastic Laplace Measure).** A random independently scattered measure \(\Lambda\) on \(\mathbb{R}^d\) with values in \(\mathcal{F}_d\) is called the Laplace measure, with parameters \(\nu > 0\), \(\mu \in \mathbb{R}^d\), a positive definite symmetric \(d \times d\) matrix \(\Sigma\) and controlled by a measure \(m\), if for \(A \subseteq \mathbb{R}^d\), \(m(A) < \infty\), the random value \(\Lambda(A) \sim GAL_d(\Sigma, \mu, m(A)/\nu)\).

The Laplace motion \(\Lambda\) can be identified with a stochastic measure through
\[
\Lambda([a, b]) = \Lambda(b) - \Lambda(a),
\]
which extends to an arbitrary Borel set and to the entire line by using an independent copy of Laplace motion on the negative half-line.
3.2. Vector valued Laplace moving averages

We start with the most general multivariate model and then discuss several of its specifications. The vector valued Laplace moving average $X$ with a vector of kernels $f = (f_1, \ldots, f_d)$ being square integrable real functions on $\mathbb{R}^n$, and the distributional parameters $\Sigma, \mu$ and $v$ is defined as

$$X(p) = \left( \int_{\mathbb{R}^n} f_1(s - p) \Lambda_1(ds), \ldots, \int_{\mathbb{R}^n} f_d(s - p) \Lambda_d(ds) \right),$$

where $\Lambda_i$ are the coordinates of a multivariate Laplace stochastic measure $\Lambda$ with values in $\mathcal{F}_d$.

The fundamental properties of this class of models are listed in the following result.

**Theorem 2.** The moving average process $X(p)$, $p \in \mathbb{R}^n$, defined by (20), is a stationary vector valued stochastic field with the following mean and covariance function:

$$[\mathbb{E}X(p)]_i = \int_{\mathbb{R}^n} f_i(s) \, ds \cdot \mu_i/v,$$

$$[\text{Cov}(X(p), X(0))]_{ij} = \int_{\mathbb{R}^n} f_i \ast f_j \cdot \sigma_{ij} + \mu_i\mu_j/v,$$

where $\ast$ is the convolution and $\tilde{f}(s) = f(-s)$.

The marginal distributions of $X(p)$ are given through the ChFs of $Y = \sum_{i=1}^n a_i X(p_i)$ for each $r \in \mathbb{N}$, $a_i \in \mathbb{R}$, and $p_i \in \mathbb{R}^n$.

$$\phi_r(u) = \exp \left( -\frac{1}{v} \int_{\mathbb{R}^n} \log \left( 1 - i u' g(p, u) + \frac{g(p, u)' \Sigma g(p, u)}{2} \right) \, dp \right),$$

where $g(p, u) = (\sum_{i=1}^n a_i f_i(p - p_i) u_1, \ldots, \sum_{i=1}^n a_i f_i(p - p_i) u_d)$ and all vectors are treated as column matrices.

Moreover, any vector process $X(p)$ obtained by subsetting the coordinates of $X(p)$ is again a moving average process with the corresponding subset of kernels and with respect to a Laplace measure with parameters $v, \tilde{\mu}$, and $\tilde{\Sigma}$, where the latter two are made of those entries in $\mu$ and $\Sigma$, indices of which were taken in $X(p)$.

**Proof.** By the standard extension argument it is enough to show the result for $f_k$ being simple function, $f_k(p) = \sum_{j=1}^m c_k A_k(p)$, where the $(A_j)$ are disjoint subsets of $\mathbb{R}^n$ and $I_A$ stands for the indicator function of a set $A$. By the definition of the stochastic integral,

$$Z = \left( \int_{\mathbb{R}^n} f_k(p - s) \, d\Lambda_k(s) \right)^d = \left( \sum_{j=1}^m c_k \Lambda_k(p - A_j) \right)^d,$$

which, for a fixed $p$, is a linear combination of the coordinates of independent generalized Laplace vectors. The ChF of such a vector is given by

$$\phi_Z(u) = \prod_{j=1}^m \left( 1 - i u' c_j(u) + \frac{c_j(u)' \Sigma c_j(u)}{2} \right)^{-m(A_j)/v},$$

where $c_j(u) = (c_{kj} u_k)^d$. This renders the characteristic function formula and, as a consequence, yields (strict) stationarity.

