On generalized Newton method for solving operator inclusions

D.R. Sahu, Krishna Kumar Singh
Department of Mathematics, Faculty of Science, Banaras Hindu University, Varanasi-221005, India

Abstract. In this paper, we study the existence and uniqueness theorem for solving the generalized operator equation of the form
\[ F(x) + G(x) + T(x) \ni 0, \]
where \( F \) is a Fréchet differentiable operator, \( G \) is a maximal monotone operator and \( T \) is a Lipschitzian operator defined on an open convex subset of a Hilbert space. Our results are improvements upon corresponding results of Uko [Generalized equations and the generalized Newton method, Math. Programming 73 (1996) 251-268].

1. Introduction

Let \( H \) be a real Hilbert space endowed with a norm \( \| \cdot \| \) and an inner product \( \langle \cdot, \cdot \rangle \), respectively. Throughout the paper, we denote the set \( D \) a closed convex subset of \( H \) with nonempty interior \( D_0 \). For a set valued map \( G \) from \( H \) to \( H \), the set \( D(G) \) defined by \( D(G) = \{ x \in H : Gx \neq \emptyset \} \) is called the domain of \( G \), the set \( \mathcal{R}(G) \) defined by \( \mathcal{R}(G) = \bigcup_{x \in H} Gx \) is called the range of \( G \), the set \( G(G) \) defined by \( G(G) = \{(x, y) \in H \times H : x \in D(G), y \in Gx \} \) is called the graph of \( G \), the set \( G^{-1}x \) defined by \( G^{-1}x = \{ y \in H : x \in \bigcup G y \} \) is called the inverse image of \( x \in \mathcal{R}(G) \) under \( G \) and \( B_r[x] \) will designate the set \( \{ y \in H : \| y - x \| \leq r \} \).

Let us recall some basic definitions.

Definition 1.1. Let \( H \) be a Hilbert space. An operator \( T : H \to H \) is said to be

(i) monotone if
\[ \langle Tx - Ty, x - y \rangle \geq 0, \forall x, y \in H; \]

(ii) strongly monotone if
\[ \langle Tx - Ty, x - y \rangle \geq k_1 \| x - y \|^2, \forall x, y \in H \text{ and for some } k_1 > 0; \]

(iii) Lipschitz continuous if
\[ \| Tx - Ty \| \leq k_2 \| x - y \|, \forall x, y \in H \text{ and for some } k_2 > 0; \]

Definition 1.2. A multi-valued operator \( G : H \to 2^H \) is said to be

2010 Mathematics Subject Classification. Primary 49M15
Keywords. Fréchet derivative, generalized Newton method, generalized equations, Lipschitzian mappings, maximal monotone operators, S-iteration process

Received: 05 January 2012; Accepted: 15 April 2012
Communicated by Qamrul Hasan Ansari and Ljubiša D.R. Kočinac
Email addresses: drsahudr@gmail.com (D.R. Sahu), kumarkrishna.bhu@gmail.com (Krishna Kumar Singh)
(i) monotone if 
\[ \langle y_2 - y_1, x_2 - x_1 \rangle \geq 0, \forall x_1, x_2, y_1, y_2 \in H, \]

(ii) maximal monotone if it is monotone and there is no other monotone operator whose graph contains strictly the graph \( G(G) \) of \( G \);

(iv) strongly monotone if there is some \( \alpha > 0 \) such that 
\[ \langle y_2 - y_1, x_2 - x_1 \rangle \geq \alpha \| x_1 - x_2 \|^2, \forall x_1, x_2, y_1, y_2 \in H. \]

In the sequel, we will regard the statements \( [x, y] \in G, G(x) \ni y, -y + G(x) \ni 0 \) and \( y \in G(x) \) as synonymous. In [7], it is given that if \( G \) is maximal monotone then \( G(\cdot) : H \to 2^H \) is closed in the sense that 
\[ \lim_{m \to \infty} x_m = x \quad \text{and} \quad \lim_{m \to \infty} y_m = m \Rightarrow [x, y] \in G. \]

A well-known example (see [7, 8]) of a maximal monotone operator is the subgradient 
\[ \partial \phi(t) = \{ z \in H : \phi(t) - \phi(s) \leq \langle z, t - s \rangle, \forall s \in H \} \]
of a proper lower semicontinuous convex function \( \phi : H \to (-\infty, \infty] \).

