Homothetic Polygons and Beyond: Intersection Graphs, Recognition, and Maximum Clique

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Abstract

We study the Clique problem in classes of intersection graphs of convex sets in the plane. The problem is known to be NP-complete in convex-set intersection graphs and straight-line-segment intersection graphs, but solvable in polynomial time in intersection graphs of homothetic triangles. We extend the latter result by showing that for

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every convex polygon P with sides parallel to k directions, every n-vertex graph which is an intersection graph of homothetic copies of P contains at most n^k inclusion-wise maximal cliques. We actually prove this result for a more general class of graphs, the so called k_{DIR}-CONV, which are intersection graphs of convex polygons whose sides are parallel to some fixed k directions. Moreover, we provide some lower bounds on the numbers of maximal cliques, discuss the complexity of recognizing these classes of graphs and present a relationship with other classes of convex-set intersection graphs. Finally, we generalize the upper bound on the number of maximal cliques to intersection graphs of higher-dimensional convex polytopes in Euclidean space.

1 Introduction

Geometric representations of graphs, and intersection graphs in particular, are widely studied both for their practical applications and motivations, and for their interesting theoretical and structural properties. It is often the case that NP-hard optimization problems for general graphs can be solved, or at least approximated, in polynomial time on such graphs. Classical examples are the Independent set, Clique or Coloring problems for interval graphs, one of the oldest intersection-defined classes of graphs [6]. The former two problems remain polynomially solvable in circle and polygon-circle graphs, while the last one already becomes NP-complete. For definitions and more results about these issues, as well as some possible applications, the interested reader is referred to the work of Golumbic [7], McKee and McMorris [19], and Spinrad [22].

In this paper we investigate subclasses of the class of intersection graphs of convex sets in the plane, denoted by CONV, and the computational complexity of the problem of finding a maximum clique in such graphs. This has been motivated by three arguments. First, the Clique problem was shown to be polynomial time solvable for intersection graphs of homothetic triangles in the plane by Kaufmann *et al.* [11]. (These graphs have been shown to be equivalent to the so called max-tolerance graphs, and as such found direct application in DNA sequencing.) Secondly, the Clique problem is known to be NP-complete in CONV graphs [15], and so it is interesting to inspect the boundary between easy and hard instances more closely. Thirdly, straight line segments are the simplest convex sets, and it is thus natural to ask how difficult Clique is in intersection graphs of segments in the plane (this class is denoted by SEG). Kratochvíl and Nešetřil posed this problem in [17] after they observed that if the number of different directions of the segments is bounded by a constant, say k, a maximum clique can be found in time $O(n^{k+1})$ (this class of graphs is denoted by k-DIR, see [16] for more details). This question has been answered very recently by Cabello *et al.* [5] who showed that Clique is NP-complete in SEG graphs. Maximal and maximum cliques in intersection graphs of convex sets have also been considered by Ambühl and Wagner [1] (ellipses and triangles), Brimkov *et al.* [3] (trapezoids) and Imai and Asano [9] (rectangles).

In [18], Kratochvíl and Pergel initiated a study of P_{hom} graphs, defined as intersection graphs of convex polygons homothetic to a single polygon P. They announced that for every convex polygon P, recognition of P_{hom} graphs is NP-hard, and asked in Problem 3.1 if P_{hom} graphs can have a superpolynomial number of maximal cliques. Our main result shows that for every convex p-gon P, every P_{hom} graph with n vertices contains at most n^p maximal cliques, and hence Clique is solvable in polynomial time on P_{hom} graphs for every fixed polygon P. For the sake of completeness, we will also present the proof of NP-hardness of P_{hom} recognition.

E.J. van Leeuwen and J. van Leeuwen [24] considered a more general class of graphs based on affine transformations of one (or more) master objects. These were called $\mathcal{P}$-intersection graphs, where $\mathcal{P} = (S, T)$ is a signature consisting of a set S of master objects and a set T of transformations. They proved that if all objects in the signature are described by rational numbers, such graphs have representations of polynomial size and the recognition problem is in NP. As a corollary, recognition of P_{hom} graphs is in NP (and hence NP-complete) for every rational polygon P. In [20], van Leeuwens and T. Müller proved tight bounds on the maximum sizes of representations (in terms of coordinate sizes) of $P_{translate}$ and P_{hom} graphs.

In proving the main result of our paper, the polynomial bound on the number of maximal cliques, we go beyond the homothetic polygon intersection graphs. We observe that in any representation by polygons homothetic to a master one, the sides of the polygons are parallel to a bounded number of directions in the plane. So if we relax the requirement on the homothetic relation of the polygons in the representation, we simply consider a set of k directions and look after graphs that have intersection representations by convex polygons whose every side is parallel to one of those k directions (see Figure 1). We call this class k_{DIR}-CONV graphs.

We prove that every such graph has at most n^k maximal cliques, where

k is the number of directions parallel to at least one side (since we may have two parallel sides). We find this fact worth emphasizing, as it also covers van Leeuwens' $\mathcal{P}$-intersection graphs for transformations without rotations. So we further investigate the class of k_{DIR}-CONV graphs, discuss the complexity of its recognition and relationship to other relevant graph classes (SEG, k-DIR, and P_{hom}). The immediate complexity impact of this result is that, for every convex polygon P, the Clique problem can be solved in polynomial time in k_{DIR}-CONV graphs, even when a representation of the input graph is not given. (It is well known that all maximal cliques of an input graph can be enumerated with polynomial delay, see Tsukiyama *et al.* [23].) The exponent of the polynomial of course depends on k.