The mean and covariance of a multivariate GAL distribution as given in Definition 1 are $\mu/v$ and $(\Sigma + \mu \mu')/v$, respectively; see Section 2.2. The formula for the mean for kernels $f_i$ that are simple functions can be deduced from the linearity of the expectation but it also easily follows from the fact that coordinate-wise, both $X$ and $\Lambda$, are univariate Laplace moving averages that have been studied for example in [1]. The formula for the covariance follows from the independence of the increments for $\Lambda$. The extensions to arbitrary functions $f_i$ are standard. The fact that $Y_p$ is again a Laplace moving average process with the indicated parameters is owed to the fact that a vector made of some coordinates of a multivariate GAL distribution is again GAL (see Section 2.3).

In the following, we list some special subclasses of the vector valued Laplace moving averages. By a proper choice of the scale parameter for $\Lambda$, we may always assume without losing generality that the functions $f_i^2$ integrate to one.

3.2.1. Vectors of Laplace moving averages

By taking the same one dimensional Laplace stochastic measure $\Lambda$ for all coordinates in $\Lambda$ (the case $\Sigma$ having all entries equal to one) leads to a vector valued field

$$X(p) = \int_{\mathbb{R}^n} f(s - p) \, d\Lambda(ds) = \left( \int_{\mathbb{R}^n} f_1(s - p) \, d\Lambda(ds), \ldots, \int_{\mathbb{R}^n} f_d(s - p) \, d\Lambda(ds) \right).$$
Formulas for the ChFs and cross-correlations can be reduced as follows. Consider the $d \times r$ matrix $F(s) = (f_i(s - p))$ and an $r \times d$ matrix $u$. We denote $(F, u) = \sum_{i=1}^{d} (F_i, u_i)$, i.e. the sum of inner products between the rows of $F$ and the columns of $u$. If the parameters of $\Lambda$ are $\sigma, \mu$ and $\nu$, then we can write the ChF of $(X(p_1), \ldots, X(p_r))$ as follows:

$$
\phi(u) = \exp \left( -\frac{1}{\nu} \int_{\mathbb{R}^n} \log \left( 1 - i\mu \langle F(s), u \rangle + \frac{\sigma^2}{2} \langle F(s), u \rangle^2 \right) ds \right).
$$

(23)

The moments of the coordinates for this stochastic vector field are readily available as discussed in [1], as are the cross-moments

$$
\text{Corr}(X_i(t), X_j(0)) = f_i \ast f_j(t), \quad i, j = 1, \ldots, d,
$$

(24)

where $\ast$ is the convolution and $f_i(s) = f( -s )$, while the integration, if not shown otherwise in the notation, is always understood over $\mathbb{R}^n$ and with respect the Lebesgue measure. The model discussed here, together with the formula for the cross-correlation, is used in the application discussed in Section 4.

### 3.2.2. Vector of moving averages with different noise parameters

In the above model, the noise used for all coordinates of $X(p)$ had the same parameters and in the result the covariance structure depended only on the kernel $F$ and not on the noise. A natural extension is to consider different but dependent noise measures $\Lambda_i$ that are defined using the well known representation (see [25]) of stochastic Laplace measure

$$
\Lambda_i(A) = a_i \Gamma_i^1(A) - b_i \Gamma_i^2(A), \quad i = 1, \ldots, d,
$$

where $a_i = (\sqrt{2\sigma_i^2 + \mu_i^2 + \mu_i^2})/2, b_i = (\sqrt{2\sigma_i^2 + \mu_i^2 - \mu_i^2})/2$, and the independent Gamma measures $\Gamma_i^1$ and $\Gamma_i^2$ are controlled by the Lebesgue measure so the variance of $\Gamma_i^j(A)$ is equal to $m(A) / \nu$.