The following definition will play a crucial role in the paper.

**Definition 1.3.** ([2]) Let \( C \) be a nonempty convex subset of a normed space \( X \) and \( T : C \to C \) an operator. The operator \( P : C \to C \) is said to be \( S \)-operator generated by \( \alpha \in (0, 1) \) and \( T \) if 
\[ P = T[(1 - \alpha)I + \alpha T], \]

where \( I \) is the identity operator.

In this paper, we consider the following problem:

Let \( F : D \to H \) be an operator which is Fréchet differentiable at each point of \( D_0 \) and \( G : H \to 2^H \) a maximal monotone operator. Consider the problem:

\[ \text{find } x \in H \text{ such that } Fx + Gx + Tx \ni 0 \tag{1} \]

where \( T : D \to H \) is a given operator.

Examples of the variational inclusion (1):

1. If \( G = \partial \phi \) and \( T = 0 \), then problem (1) reduces to the following problem: find \( x \in H \) such that 
\[ \langle Fx, y - x \rangle \geq \phi(x) - \phi(y), \forall y \in H, \]

which is called a nonlinear variational inequality and has been studied by many authors, see, for example, [3, 4, 9].

2. Let \( G = \partial \delta_K \) and \( T = 0 \), where \( \partial \delta_K \) is the indicator function of a nonempty, closed and convex subset \( K \) of \( H \) defined by 
\[ \partial \delta_K(x) = \begin{cases} 0, & x \in K; \\ \infty, & \text{otherwise}. \end{cases} \]

In this case problem (1) reduces to the following problem:

\[ \text{find } x \in K \text{ such that } \langle Fx, y - x \rangle \geq 0, \forall y \in K, \]

which is the classical variational inequality, see ([5, 14]).
The generalized Newton method is given by
\[ F'_w w_{n+1} + Gw_{n+1} \ni F'_w w_n - Fw_n, \quad n = 0, 1, \ldots, \]
where \( F'_w \) denotes the Fréchet derivative of \( F \) at the point \( w \in D_0 \). Many results on the convergence of the generalized Newton method (2) can be found in [10, 11, 16, 17].

In Newton method (2), functional value of inverse of derivative is required at each iteration. This brings us a natural question how to modify generalized Newton iteration process (2), so that the computation of the inverse of derivative at each step in Newton method (2) can be avoided.

In [15], Uko approximated the solution of (1) for \( T \equiv 0 \) by the generalized Newton method
\[ F'_w w_{n+1} + Gw_{n+1} \ni F'_w w_n - Fw_n, \quad n = 0, 1, \ldots, \]
and gave the following theorem for semi-local convergence analysis of (3) to solve the operator equation (1) for \( T = 0 \).

**Theorem 1.4.** Let \( F : D \to H \) be a continuous operator which has Fréchet derivative at each point of \( D_0 \) and \( G : H \to 2^H \) be a maximal monotone operator. For some \( x_0 \in D \), assume the operators \( F \) and \( G \) satisfy the following conditions:

(i) there exists \( y_0 \in H \) such that \( y_0 \in G(x_0) \) and \( \|F(x_0) + y_0\| \leq \beta \), for some \( \beta > 0 \);
(ii) \( (F'_w x, x) \geq \eta \|x\|^2 \), for some \( \eta > 0 \);
(iii) \( \|F'_w - F'_y\| \leq K \|x - y\| \) for all \( x, y \in D_0 \) and for some \( K > 0 \);
(iv) \( \gamma y_1 - y_1, x_2 - x_1 \| \geq K_0 \|x_1 - x_2\|^2 \), \( \forall \gamma, x_1, x_2 \in H, y_1 \in Gx_1, y_2 \in Gx_2 \) and for some \( K_0 > 0 \).