We also pay closer attention to maximal cliques in P_{hom} graphs for specific polygons P. If P is a $2k$-gon with k pairs of parallel sides, then we can construct a P_{hom} graph with $\Omega(n^{k(1-\epsilon)})$ maximal cliques (where ϵ is an arbitrarily small positive constant). Moreover, for every fixed polygon but parallelograms we present a construction of a P_{hom} graph with $\Omega(n^3)$ maximal cliques (by a modification of a construction for triangles from [11]). It is worth noting that also for the max-coordinate results of [20], parallelograms play an exceptional role.

In the last section we generalize the upper bound on the number of maximal cliques to intersection graphs of convex polytopes of higher dimensions. Intersection graphs of higher-dimensional polytopes (namely, high-dimensional boxes) have been considered earlier (see for example [2, 12, 21]).

2 Preliminaries

2.1 Definitions and basic properties

In this paper we deal with intersection graphs of subsets of the Euclidean plane $\mathbb{R}^2$. The following concepts are standard and we only briefly overview them to make the paper self-contained. For a collection R of sets the *intersection graph* of R is denoted by $IG(R)$; its vertices are in 1-1 correspondence with the sets and two vertices are adjacent if and only if the corresponding sets are non-disjoint. In such a case the collection R is called an *(intersection) representation* of G, and the set corresponding to a vertex $v \in V(G)$ is called *the representative* of v and denoted by R_v.

The intersection graphs of straight-line segments are called the SEG

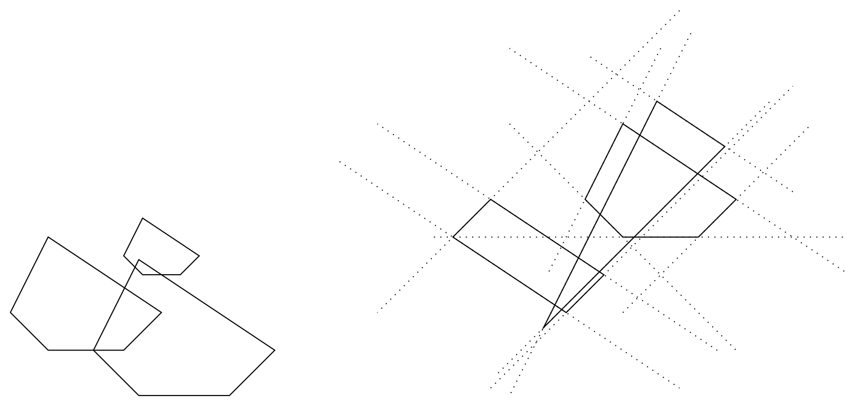

Figure 1: Homothetic pentagons (left) and polygons with 5 directions of sides (right).

graphs, of convex sets the CONV graphs, and k-DIR is used for SEG graphs having a representation with all the segments being parallel to at most k directions (thus 1-DIR are exactly the interval graphs). For a fixed set P (in most cases a convex polygon), the class of intersection graphs of sets homothetic to P is denoted by P_{hom} (two sets are homothetic if one of them can be obtained from the other by scaling and/or translating). If P is a disk, we get disk-intersection graphs, a well studied class of graphs. Pseudodisk intersection graphs are intersection graphs of collections of closed planar regions (bounded by simple Jordan curves) that are pairwise in the pseudodisk relationship, i.e., both differences $A \setminus B$ and $B \setminus A$ are arc-connected. It is easy to observe that the borders of any two sets in pseudodisk relation may intersect at most twice.

Throughout the paper, a *polygon* means a closed convex polygon in the plane. Let be the set of all distinct lines in $\mathbb{R}^2$ that contain the point $(0,0)$. For a k-tuple of lines $L = \{\ell_1, .., \ell_k\} \in \binom{}{k}$, we denote by $\mathcal{P}(L)$ the family of all polygons P such that every side of P is parallel to some $\ell \in L$. Moreover, by $\mathcal{P}(k)$ we denote $\bigcup_{\binom{}{k}} \mathcal{P}(L)$.

Now we introduce the main characters of the paper. By $k_{\mathrm{DIR}(L)}$-CONV we denote the class of intersection graphs of polygons of $\mathcal{P}(L)$. Finally, we define k_{DIR}-CONV $= \bigcup_{L \in \binom{}{k}} k_{\mathrm{DIR}(L)}$-CONV. Fig. 1 shows examples of representations of the same graph P_3 by intersections of homothetic pentagons and as a 5_{DIR}-CONV graph.

Note that polygons in $\mathcal{P}(L)$ do not have to be in the pseudodisk relation while homothetic copies of the same polygon always are. Moreover, observe that 2_{DIR}-CONV are the intersection graphs of isothetic rectangles, which are exactly the graphs of boxicity at most 2 (see [21] for more details on boxicity of graphs).

The following property of convex polygons is well known.