One can notice that the vector of one dimensional Laplace measures, $\Lambda(A) = (\Lambda_1(A), \ldots, \Lambda_d(A))$ constitutes a vector Laplace measure (in the sense of Definition 3) with the parameters $\nu, \mu$, and the symmetric positive definite matrix $\Sigma = ab' + ba'$, where $a = (a_1, \ldots, a_d)$ and $b = (b_1, \ldots, b_d)$ are treated as column matrices. Thus our general model has the following form

$$
X(p) = \left( \int_{\mathbb{R}^n} f_1(s - p) \Lambda_1(ds), \ldots, \int_{\mathbb{R}^n} f_d(s - p) \Lambda_d(ds) \right).
$$

The cross-correlation for this model now involves a distributional parameter. Namely, for $i, j = 1, \ldots, d$, we have

$$
\text{Corr}(X_i(t), X_j(0)) = \frac{(a_i a_j + b_i b_j)}{\sqrt{(a_i^2 + b_i^2)(a_j^2 + b_j^2)}} \cdot f_i \ast f_j(t).
$$

(25)

### 3.2.3. Single kernel case

The selection and estimation of kernels is a difficult task for non-Gaussian processes, as will be seen in Section 4. Therefore, we shall discuss simplified vector valued fields with a single kernel function.

For a Laplace measure $\Lambda$ with values in $\mathcal{F}_d$ and a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$, the following process will be called a moving average:

$$
X(p) = \int_{\mathbb{R}^n} f(s - p) d\Lambda(s).
$$

(26)

**Theorem 2** yields the following mean and covariance function:

$$
\text{EX}(p) = \int_{\mathbb{R}^n} f(s) ds \cdot \frac{\mu}{\nu}, \quad \text{Cov}(X(p), X(0)) = f \ast \tilde{f}(p) \cdot \frac{\Sigma + \mu'\mu}{\nu}.
$$

(27)

Moreover, the marginal distributions of $X(p)$ are given through the ChFs of $Y = \sum_{i=1}^{r} a_i X(p_i)$ for each $r \in \mathbb{N}, a_i \in \mathbb{R}$, and $p_i \in \mathbb{R}^n$.

$$
\phi_Y(u) = \exp \left( -\frac{1}{\nu} \int_{\mathbb{R}^n} \log \left( 1 - i\mu \cdot \mu' u + g^2(p) \cdot \frac{u'\Sigma u}{2} \right) dp \right),
$$

(28)

where $g(p) = \sum_{i=1}^{r} a_i f(p - p_i)$.

### 3.3. Model fitting and estimation

Estimation for the discussed models is a complex problem, and is currently a subject of active investigations. While a complete treatment of fitting methods would go beyond the scope of this paper, we provide an overview of main estimation strategies, their cons and pros as well as references to some recent developments.
All models discussed above have stochastics driven by a noise having generalized asymmetric Laplace distributions. Thus, the parameters \((\Sigma, \mu, \nu)\) of such a distribution are referred to as the distributional parameters of the model. It is also natural to consider an additional location parameter, say \(\delta \in \mathbb{R}^d\), which, for the sake of our discussion, is also treated as a distributional parameter. The remaining parameters are related to the kernels used in the moving averages, and so are more related to the structure of the model. For this reason, they are referred to as the model parameters.

The estimation of the parameters is a challenging problem even in the case of an independent sample from \(\text{GAL}_d(\delta, \Sigma, \mu, \nu)\) distribution. Despite the explicit likelihood for this case based on (4), maximum likelihood estimators (MLEs) are generally not known explicitly, with the notable exception of the one dimensional case with \(\nu = 1\), i.e. asymmetric Laplace distribution, for which their asymptotic covariance is well-known; see [17]. Therefore, one has to either resort to different estimation methods, like the method of moment estimators (MME), or apply numerical methods to effectively find the MLEs.

The problem of estimation for dependent data, which is the case for moving averages, is even more difficult, since the likelihood function is not given explicitly. Thus, the MLEs of the distributional parameters cannot be approached directly. However, for parametric kernels being Green functions of a linear differential equation, one can use the EM algorithm for computing the MLEs. This approach is the subject of an ongoing study; see [24]. Alternatively, one can resort to the method of moments estimation, to which we shall turn next.