Assume that \( \eta + K_0 > 0 \) and denote \( d = \frac{\beta}{\eta + K_0} \) and \( h = \frac{2d}{\eta + K_0} \). Assume further that \( h < \frac{1}{2} \) and \( B_r[x_0] \subseteq D_0 \), where \( r = \frac{2d}{1 - \sqrt{1 - 2h}} \). Then, we have the following:

(a) The operator equation (1) with \( T = 0 \) has a unique solution \( x^* \) in \( B_r[x_0] \cap D \), where \( r_1 = \frac{2d}{1 - \sqrt{1 - 2h}} \).
(b) The sequence \( \{x_n\} \) generated by (3) is in \( B_r[x_0] \) and it converges to \( x^* \).
(c) The following error estimate holds:
\[ \|x_{n+1} - x^*\| \leq \frac{d}{\eta} \gamma^{n+1}, \quad \forall n \in \mathbb{N}, \]
where \( \gamma = 1 - \sqrt{1 - 2h} \).

Recently, Agarwal, O'Regan and Sahu [2] have introduced the S-iteration process (SIP) as follows: Let \( X \) be a normed space, \( C \) a nonempty convex subset of \( X \) and \( A : D \to C \) an operator. Then, for arbitrary \( x_0 \in C \), the S-iteration process is defined by
\[
\begin{cases}
  x_{n+1} = (1 - \alpha_n)Ax_n + \alpha_nAy_n, \\
  y_n = (1 - \beta_n)x_n + \beta_nAx_n, \quad n \in \mathbb{N},
\end{cases}
\]
where \( \{\alpha_n\} \) and \( \{\beta_n\} \) are sequences in \((0, 1)\).

In [12], motivated by S-iteration process, the first author has introduced the normal S-iteration process as follows: Let \( X \) be a normed space, \( C \) a nonempty convex subset of \( X \) and \( A : C \to C \) an operator. Then, for arbitrary \( x_0 \in C \), the normal S-iteration process is defined by
\[
x_{n+1} = A((1 - \alpha_n)x_n + \alpha_nAx_n), \quad n \in \mathbb{N},
\]
where \( \{\alpha_n\} \) be a sequence in \((0, 1)\). Noticing that the normal S-iteration process is applicable for finding solutions of constrained minimization problems and split feasibility problems (see Sahu [12]).
Following [12, Theorem 3.6], we remark that the normal S-iteration process is faster than the Picard and Mann iteration processes for contraction mappings.

In [13], following the ideas of normal S-iteration process first author introduced the S-iteration process of Newton-like for a real-valued function \( f \) defined on an open interval \( I \) as follows: For arbitrary \( x_0 \in I \), the S-iteration process of Newton-like is defined by

\[
\begin{align*}
x_{n+1} &= z_n - \frac{f'(x_n)}{f''(x_n)}, \\
z_n &= (1 - \alpha_n)x_n + \alpha_n y_n; \\
y_n &= x_n - \frac{f(x_n)}{f'(x_n)},
\end{align*}
\]

where \( \{\alpha_n\} \) is a sequence of real numbers in \( (0, 1) \).

In the present paper, motivated by normal S-iteration process, we introduce the S-iteration processes of Newton-like (7) for finding the solution of operator equation (1) in Hilbert space setting.

**Algorithm 1.5.** Let \( \alpha \in (0, 1) \). Starting with \( x_0 \in X \) and after \( x_n \in X \) is defined, we define the next iterate \( x_{n+1} \) as follows:

\[
\begin{align*}
F'_{x_n} x_{n+1} + Gx_{n+1} &\supseteq F'_{x_{n+1}} z_n - Fz_n - Tz_n, \\
z_n &= (1 - \alpha)x_n + \alpha y_n; \\
F'_{x_n} y_n + Gy_n &\supseteq F'_{x_n} x_n - Fx_n - Tx_n.
\end{align*}
\]

2. Convergence analysis

Before presenting main result, we need the following technical lemmas:

**Lemma 2.1.** ([15]) Let \( U : H \to H \) be a linear operator such that

\[
\langle Ux, x \rangle \geq \eta \|x\|^2, \forall x \in H,
\]

where \( \eta \) be a real number and let \( V : H \to 2^H \) a strongly and maximal monotone operator with constant \( \varphi \). Suppose \( u + \varphi > 0 \). Then the set \( (U + V)^{-1}x \) contains exactly one element for all \( x \in R(A + B) \).

**Lemma 2.2.** ([1, 6]) Let \((X, d)\) be a complete metric space and \( F : X \to X \) a contraction mapping. Then \( F \) has unique fixed point in \( X \).