Lemma 1 (Folklore)**.** *Any two disjoint convex polygons in $\mathcal{P}(L)$ can be separated by a line parallel to a line from L.*

2.2 Relations between graph classes

In this section we investigate the relations between the graph classes considered in this paper, i.e. k_{DIR}-CONV, P_{hom} and k-DIR. We first observe that for every $k \geq 2$, each k-DIR graph is also in k_{DIR}-CONV.

Theorem 1. *For every $k \geq 2$, it holds that k-DIR $\subseteq k_{DIR}$-CONV.*

Proof. Let G be a k-DIR graph and let $R = \{S_i \colon i \in \{1, .., n\}\}$ be a segment representation of G. Let $L = \{\ell_1, .., \ell_k\}$ be a set of lines such that every segment in R is parallel to some line in L. We will define a family R' of parallelograms from $\mathcal{P}(L)$, such that the intersection graph of R' is isomorphic to G. This will show that $G \in k_{\mathrm{DIR}(L)}$-CONV and therefore $G \in k_{\mathrm{DIR}}$-CONV. The idea of constructing R' is to extend every segment from R to a very narrow parallelogram in such a way that no new intersection appears (see Figure 2).

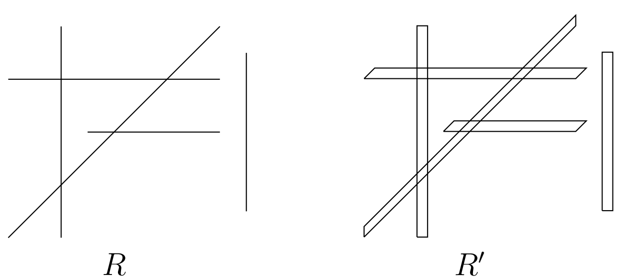

Figure 2: Transformation from a segment representation (left) to a polygon representation (right).

Let $d = \min\{\mathrm{dist}(S_i, S_j) : S_i \cap S_j = \emptyset, i, j \in \{1, .., n\}\}$ (where $\mathrm{dist}(S_i, S_j)$ denotes the length of the shortest segment with one end in S_i and the other in S_j) and let x_i and y_i be the endpoints of the segment S_i for $i \in \{1, .., n\}$. Consider S_i for $i \in \{1, .., n\}$. The segment S_i is parallel to a line from L, say to ℓ_j. Let b_i be a unit length vector parallel to $\ell_{j+1 \bmod k}$. We set P_i to be the parallelogram with corners $x_i + \frac{d}{3}b_i, x_i - \frac{d}{3}b_i, y_i - \frac{d}{3}b_i, y_i + \frac{d}{3}b_i$, and we set $R' = \{P_i : i \in \{1, .., n\}\}$.

Obviously, if $S_i \cap S_j \neq \emptyset$, then $P_i \cap P_j \neq \emptyset$, since S_i is contained in P_i for every $i \in \{1, .., n\}$. On the other hand notice that every point of P_i is at distance at most $\frac{d}{3}$ from S_i. Assume $P_i \cap P_j \neq \emptyset$ and consider a point $z \in P_i \cap P_j$. Such a z is at distance at most $\frac{d}{3}$ from S_i and S_j. By the triangle inequality, S_i and S_j are at distance at most $\frac{2d}{3}$ and by the definition of d they intersect each other. □

Now let us turn our attention to P_{hom} graphs. Clearly $P_{hom} \in k_{\mathrm{DIR}}$-CONV for any convex polygon P with sides parallel to at most k directions. In particular, $P_{hom} \in k_{\mathrm{DIR}}$-CONV for any k-gon P.

Now let us prove that P_{hom} graphs are pseudodisk intersection graphs for any P.

Lemma 2. *Homothetic convex bodies in the plane in general position (i.e., no two bodies have an infinite number of common points on their boundaries) form an arrangement of pseudodisks. Thus, for every convex polygon P, the class P_{hom} is a subclass of the class of pseudodisk graphs.*

Proof. Let us take two homothetic convex bodies with the required property. First we consider the case where one of them is smaller than the other. We use the Banach fixed point theorem for a linear mapping which maps the bigger polygon to the smaller one (such a mapping can be obtained as a composition of shift and scaling). The Banach theorem implies that such a fixed point exits (note that we do not need an efficient algorithm to find one). Starting from that fixed point, we can then draw boundaries of these two convex bodies on rays. Thus we start from the fixed point and pass along a straight line until we reach boundary of some of these two polygons. If the fixed point is inside the two bodies, then clearly no intersection of the boundaries would appear. Now consider the case where the fixed point lies outside the bodies. Under this condition, in the drawing procedure rays emanate from the fixed point to each direction. Informally, this can be interpreted as a radar which is scanning the fixed point's neighborhood. In such a "radar-like strategy,"

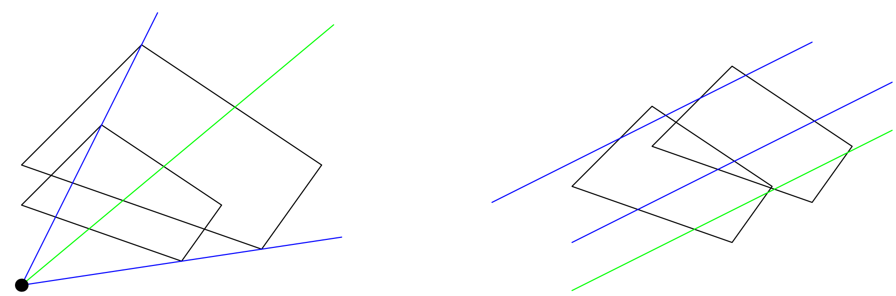

Figure 3: Illustration to the procedure of finding a fixed point (left) and to the behaviour of that fixed point “in infinity” for bodies of the same size (right).

on each ray points are placed/reached of the boundaries of the convex bodies in the following order:

1. The start-point of the smaller body;
2. The start-point of the bigger body or the endpoint of the smaller body;
3. The second from the previous point, i.e., again, the start-point of the bigger body or the endpoint of the smaller body;
4. Endpoint of the bigger body.