### 3.3.1. Method of moments

Since the moments of the models are given in terms of the kernel functions and the distributional parameters \(\Theta\) (see for example [24] and Theorem 2), the MME can be adopted for any given kernel. The foundations of the MME have been laid down in [1,25] for the case of one dimensional Laplace distributions. The multivariate case was also considered in [28]. These methods seem to work quite well, and, for a certain range of parameter \(\nu\), are known to be comparable to the likelihood based methods; see [25]. However, they are hinged on an estimate of the kernel, i.e. the model parameters \(\Theta_M\). There is no obvious way to estimate \(\Theta_M\) in a general case. However, for symmetric kernels, non-parametric estimates are readily available and the MME becomes fairly straightforward.

For the purpose of the example of Section 4, below we discuss further details related to the MMEs in fitting two multivariate vector field models that were discussed in the previous section. For practical reasons, it is natural to assume that the models should have an additional location parameter \(\delta\). Thus, we shall consider \(\delta + X(\mathbf{p})\), for which from now on we use the same \(X(\mathbf{p})\).

Let us start with the models of Section 3.2.1. This model assumes a special structure of the cross-correlation, given by \(\mathbf{f} = \tilde{\mathbf{f}}\), where we assume that the integrals of the squared coordinates of \(\mathbf{f}\) are equal to one (normalized kernels) and \(\mathbf{f}(t) = f(-t)\). In a more general model of Section 3.2.2, the elements of this matrix of cross-correlations are scaled by \((a_i a_j + b_i b_j)/\sqrt{(a_i^2 + b_i^2)(a_j^2 + b_j^2)}\). As a result, the covariance of the \(i\)th coordinate is given by \(r_i = f_i \ast \tilde{f}_i\), which in the symmetric case (i.e. when \(\mathbf{f} = \tilde{\mathbf{f}}\)) leads to the identification of kernels from the covariances \(r_i\)’s through

\[
\hat{f}_i(\omega) = (2\pi)^{1/2} \mathcal{F}^{-1} \sqrt{s_i(\omega)}.
\]

Here, the quantity \(s_i(\omega)\) is the Fourier transform of the covariance \(r_i\), i.e. the spectrum of the \(i\)th coordinate of the process. Consequently, a non-parametric estimate \(\hat{f}_i\) can be obtained by substituting an estimate \(\hat{s}_i\) of the (two-sided) spectral density function \(s_i\), which is typically obtained as a smoothed Fourier transform of the sample covariance.

The coordinates of each of these models are one dimensional LMAs and the univariate estimation methods can be applied using the relation for the first–fourth moments as presented in [25]:

\[
\begin{align*}
\mathbb{E} X_i &= \delta_i + (\mu_i / \nu) \cdot \int f_i, \\
\mathbb{E} (X_i - \mathbb{E} X_i)^2 &= (\mu_i^2 + \sigma_i^2) / \nu, \\
\mathbb{E} (X_i - \mathbb{E} X_i)^3 &= \mu_i (2\mu_i^2 + 3\sigma_i^2) \cdot \int f_i^3 / \nu, \\
\mathbb{E} (X_i - \mathbb{E} X_i)^4 &= 3 (\mu_i^2 + \sigma_i^2)^2 / \nu^2 + 3 (2\mu_i^4 + 4\mu_i^2 \sigma_i^2 + \sigma_i^4) \cdot \int f_i^4 / \nu.
\end{align*}
\]

These equations lead to estimates of \(\delta_i, \mu_i, \sigma_i, \nu\) based on sample moments of the observed data, provided that the estimates of kernels \(\hat{f}_i\) are used to obtain the integrals listed above. See [25] for the details. It should be noted that the procedure described there gives a different \(\nu\) for each coordinate, which is not possible within the considered class of models. Thus, the average of the so obtained \(\nu_i\) can be taken as a common value \(\nu\).