**Lemma 2.3.** Let \( F : D \to H \) be a continuous operator such that \( F \) has Fréchet derivative at each point of \( D_0 \) and \( T : D \to H \) be an operator. Further, let \( G : H \to 2^H \) be a maximal monotone operator. For some \( x_0 \in D_0 \), assume the following:

(i) \( \langle F'_{x_0} x, x \rangle \geq \eta \|x\|^2 \) for some real number \( \eta \);

(ii) \( \langle y_2 - y_1, x_2 - x_1 \rangle \geq K_0 \|x_1 - x_2\|^2 \) for all \( x_1, x_2 \in H, y_1 \in Gx_1, y_2 \in Gx_2 \) and for some \( K_0 \geq 0 \).

Assume that \( \eta + K_0 > 0 \). Then, the operator \( A : D \to H \) defined by

\[
A(x) = (F'_{x_0} + G)^{-1}(F'_{x_0} x - Fx - Tx), x \in D
\]

is well defined on \( D \).

**Proof.** It follows from Lemma 2.1 by taking \( U = F'_{x_0}, V = G, u = \eta \) and \( \varphi = K_0 \).  \( \Box \)
First, we establish the following fixed point theorem to study convergence analysis of Algorithm 1.5, for the existence of unique solution of the operator equation (1).

**Theorem 2.4.** Let \( F : D \to H \) be a continuous operator such that \( F \) has Fréchet derivative at each point of \( D_0 \) and \( T : D \to H \) be a Lipschitz continuous operator with Lipschitz constant \( L \). Further, let \( G : H \to 2^H \) be a maximal monotone operator. For some \( x_0 \in D \), assume for the following:

(i) There exists \( y_0 \in H \) such that \( y_0 \in G(x_0) \) and \( \|F(x_0) + T(x_0) + y_0\| \leq \beta \), for some \( \beta > 0 \);

(ii) \( \langle F'_s(x), x \rangle \geq \eta \|x\|^2 \), for some real number \( \eta \);

(iii) \( \|F'_y - F'_y\| \leq K \|x - y\| \), for all \( x, y \in B_r[x_0] \) and for some \( K > 0 \);

(iv) \( (y_2 - y_1, x_2 - x_1) \geq K_0\|x_1 - x_2\|^2 \), \( \forall x_1, x_2 \in H, y_1 \in Gx_1, y_2 \in Gx_2 \) and for some \( K_0 \geq 0 \).

Assume that \( \eta + K_0 > 0 \) and denote \( c = \frac{L}{\eta + K_0} + \frac{\beta}{\eta + K_0} \) and \( h = \frac{Kd}{\eta + K_0} \). Assume further that \( h < \frac{(1-c^3)}{2} \), and \( B_r[x_0] \subseteq D_0 \), where \( r = \frac{1}{(1-c)^2 L} \). Then, for fixed \( \alpha \in (0, 1) \), we have the following:

(a) The operator \( A \) defined by (9) is well defined and a contraction self-operator on \( B_r[x_0] \) with Lipschitz constant \( \gamma \), where \( \gamma = \frac{h^2 + K}{\eta + K_0} \), and the operator equation (1) has a unique solution in \( B_r[x_0] \).

(b) The S-operator \( A_A : B_r[x_0] \to H \) generated by \( \alpha \) and \( A \) is a contraction self-operator on \( B_r[x_0] \) with Lipschitz constant \( \gamma (1 - \alpha + \alpha y) \).

**Proof.** (a) By Lemma 2.3, it is clear that the operator \( A \) is well defined on \( B_r[x_0] \). First note that \( \gamma = 1 - \sqrt{(1-c^3 - 2L)^2} < 1 \). For \( x \in B_r[x_0] \), by monotonicity of \( G \) we have

\[
K_0\|Ax - x_0\|^2 \leq \langle y_0 + F(x) + T(x) + F'_x(Ax - x), x_0 - Ax \rangle \\
\leq \langle F(x_0) + y_0 + F(x) + T(x) - F(x_0) - F'_x(x - x_0) + F'_x(Ax - x_0), x_0 - Ax \rangle \\
= \langle F(x_0) + T(x_0) + y_0 + F(x) - F(x_0) - F'_x(x - x_0) + T(x) - T(x_0), x_0 - Ax \rangle - \langle F'_x(Ax - x_0), Ax - x_0 \rangle \\
\leq \|F(x_0) + T(x_0) + y_0\|\|Ax - x_0\| + K_2\|x - x_0\|^2\|Ax - x_0\| \\
- \eta\|Ax - x_0\|^2 + \frac{K}{2\|x - x_0\|^2}\|Ax - x_0\|.
\]