Suppose that there are at least three rays where the second and the third point are the same. Assume first that these three points lie on a straight line. Then, in order to preserve the convexity of both polygons between these three points, the line between these three points has to be a common border, i.e. the boundaries of the polygons would contain infinitely many common points, which is a contradiction. Otherwise, if these three points do not lie on the same straight line, we may pick them in the clockwise ordering starting with the direction where both polygons are touched only once by the ray (beginning of the polygons) up to the point where the same happens once again (ends of the polygons). Suppose that we pick two consecutive segments both stemming from the middle intersection-point, as one of them is ending in the 1st intersection-point while the other one is ending in the 3rd intersection-point. Now if we consider the segment between the 1st and 3rd intersection point (that must be contained in both polygons because of their convexity), then the construction of the middle intersection-point witnesses that at least one of these points must not belong to at least one of these polygons (as their common boundary intersection does not lie on this segment), and thus at least one of these polygons cannot be convex.

If the fixed point is on the boundary, then we start "distributing" the borders of convex bodies by placing the start-point of the smaller and the start-point of the larger. Then we only have to place the endpoints. Note that as one body is smaller and the other is bigger, the endpoints must be distinct. Thus we obtain that the borders of the polygons share exactly one common point (as, otherwise, we would once again have infinitely many common points on the borders).

The case of two convex bodies of the same size can be handled analogously. The only somewhat more essential difference is that instead of rays one can use parallel lines in the appropriate direction. All arguments remain essentially the same, up to the the fact that the "fixed point" would lie in the infinity. This completes the proof. □

Next we discuss the relationships between classes P_{hom} and k-DIR.

Lemma 3. *Graph classes k-DIR and P_{hom} are essentially distinct, i.e., for any polygon P and $k \geq 2$,*

1. *k-DIR $\not\subseteq P_{hom}$;*
2. *$P_{hom} \not\subseteq k$-DIR.*

Proof. 1. Let us consider the graph $K_{3,3}$. Figure 4 shows that it is in 2-DIR (and thus in k-DIR for any $k \geq 2$). On the other hand, Kratochvíl [14] proved that triangle-free pseudodisk intersection graphs are planar. Since by Lemma 2 all P_{hom} graphs are pseudodisk intersection graphs, $K_{3,3}$ is not a P_{hom} graph for any polygon P.

2. Let k be fixed and let S_k be a graph consisting of a $(2k+1)$-clique, whose every vertex has a private neighbor. Figure 5 shows S_k and its geometric representation as a P_{hom} graph for P being a square and a triangle. Note that this construction can be easily generalized for any convex polygon P. Thus $S_k \in P_{hom}$ for any P.

   Suppose now that $S_k \in k$-DIR. Since we have $2k+1$ vertices in the clique and only k directions available, there exists a direction with at least three segments in the clique. Since those segments are pairwise

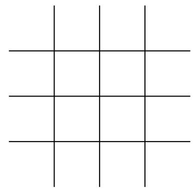

Figure 4: A 2-DIR representation of $K_{3,3}$. It can be clearly generalized for any $K_{n,n}$.

parallel and intersect each other, they lie on the same line. But in such a case at most two of them may have private neighbors. Therefore $S_k \notin k$-DIR.

□

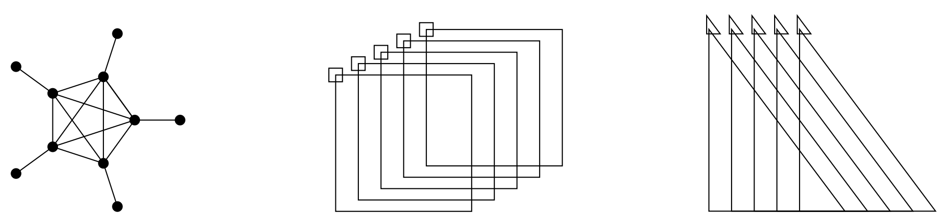

Figure 5: Graph S_2 and its representation by squares and by homothetic triangles.

The construction from the second part of Lemma 3 exploited the fact that the number of directions of segments is fixed. For k_{DIR}-CONV graphs we can improve the result by constructing a graph that is in k_{DIR}-CONV for any $k \geq 2$, but cannot be represented by any configuration of segments (so it not in SEG).

Theorem 2. *For any $k \geq 2$ there exists a graph in k_{DIR}-CONV which is not a SEG graph.*

Proof. Let us consider the graph G in Figure 6 which is inspired by construction of Kratochvíl and Matoušek from [16]. Suppose that G is a SEG graph.

In any geometric representation the white cycle is represented by a closed Jordan curve. We will refer to it as the outer circle. It divides the plane into two faces – an interior and an exterior.