In the model discussed above, once the kernels and distributional parameters are estimated, the cross-correlation is automatically determined. Model (26) is in this respect different: since it uses a single kernel, the cross-correlations are essentially controlled by the off-diagonal terms in \(\Sigma\), allowing for some cross-correlation fitting to data. Indeed, assuming that the correlations for each coordinate are the same, we can estimate \(\tilde{f}\) by averaging \(\tilde{f}_i, i = 1, \ldots, d\), obtained from the estimated spectra by means of (29). Then, we can use (30), with either individual \(\tilde{f}_i\) or the averaged common \(\tilde{f}\), to get the...
MMEs $\hat{v}, \hat{\delta}_i, \hat{\mu}_i$ and $\hat{\sigma}_i$, $i = 1, \ldots, d$. Now, the model is capable of fitting some cross-covariance structure by estimation of the $(\sigma_{ij})$ (for $i \neq j$) through (21):

$$\hat{\sigma}_{ij} = \hat{v} \cdot \frac{\int_{\mathbb{R}^n} \text{Cov} (X_i(p), X_j(0)) \cdot w(p) \, dp - \hat{\mu}_i \hat{\mu}_j}{\hat{f} \ast \hat{f}(p)}.$$  

Here, $\text{Cov}(X_i(p), X_j(0))$ is the standard non-parametric estimate of the cross-covariance, $w$ is a density function over $\mathbb{R}^n$ representing an appropriate weight to account for different accuracy of this non-parametric estimate at various values of $p$. For example, one choice of $w(p)$ is taking it inversely proportional to the standard deviation of $\text{Cov}(X_i(p), X_j(0))$.

It is clear how one could proceed to adopt the above estimation technique to the most general model (20), in which different kernels are used. We skip the technical details of such a procedure.

We conclude the MMEs section by noting that the above procedures work only when estimates of the kernels are available (like for symmetric kernels). However, for asymmetric kernels there may be no obvious choice for an estimator even in the case when the kernels are fully parameterized. In [29], moments based on the joint distribution of the process and its derivative were shown to be able to estimate the asymmetry parameter for certain families of kernels. Accounting for kernel’s asymmetry is an important issue, since in contrast with the Gaussian case, asymmetry may greatly affect the properties observed in the data even if it does not change covariances in the process. We discuss the importance of asymmetry in Section 4, where we present an illustrative example based on the models of Section MLMA.

### 3.3.2. Maximum likelihood and EM algorithm

Improving efficiency is the main reason behind pursuing the maximum likelihood estimators (MLEs). Additionally, for certain models with parameterized asymmetric kernels, the EM algorithm can be conveniently adopted to find the MLEs of both the model and distributional parameters. In numerical studies, it was observed that for certain ranges of parameters they dramatically outperform the MMEs.

Since the likelihood is expressed by the Bessel function, any direct optimization method would be slow. However, due to the special representation (7), one can implement the expectation–maximization (EM) algorithm while treating the unobservable random variance $Z$ as a missing value and exploiting the fact that this random variable given the observed value $Y$ has the generalized inverse Gaussian distribution. This method proved to efficiently produce the MLEs for some classes of distributions related to generalized Laplace ones; see [2,14,20]. The method can be adopted for a wide range of linear models involving the generalized multivariate Laplace laws. This topic is currently under further investigation and their results will be the subject of some separate work; see [7,24].

### 4. An illustration: parallel road tracks roughness

Modeling roads’ profiles is an important area of transportation engineering as durability studies of vehicle components often require a customer or market specific load description. One often assumes here that road roughness is homogeneous for sufficiently long distance, so that the loads caused by changes in roughness properties can be neglected in fatigue life predictions. The most desired properties of the models are robustness and simplicity, so that only a small number of parameters is used to describe short homogeneous parts of the road. For example, the ISO 8608 standards recommend to fit a single parameter spectrum $S(\omega) = C \omega^{-2}$, where $C$ is a measure of road quality.

Although models based on Gaussian distribution are standard in the field (see, e.g., [27] and also [21] for some recent studies), most experts of vehicle engineering agree that road surfaces are not really accurately represented by a Gaussian process; see [12]. The reason is that the actual roads contain short sections with above-average irregularity. As shown in [4], such irregularities cause most of the vehicle fatigue damage.

In [6], two models have been proposed for a single track case. The first one involved a gamma-distributed $C$, leading to a non-homogeneous LMA process. While this model fitted single track data reasonably well, the second model, involving a homogeneous LMA process, appeared to represent the road roughness observed in real data quite well. Consequently, only one additional parameter (excess kurtosis) besides the power spectral density is needed to describe the road surface roughness.