Therefore, we get

\[
\|Ax - x_0\| \leq \frac{1}{\eta + K_0} \|F(x_0) + T(x_0) + y_0\| \\
+ \frac{K}{2(\eta + K_0)}\|x - x_0\|^2 + \frac{L}{\eta + K_0}\|x - x_0\| \\
\leq \frac{\beta}{\eta + K_0} + \frac{K_2^2}{2(\eta + K_0)} + \frac{L}{\eta + K_0} \leq r.
\]

For \( x, y \in B_r[x_0] \), by monotonicity of \( G \), we have

\[
K_0\|Ax - Ay\|^2 \leq \langle F'_x(x - F(x) - T(x) - F'_x(Ax - x), y + F(y) + T(y) + F'_x(Ay, Ax - Ay) \rangle \\
\leq \langle F(y) - F(x) - F'_x(y - x), y + F(y) + T(y) \rangle \\
= \langle F(y) - F(x) - F'_x(y - x), Ax - Ay \rangle \\
+ \langle T(y) - T(x), Ax - A(y) - F'_x(Ax - Ay), Ax - Ay \rangle \\
\leq \|F(y) - F(x) - F'_x(y - x)\|\|Ax - Ay\| + L\|x - y\|\|Ax - A(y)\| \\
- \eta\|Ax - Ay\|^2.
\]
Consequently, we get
\[ \|Ax - Ay\| \leq \frac{K}{\eta + K_0} \|x - y\| \max(\|x - x_0\|, \|y - x_0\|) + \frac{L}{\eta + K_0} \|x - y\| \]
\[ \leq \frac{Kr + L}{\eta + K_0} \|x - y\| \]
\[ = \gamma \|x - y\|. \]

Hence, the operator \( A \) is contraction self-operator on \( B_r[x_0] \) with Lipschitz constant \( \gamma \).

(b) For \( x, y \in B_r[x_0] \), observe that
\[ \|S_\alpha x - S_\alpha y\| = \|A[(1-\alpha)x + \alpha Ax] - A[(1-\alpha)y + \alpha Ay]\| \]
\[ \leq \gamma \|(1-\alpha)(x - y) + \alpha(Ax - Ay)\| \]
\[ \leq \gamma(1 - \alpha + \alpha \gamma) \|x - y\|. \]

Hence, the S-operator \( A_\alpha \) generated by \( \alpha \) and \( A \) is a contraction self-operator on \( B_r[x_0] \) with Lipschitz constant \( \gamma(1 - \alpha + \alpha \gamma) \).

Now, we are ready to present the semilocal convergence analysis of (8).

**Theorem 2.5.** Let \( F : D \to H \) be a continuous operator such that \( F \) has Fréchet derivative at each point of \( D_0 \) and \( T : D \to H \) be a Lipschitz continuous operator with Lipschitz constant \( L \). Further, let \( G : H \to 2^H \) be a maximal monotone operator. For some \( x_0 \in D \), assume following:

(i) There exists \( y_0 \in H \) such that \( y_0 \in G(x_0) \) and \( \|F(x_0) + y_0\| \leq \beta, \) for some \( \beta > 0; \)
(ii) \( (F'_\alpha x, x) \geq \eta \|x\|^2, \) for some real number \( \eta; \)
(iii) \( \|F'_\alpha - F'_\gamma\| \leq K \|x - y\|, \) for all \( x, y \in B_r[x_0] \) and for some \( K > 0; \)
(iv) \( \langle y_2 - y_1, x_2 - x_1 \rangle \geq K_0 \|x_1 - x_2\|^2, \forall x_1, x_2 \in H, y_1 \in Gx_1, y_2 \in Gx_2 \) and for some \( K_0 \geq 0. \)