The outer circle cannot be crossed by the representative of any black vertex. Moreover, as two black vertices are adjacent and therefore their representatives intersect each other, they have to be represented in the same face (with respect to the outer circle). Therefore, along this circle the representatives of gray vertices appear in a prescribed ordering. This implies the ordering in which some part of representatives of the black vertices occur.

The vertices a and b are represented as two mutually intersecting segments. The segment representing the vertex c must cross both of them and it must do it in such a way that it has a part in each of the four quadrants (with respect to representatives of a and b). This is impossible as a single segment may meet at most three quadrants.

Thus G is not a SEG graph. On the other hand G is a 2_{DIR}-CONV-graph (see Figure 6) and therefore a k_{DIR}-CONV graph for any $k \geq 2$. $\square$

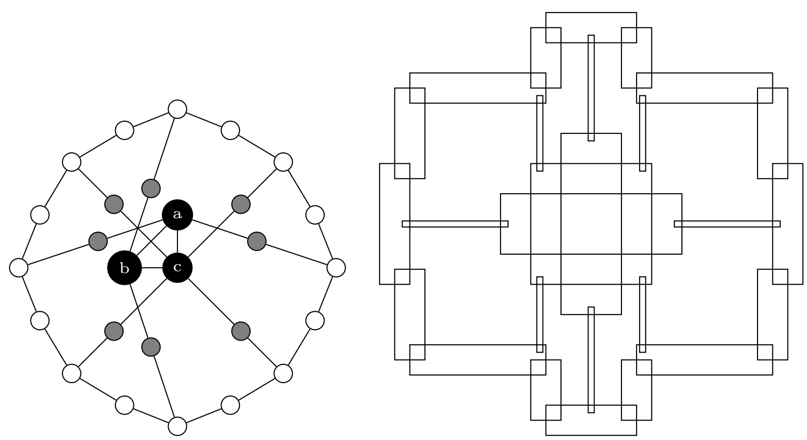

Figure 6: A graph G and its representation as a 2_{DIR}-CONV graph.

We conclude this section by exhibiting the relationship of P_{hom} and k_{DIR}-CONV graphs.

Lemma 4. *k_{DIR}-CONV $\not\subseteq P_{hom}$ for any convex polygon P and $k \geq 2$.*

Proof. Let us again consider the graph G depicted in Figure 6 and a very similar argument to what we already used. Any intersection representation of this graph by convex polygons requires two polygons whose boundaries intersect at least four times. As we can see in Figure 6, this is not a problem for 2_{DIR}-CONV graphs, but our Lemma 2 shows that any representation by homothetic convex polygons forms an arrangement of pseudodisks. Since the boundaries of two pseudodisks may intersect at most twice, our graph is not a P_{hom} graph for any P. $\square$

3 Recognition

In this section we prove some results concerning the hardness of recognition of k_{DIR}-CONV and P_{hom} graphs.

Theorem 3. *For every fixed $k \geq 2$, it is NP-complete to recognize*

1. *$k_{DIR(L)}$-CONV graphs for any $L \in \binom{}{k}$,*
2. *k_{DIR}-CONV graphs.*

Proof. As 2_{DIR}-CONV graphs are exactly graphs of boxicity at most 2, they are NP-complete to recognize [13].

For $k > 2$, the class of k_{DIR}-CONV graphs contains the class of 3-DIR graphs and at the same time is contained in CONV. Thus, to prove NP-hardness we may apply the reduction from [13]. This is a unified reduction that implies NP-hardness of k-DIR graphs (for any $k \geq 3$) and CONV graphs in one step (reducing from satisfiability and showing that the obtained graph is in 3-DIR if the initial formula is satisfiable, but is not in CONV otherwise). Since for $k = 3$ all triples of directions are equivalent under an affine transformation of the plane, this shows that for $k > 2$, recognition of k_{DIR}-CONV, and also of $k_{\text{DIR}(L)}$-CONV or of any k-tuple of directions, is NP-hard.

Similar to [16], we show that the recognition problem for both $k_{\text{DIR}(L)}$-CONV and k_{DIR}-CONV is in NP. We have to establish a polynomially-large certificate. For this we use a combinatorial description of the arrangement, i.e., we guess a description specifying in what order individual sides of individual polygons get intersected. We also need the information about particular corners of individual polygons. For individual sides of polygons we also need to know their directions (for this it is sufficient to keep the index of the direction, i.e. a number in $\{1, .., k\}$). To make the situation formally simpler, instead of segments we consider a description of the whole underlying lines. Note that a corner of a polygon and the intersection of boundaries appear here as intersection of two lines.

Now we have to verify the realizability of such an arrangement (once again, analogously to [16]). For this, we construct a linear program consisting of inequalities describing the ordering of the intersections along each side of each polygon. For a line p described by the equation $y = a_p x + b_p$ the intersection with q precedes the intersection with r ("from the left to the right") if $\frac{b_q - b_p}{a_p - a_q} < \frac{b_r - b_p}{a_p - a_r}$. In the case of prescribed directions (a_p, a_q, a_r), we have a linear program whose variables are the b coefficients. This linear program can be solved in a polynomial time, which shows the NP-membership for $k_{\mathrm{DIR}(L)}$-CONV.