Accounting for just a single path along the road is an oversimplification, as any four-wheeled vehicle is subjected to excitations due to road roughness in the left as well as the right wheel paths. Accordingly, accounting for both paths should be an important aspect of heavy vehicle fatigue assessment. Hence, it is natural to propose a bivariate stochastic model corresponding to parallel road tracks. Such a model is discussed in this section. We demonstrate that a bivariate version of the LMA used for parallel tracks modeling provides a fairly accurate statistical description of road surface irregularities. Our model is a multivariate extension of the homogeneous LMA model for a single track proposed in [6], which represented the road roughness observed in real data quite well.

#### 4.1. Description of the model

Let $Z_l(x)$ and $Z_r(x)$ denote the right and the left track elevations, respectively, at the location $x$. We assume a homogeneous road section, and require that the right and the left tracks have the same distribution. We suppose these two processes are of the second order, and start with the description of the relevant spectra.
4.1. Second order properties

Although a simple parametric spectral density may not accurately approximate the road roughness for the whole range of wave numbers, it is important that it correctly estimates the energy in the range which may excite the vehicle response. While the industry’s standard is the MIRA spectrum (see [12,18]), our example involves the Matérn spectra,

\[ S(\omega) = \frac{1}{(c^2 + (\omega/\omega_0)^2)^{\nu/2}} + \frac{1}{(c^2 + (\omega/\omega_0)^2)^{\nu/2}}. \]  

which are analytically more trackable. This spectrum, with a proper choice of parameters, gives similar values to the MIRA spectrum used in [6] for the range of frequencies that are important for a traveling vehicle. The exponent \( \nu \) describes energy distribution for wavelengths between 100 and 5 m, while \( \nu_2 \), with wavelengths between 5 and 0.1 m, describes the state of road deterioration. For simplicity, spectrum (32) is normalized so that the variances of \( Z_R(x) \) and \( Z_L(x) \) are one.

Upon defining the covariance for each track through Fourier transform of spectrum (32), we consider the cross-covariance

\[ r_{LR}(\tau) = E[Z_L(x + \tau) \cdot Z_R(x)], \]

which is also defined through the cross-spectrum,

\[ S_{LR}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} r_{LR}(\tau) e^{-i\omega \tau} d\tau, \]

and let

\[ K(\omega) = \frac{S_{LR}(\omega)}{S(\omega)}. \]  

As shown in [5], the function \( K(\omega) = \exp(-\rho |\omega|) \) describes the correlation between the tracks in many measured signals rather well, where the values of \( \rho \) for the road data are typically in the interval [1, 7].

4.1.2. Gaussian model

Knowledge of the spectrum and cross-spectrum is sufficient to define a bivariate Gaussian model for road tracks. This can be done in several ways. Following [8], let us define three kernel functions, \( f_R, f_L \) and \( f_{LR} \), by the relations

\[ (\mathcal{F} f_R)(\omega) = \sqrt{S(\omega)}, \]  

\[ (\mathcal{F} f_L)(\omega) = \sqrt{S(\omega) \cdot 1 - |K(\omega)|^2}, \]  

\[ (\mathcal{F} f_{LR})(\omega) = \sqrt{S(\omega) \cdot K(\omega)}, \]

where \( \mathcal{F} \) stands for Fourier transform, and consider a trivariate Gaussian moving average (GMA) model \( X = (X_1, X_2, X_3) \) defined as

\[ X(x) = \left( \int f_R(x - u) \, dB_R(u), \int f_L(x - u) \, dB_L(u), \int f_{LR}(x - u) \, dB_{LR}(u) \right). \]  

Here, \( B_R(x) \) and \( B_L(x) \) are independent Brownian motions. Then, the GMA model for the two tracks roughness is \( Z_L = X_1, Z_R = X_2 + X_3 \). It is clear from the construction that the tracks \( Z_L \) and \( Z_R \) have the same spectrum, and thus their (Gaussian) distributions are the same as well.