Assume that \( \eta + K_0 > 0 \) and denote \( c = \frac{1}{\eta + K_0}, d = \frac{\beta}{1 + K_0}, h = \frac{d}{\eta + K_0} \). Assume further that \( h < \frac{(1-c^2)}{2} \) and \( B_r[x_0] \subseteq D_0, \)

where \( r = \frac{2d}{1 - c^2 \sqrt{1-\gamma^2 - 2h}}. \) Then, for fixed \( \alpha \in (0, 1) \), we have the following:

(a) The operator equation (1) has a unique solution \( x^* \) in \( B_r^\alpha[x_0] \cap D \), where \( r_1 = \frac{2d}{1 - c^2 \sqrt{1-\gamma^2 - 2h}}. \)
(b) The sequence \( \{x_n\} \) generated by (3) is in \( B_r^\alpha[x_0] \) and it converges to \( x^*. \)
(c) The following error estimate holds:
\[ \|x_{n+1} - x^*\| \leq \frac{d}{h} r^{n+1}, \forall n \in N_0, \]  

where \( \rho = \gamma(1 - \alpha + \alpha \gamma). \)

**Proof.** (a) By definition of S-operator \( A_\alpha \), it is clear that a point is fixed points of the operator \( A_\alpha \) if and only if it is a solution of the operator equation (1). Since the operator \( A_\alpha \) is contraction self-operator on \( B_r^\alpha[x_0] \), it follows from Lemma 2.2 that there exists a unique solution \( x^* \) of operator equation (1) in \( B_r^\alpha[x_0]. \) Now, we show that this solution \( x^* \) is unique in \( B_r^\alpha[x_0]. \) Suppose, for contradiction, that \( y^* \) is another solution of (1) in \( B_r^\alpha[x_0]. \) Then, \( y^* \) is another fixed point of \( A_\alpha. \) Using (10), we have
\[ \|y^* - x_0\| \leq \frac{\beta}{\eta + K_0} + \frac{K}{2(\eta + K_0)} \|y^* - x_0\|^2 + \frac{L}{\eta + K_0} \|y^* - x_0\|. \]

From inequality (12), we must have either \( \|y^* - x_0\| \geq r_1 \) or \( \|y^* - x_0\| \leq r. \) Hence, if \( y^* \in B_r^\alpha[x_0] \) then \( y^* \in B_r^\alpha[x_0]. \) Therefore, by uniqueness of fixed points of \( A_\alpha \) in \( B_r^\alpha[x_0], \) \( y^* = x^*. \)
By (8), we can write
\[
\|x_{n+1} - x^*\| = \|A_\alpha(x_n) - Ax^*\|
\leq \gamma(1 - \alpha + \gamma\alpha\alpha)\|x_n - x^*\|
\leq \gamma^n(1 - \alpha + \gamma\alpha\alpha)^{n+1}\|x_0 - x^*\|
\leq \frac{d}{\eta^{n+1}}.
\]
Therefore, \(x_n \to x^*\) as \(n \to \infty\).

(c) It follows from the part (b). □

In order to compare our Theorem 2.5 with the Theorem 1.4, we have the following corollary.

**Corollary 2.6.** Let \(F: D \to H\) be operators such that \(F\) has Fréchet derivative at each point of \(D_0\) and \(G: H \to 2^H\) be a maximal monotone operator. For some \(x_0 \in D\), assume following:

(i) There exists \(y_0 \in H\) such that \(y_0 \in G(x_0)\) and \(\|F(x_0) + y_0\| \leq \beta\), for some \(\beta > 0\);
(ii) \((F^\alpha F, x, y) \geq \eta \|x\|^2\), for some real number \(\eta\);
(iii) \(\|F(x) - F(y)\| \leq K\|x - y\|\), for all \(x, y \in B_1(x_0)\) and for some \(K > 0\);
(iv) \((y_2 - y_1, x_2 - x_1) \geq K_0\|x_1 - x_2\|^2\), \(\forall x_1, x_2 \in H, y_1 \in Gx_1, y_2 \in Gx_2\) and for some \(K_0 > 0\).