For k_{DIR}-CONV, one can use an argument from [16], which is that the directions obtained as solutions of the considered linear program are of polynomial size and thus they may also be a part of a polynomial certificate. □

Concerning the class of P_{hom} graphs, our aim is to obtain the following result announced in [18].

Theorem 4. *For every convex polygon P, the recognition problem of P_{hom} graphs is NP-hard.*

Before proving the theorem, we observe the following. The convex polygons are required either to properly intersect or to not intersect, i.e., they are not allowed to only touch each other at border points. Therefore, we may consider that any representation is perturbation resistant, i.e., we may slightly move any polygon in any direction and obtain a topologically equivalent configuration. Thus we may consider only representations satisfying the assumptions of Lemma 2.

Proof of Theorem 4. As the first step we refer to [11] which proves the NP-hardness for homothetic triangles.

For polygons with more corners we use the construction introduced in [8]. It is known that for an unsatisfiable formula the graph (obtained from the construction) cannot be represented by pseudo-disks. By Lemma 2 we have that the graphs representable by the homothetic polygons form a subset of graphs representable by pseudo-disks. Thus it suffices to show that the graph obtained from any satisfiable formula can be represented by homothetic polygons in a plane. For this, we need to introduce some terminology and denotations.

For a convex polygon P we choose an orthogonal basis b_1, b_2 such that all sides of the polygon P are not parallel to b_1 or to b_2. Given such a basis b_1, b_2, we consider the smallest axis-aligned rectangle containing P (its bounding box) and denote it by $BB(P)$. One can easily see that we may choose even such a basis that P touches $BB(P)$ inside $BB(P)$'s edges rather than at corners.

For an arrangement of homothetic convex polygons we may pick up such a basis b_1, b_2. As we will not be interested in the basis itself but only in the bounding boxes, we will not require that $BB(P)$ must be taken with respect to a certain basis. The basis will be fixed in a way to secure the condition that the corners of $BB(P)$ are not elements of P.

Now we recall the reduction. As it is described in detail in in [8], we focus only on the main points:

We use the E3-Nae-Sat(4) which is known to be NP-complete. The instance of Nae-Sat is a boolean formula in conjunctive normal form and we ask whether there exists a (satisfying) assignment such that in no clause are all the literals evaluated true. The version E3-Nae-Sat(4) is a restriction of Nae-Sat to formulae with each clause consisting of **exactly** three literals and each variable occurs at most four times.

We reduce E3-Nae-Sat(4) to representability by homothetic polygons of any shape. The graph consists of gadgets for clauses, gadgets for variables and connections between them. The gadget for a variable is C_8 (with vertices $c_1, \ldots c_8$, which can clearly be represented by homothetic polygons in a plane. Each occurrence of the variable is represented by two consecutive vertices (c_{2k-1}, c_{2k}). If the variable's occurrence is negated, we swap the labels of c_{2k-1} and c_{2k}. The truth assignment in the representation is determined by the orientation of polygons $P_{c_1}, \ldots P_{c_8}$ (they may go either clockwisely or counter-clockwisely). The connections are represented by a ladder (see Figure 7) and crucial for the construction is the fact (proved in [8]) that the ladder cannot distort (*i.e.*, swap vertices "from the left to the right"). The path that begins on the left side never crosses the path on the right side.

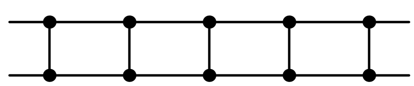

Figure 7: The ladder

The ladders representing the first and the second occurrence of the particular variable are connected by a "cross-over"-gadget. The same applies to the third and the fourth occurrence. We want to obtain the ladders representing particular occurrences of the variable with respect to clockwise orientation for positive and negative variable to appear around the variable-gadget always in the same order (*i.e.*, ladder 1, ladder 2, ladder 3, and ladder 4). This is secured by implementation of the cross-over. If the variable is assigned a value “true," we just make respective ladders to touch; if the variable is assigned a value “false," we cross them. A crucial fact (proved in [8]) is that twisting a ladder does not occur (as there is always “the left” row and “the right” row). The cross-over gadget is depicted in Figure 8.

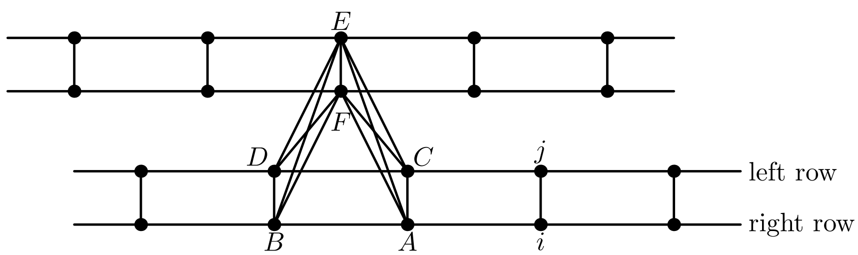

Figure 8: This figure shows one cross-over on two ladders. One raw in each ladder is called the left one, the other the right one. Vertex labels correspond to the labels on the next figure.

The two pictures in Figure 9 illustrate how two ladders touch (resp. cross). In the following paragraphs we refer to denotations from that figure.