4.1.3. Laplace moving averaged model

We propose to replace the GMA model (37) by a symmetric LMA four-variate model. To this end, we consider the four dimensional stochastic Laplace measure \( \Lambda \) with \( \mu = 0 \), scale matrix

\[ \Sigma = \sigma^2 \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \]

and shape parameter \( \nu \). This multivariate stochastic measure can be written as \( \Lambda = (\Lambda_R, \Lambda_L, A_R, A_L) \), where \( A_L(u) = B_L(\Gamma(u)) \) and \( A_R(u) = B_R(\Gamma(u)) \). Our goal is to define a stochastic model for the road tracks that mimics the GMA model. However, in order to obtain the same distributional properties for each track, here we cannot simply take a trivariate model with the same vector of kernels \( (f_R, f_{LR}, f_L) \) as above and consider a vector-valued Laplace moving average by replacing the Gaussian measure in (37) by the Laplace one. This is in sharp contrast with the Gaussian case, as it would lead to the same spectra for each track but not the same distribution of the processes, except when the tracks are uncorrelated or identical, \( K(\omega) = 0, 1 \), respectively.

For simplicity of the presentation, let us denote by \( f_0 \) the kernel for uncorrelated tracks, where \( \mathcal{F} f_0 = \sqrt{S(\omega)} \), and let \( K(\omega) \) be a real function. By considering the four-variate moving average model \( X \) given by (20) with \( \Sigma \) as above and the
kernels \( f = (f_1, f_2, f_3, f_4) \), where \( f_1 = f_0 * \tilde{f}_1, f_2 = f_0 * \tilde{f}_2 \), and
\[
\begin{align*}
(F \tilde{f}_1)(\omega) &= \left( \sqrt{1 + K(\omega)} + \sqrt{1 - K(\omega)} \right) / 2, \\
(F \tilde{f}_2)(\omega) &= \left( \sqrt{1 + K(\omega)} - \sqrt{1 - K(\omega)} \right) / 2,
\end{align*}
\]

one can define the left and the right-hand-side tracks through
\[
\begin{align*}
Z_L &= X_1 + X_2, \\
Z_R &= X_3 + X_4.
\end{align*}
\]

Note that under this construction not only the covariances (and thus the spectra) of \( Z_R \) and \( Z_L \) coincide, but also their distributions are the same as well. This is a result of the symmetry of the construction. However, these distributions depend on \( K(\omega) \) despite the marginal covariances do not.

4.2.2. Distributional parameters

of the simulated road-surface records. The choice of kernel technique introduced in [30]; see also more recent work [10]. The choice of \( p = 0.6 \) was chosen simply by a visual inspection of the simulated road surface records.

4.2.1. Model parameters

The model parameters involve \( w_1, w_2, c, \omega_0, \rho \) and \( p \), which appear in the definition of the kernels through their spectral representations. Their fitting is based on the methods that are to great extent part of the industry standards. First, there is no interest in very small frequencies, so all spectra are only considered for \( \omega \geq 0.02\pi \). The signal is typically divided into wavelengths between 100 and 5 m and those between 5 and 0.1 m, the latter describing the state of road deterioration. From this the value \( \omega_0 = 2\pi/5 = 0.4\pi \) enters the model. Next, the values \( w_1 = 3.61 \) and \( w_2 = 1.63 \) are based on the fit to the MIRA spectrum, which is defined separately for each of these two ranges of frequencies. This procedure is essentially based on fitting the ratio of the zeroth and first spectral moments of the corresponding portions of empirical spectrum, see [6].

Finally, the constant \( c = 0.071 \) for the spectrum given in (32) is chosen so that its graph closely approximates the MIRA spectrum over the range of frequencies that are of interest in road applications. The corresponding graph is shown in Fig. 1 (Right Panel), where the non-parametric spectrum is compared with the MIRA and Matérn spectra.

In order to completely define the model, the parameter \( \rho \) for the cross-spectrum and the asymmetry parameter \( p \) of the kernel \( f_{R,1} \) have to be specified. The value \( \rho = 1.6 \) has been taken from Scania database, which employs the so called WOSA technique introduced in [30]; see also more recent work [10]. The choice of \( p = 0.6 \) was chosen simply by a visual inspection of the simulated road surface records.