Denote \(d = \frac{\beta}{\eta + K_0}\) and \(h = \frac{kd}{\eta + K_0}\). Assume that \(h < \frac{1}{2}\), \(\eta + K_0 > 0\) and \(B_1(x_0) \subseteq D_0\), where \(r = \frac{2d}{1 + \sqrt{1 - 2h}}\). Then, for fixed \(\alpha \in (0, 1)\), we have the following:

(a) The operator equation (1) has a unique solution \(x^* \in B_1(x_0) \cap D\), where \(r_1 = \frac{2d}{1 - \sqrt{1 - 2h}}\).

(b) The sequence \(\{x_n\}\) generated by (3) is in \(B_1(x_0)\) and it converges to \(x^*\).

(c) The following error estimate holds:
\[
\|x_{n+1} - x^*\| \leq \frac{d}{\eta^{n+1}}, \forall n \in \mathbb{N},
\]
where \(\rho = \gamma(1 - \alpha + \gamma\alpha\alpha)\).

**Proof.** By putting \(T \equiv 0\) in the Theorem 2.5, we get the result. □

**Remark 2.7.** One can observe from (13) and (4) that
\[
\rho = \gamma(1 - \alpha + \gamma\alpha\alpha) < \gamma.
\]
The strict inequality (14) shows that the error estimate in Corollary 2.6 is sharper than that of Theorem 1.4. Therefore, Corollary 2.6 is an improvement of Theorem 1.4.

**Remark 2.8.** Theorem 1.4 cannot be applied for operator equation (1) as in Theorem 2.5, the Algorithm 1.5 can be successfully applied for solving operator equation (1).

In the following example, we show numerically that the rate of convergence of the sequence \(\{x_n\}\) generated by (8) is faster than the sequence \(\{w_n\}\) generated by (3).

**Example 2.9.** Let \(H = \mathbb{R}\) and \(D = [-1, 1]\). Consider the problem: find \(x \in \mathbb{R}\) such that
\[
x^2 + x - 1 = 0.
\]
For all \(x \in D\), let \(F(x) = x^2 - 1, G(x) = x\) and \(T \equiv 0\). Clearly, for \(x_0 = 1\), we have
\[
\eta = 2, \beta = 1, K = 2, K_0 = 1 \text{ and } c = 0.
\]
Thus, $h = 0.22$ and therefore $h < \left(\frac{1-\sqrt{25}}{2}\right)$. Hence, all the conditions of Theorem 2.5 are satisfied, hence there exists a point $x^* \in B[x_0]$, where $r = 0.3813$, such that

$$F(x^*) + G(x^*) + T(x^*) = 0$$

Particularly, for $\alpha = 0.5$, the following table shows that the rate of convergence of the sequence $\{x_n\}$ generated by (8) is faster than the sequence $\{w_n\}$ generated by (3).

<table>
<thead>
<tr>
<th>$n$</th>
<th>Newton method ${w_n}$</th>
<th>SIP of generalized Newton-like ${x_n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.6666666666666667</td>
<td>0.657407407407407</td>
</tr>
<tr>
<td>2</td>
<td>0.629629629629629630</td>
<td>0.624058725602650</td>
</tr>
<tr>
<td>3</td>
<td>0.620941929583905</td>
<td>0.61990116370879</td>
</tr>
<tr>
<td>4</td>
<td>0.618771659750809</td>
<td>0.61818655641400</td>
</tr>
<tr>
<td>5</td>
<td>0.61821650863616</td>
<td>0.618058357903157</td>
</tr>
<tr>
<td>6</td>
<td>0.618081764043566</td>
<td>0.618037881467669</td>
</tr>
<tr>
<td>7</td>
<td>0.618046153881308</td>
<td>0.618034610584751</td>
</tr>
<tr>
<td>8</td>
<td>0.618034777551723</td>
<td>0.61803404617913</td>
</tr>
</tbody>
</table>

In the following figure, graph with red color is corresponding sequence $\{w_n\}$ generated by generalized Newton method (3) and graphs with blue, black and green colors are corresponding the sequence $\{x_n\}$ generated by SIP of generalized Newton-like (8), for $\alpha = 0.25$, $\alpha = 0.5$ and $\alpha = 0.75$, respectively. Clearly, the rate of convergence of the sequence $\{x_n\}$ generated by SIP of generalized Newton-like (8) is always faster than the sequence $\{w_n\}$ generated by generalized Newton method (3).

References