More precisely, if we want to cross two ladders, we represent P_A and P_B by polygons of the same size crossing only slightly. We create P_C and P_D to be of the same size. So we obtain a quadruple of polygons of width $(2-\delta)\cdot \mathrm{width}(P_A)$ and height $(1+\varepsilon)\cdot \mathrm{height}(P_A)$. We choose a factor μ and create polygons P_E and P_F scaled to P_A by factor $(1+\mu)$ and make them cross slightly more than P_A and P_B do. Now we use the following fact about bounding boxes: Except for small intervals around the four points where the polygon touches the boundary of its bounding box, there is a small stripe inside the boundary of the bounding box which is disjoint from the polygon.

If we want the ladders to touch only, $P_A ... P_D$ get represented in the same manner. We represent P_E by a polygon obtained as follows: We take P_E as a copy of P_A (placed over P_A). Then we shift it to the left (to avoid intersection with P_i) and to the bottom. Then we scale it slightly to intersect P_D (it can be done by scaling by factor $(1+2\cdot\varepsilon)$). Now we have a proper representation

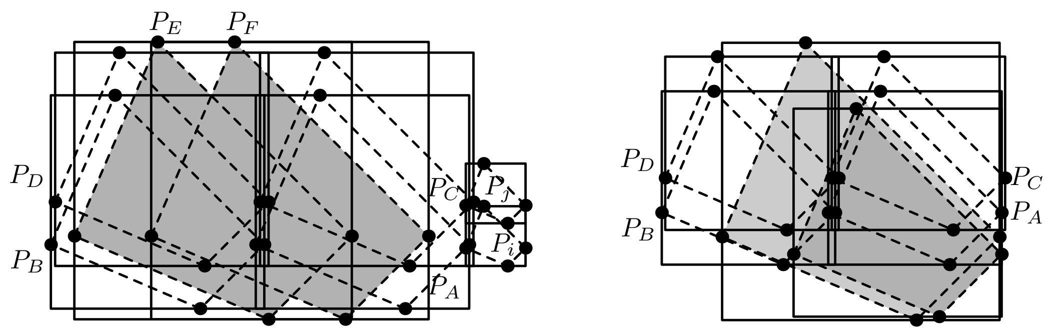

Figure 9: The picture on the left shows how to cross representations of two ladders in cross-over. The one on the right illustrates the case where they are only touched.

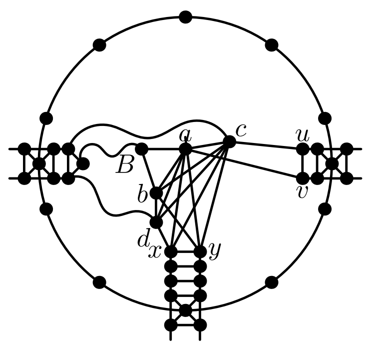

Figure 10: The clause gadget, wavy "edges" depict arbitrarily long paths.

of the whole gadget except P_E. To represent P_E, we take a copy of P_D (placed over P_D). Then we shift it slightly to the right and start scaling it up to force the right place of intersections with P_D. We certainly may stop scaling before the critical factor $(2-\delta)$ is reached, as for the factor $(2-\delta)$ we could place the polygon to intersect $P_A ... P_E$. Moreover, the rightmost corner could be placed below the bottom intersection of P_A with P_i.

Whenever two ladders leading from variables to clauses cross, we represent this crossing by the cross-over, too.

After cross-overs the ladder enters **the clause gadget**. This is represented by a surrounding circle and a structure inside it (see Figure 10).

We can easily see that certain problems with the gadget can occure only when representing vertices $abcdB$ and their neighbors. How it is done is illustrated in Figure 11. We analyze the existing possibilities depending on the positions of polygons P_a, P_c and P_x, P_y.

Case 1 – False/True: We build this representation similar to the cross-over described earlier. Note that here P_a has the y-coordinate of its left touch-point between y-coordinates of left touch-points of P_d and P_b (while the x-coordinates are the same). For the top and bottom touch-point the situation is similar. P_c is created as a copy of P_a slightly scaled down and shifted to the top and to the left, so that it is covered from the bottom by P_a and has the left and top touch-point with the same properties as P_c.

Case 2 – True/True is just a special case of the cross-over.

Case 3 – True/False works like in Case 1, but instead of P_a we start with P_c. Then we create P_a as a copy of P_c, scale P_a down slightly, move it slightly upwards, and then more slightly to the left to obtain the left touch-point of P_a's. The x-coordinate is still larger than the one of P_c, but the top touch-point has the same y-coordinate and its x-coordinate is slightly lower.

Case 4 – False/False: We proceed like in Case 1. After we add P_c, we obtain P_a as a copy of P_a slightly shifted to the right. Then we scale it up to obtain an x-coordinate of the left touch-point of P_a that is less than or equal to that of P_c, and a y-coordinate of the left touch-point of P_a that is between those of P_b and P_d. Then we scale up P_b so that it covers the intersection point of P_a and P_c, and we are done.