4.2.2. Distributional parameters

Since we assume symmetry of the distribution, and the Laplace noises used in defining \( Z_1 \) and \( Z_2 \) have the same distribution, our model, after normalizing for scaling and location, has essentially one distributional parameter \( \nu \) associated with the generalized Laplace distribution. Given a kernel \( f \) and the empirical excess kurtosis \( \kappa \), the parameter \( \nu \) is computed
using the relation

\[ \kappa = 3\nu \int f^4(x) \, dx, \]

which is valid for the LMA model defined in (26). In the estimation, we used kernel \( f_1 \) only, as it dominates over \( f_2 \). Accounting for both spectra is possible, but would require a more complex formula for the joint moments of \( Z_1 \) and \( Z_2 \). The estimates of skewness and kurtosis, based on the 5 km long measured signals, are 0.15 and 5.02, respectively, justifying the symmetry assumption and yielding the Laplace noise parameter \( \nu = 13.7 \).

4.2.3. A comparison of Laplace and Gaussian-based models

Here, we compare how our model fares against the one based on the assumption of Gaussianity, which is typically used by the industry. Fig. 2 (Left Panel) presents 1 km of simulated LMA tracks. In this type of applications, synthetic and measured roads are equivalent if they induce the same amount of vehicle fatigue damage. Fig. 2 (Right Panel) shows the accumulated damages in 20 simulated LMA and Gaussian road tracks. The damages are normalized so that the value one is assigned when the simulated damage is equal to the one observed in the measured signals. Note that all twenty damages computed for Gaussian road surface models are below 80% of the observed damage, which has the value one. The average damage is 0.45, with an estimated standard deviation of 0.12. Thus, according to the central limit theorem for the damage functional
Fig. 3. (Left) One kilometer simulated parallel tracks LMA with a symmetrical kernel. (Right) The observed number of up-crossings in 5 km measured road surface (dotted line), simulated number of up-crossings in LMA process (solid line) and in the Gaussian process (dashed dotted line).

(40) proved in [23], the observed damage is about five standard deviations above that predicted by the Gaussian road model. In contrast, the damages computed under the LMA model, are spread around the value of one. Clearly, the Gaussian model is severely underestimating the damage, while the LMA model gives quite accurate predictions.

Let us comment on why we are not using an LMA processes with symmetrical kernel to model the tracks. We have noted before that our choice \( p = 0.6 \) appears to be close to the symmetric case \( (p = 0.5) \). However, looking only at the value of \( p \) is misleading. We repeated our numerical analysis with the symmetric kernel, with the results presented in Fig. 3. Note that the estimated damage for LMA with symmetric kernels is also closed to the observed one. However, the high irregularity of the signals makes the symmetrical LMA non-physical as a model of road surfaces, see Fig. 2 (Left Panel). This observation underlines the importance of kernel asymmetries in proper modeling of real data. In particular, there is a need for efficient estimation methods dealing with asymmetry parameters of the kernels.

Finally, for the convenience of the reader, we give a definition of the damage. For a symmetric zero mean response \( Y(t) \), a simple damage accumulation model proposed in [13] (see also [26]) states that during a period of time length \( T \) the damage increment is

\[
\Delta D_T = \kappa \beta^2 \int_0^T (Y(t)^+)^{\beta-1}Y(t)^+ \, dt, \quad x^+ = \max(0, x).
\]

A failure corresponds to the damage exceeding the threshold of one. Here, \( \kappa \) and \( \beta \) are treated as deterministic material dependent constants. For vehicle components, \( \beta \) is usually in the range 3–8, making it particularly important to accurately describe the right tail of \( x(t) \). Note that for \( \beta = 1 \) the damage is the total variation of the response, i.e. depends only on the derivative.

4.3. Conclusions

In this preliminary effort to model road track by a bivariate Laplace moving average process, we have found that it is possible to tune in the parameters of the model to account for many special features of the real records. Most importantly, it is possible to obtain a damage estimate that is more accurate than that predicted by Gaussian-based models that are used as a standard in the field. Our findings for parallel tracks complement extensive comparisons of different approaches to model single track road irregularities as presented in [6], which identified the LMA approach as at least as good, or better, than other methods considered. The most common approach to model two parallel tracks employs isotropy assumption for the road surface, which is known to be incorrect. The first work using the non-isotropic model is done in [4]. In [6], we had compared the Bogsjö model with the multivariate Laplace one and found them basically equivalent, but the Laplace LMA model is simpler, with its relatively few parameters. Reliable and efficient fitting procedures for this model are currently under investigation, and will be reported elsewhere.

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