The NP-membership of the recognition of P_{hom} graphs has been shown in [24]. We briefly review an argument based on an approach of [16]. We proceed exactly in the same way as for P_{rel} graphs, but we extend the linear program with equations controlling the ratios of side-lengths (for individual polygons). For intersections of a line p with neighboring sides q and r or a polygon A, instead of the inequality checking the ordering of the intersections we add the following equation: $\frac{b_r - b_p}{a_p - a_r} - \frac{b_q - b_p}{a_p - a_q} = k_p \cdot s_A$, where the variable s_A represents the size of a polygon A. For an illustration see Figure 12. Note that the denominators are again constants, again we obtain a linear program. If the shape is fixed, we are done. But even if the polygon P is not given, we can regard the directions of its sides as variables and use the same trick as in the proof of Theorem 3. □

Thus we obtain the following strengthening of the results from [20] showing the existence of polynomial certificate for the recognition problem when the underlying polygons are rational.

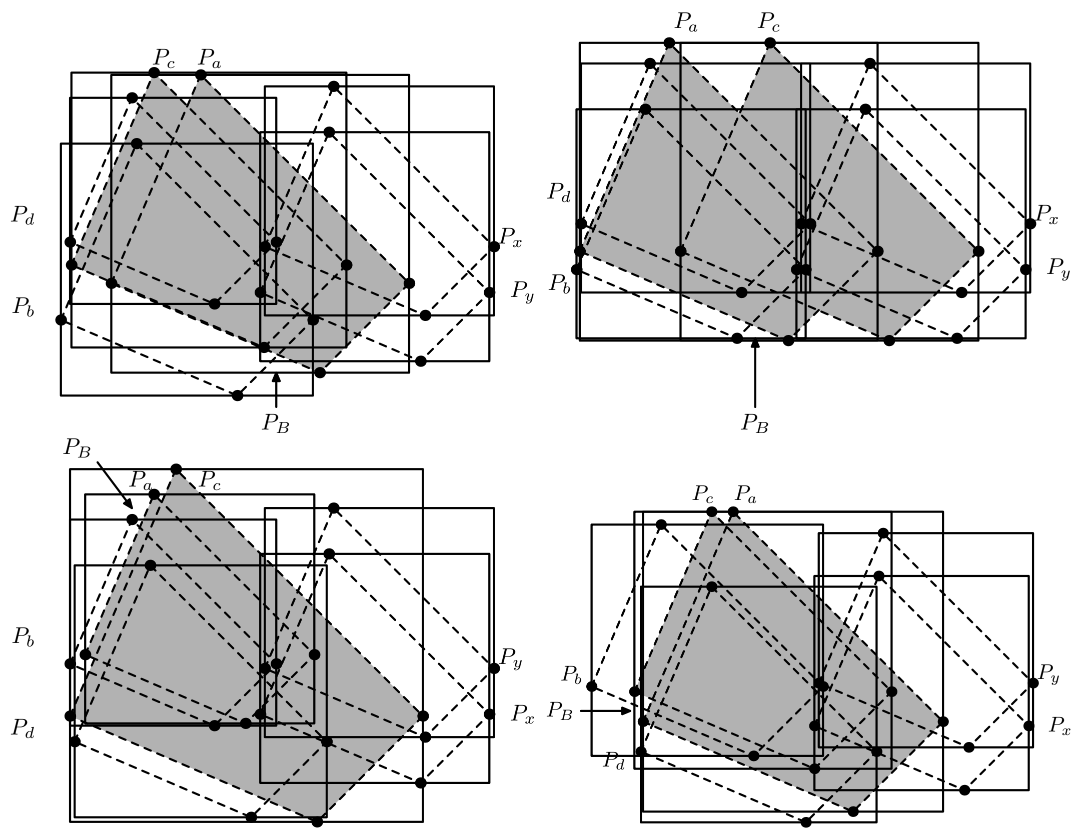

Figure 11: Clause gadget and its representation with respect to truth-assignment for individual variables.

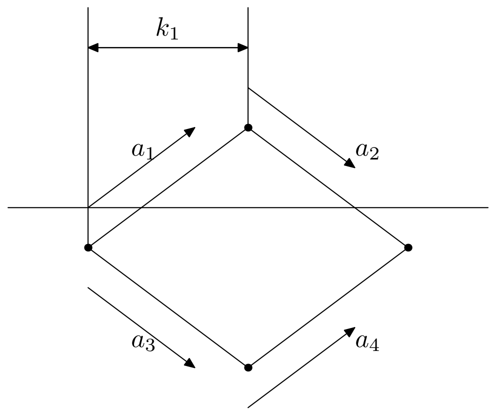

Figure 12: Illustration to the verification of the polynomial certificate of a P_{hom} graph. The parameter k_1 is assigned to the whole polygon and it is designed to verify that all the sides of the polygon were obtained by scaling the original polygon by the same factor.

Theorem 5. *For every fixed k, the problem of deciding whether there exists a convex k-gon P such that an input graph is in P_{hom} is NP-complete.*

This partially solves Problem 6.3 posed in [20], where one asks whether the recognition of intersection graphs of homothetic convex polygons is in NP for all convex polygons.

4 The number of maximal cliques

In this section we are interested in upper and lower bounds for the maximum number of maximal cliques in k_{DIR}-CONV and P_{hom} graphs.

4.1 Upper bounds

The following theorem is the main result of this section.

Theorem 6. *Let G be a k_{DIR}-CONV graph with n vertices. Then G has at most n^k maximal cliques.*

Proof. Fix a representation of $G = (V, E)$ by intersecting polygons, whose every side is parallel to one of the lines $\{\ell_1, \ldots \ell_k\}$. For every $i \in \{1, \ldots, k\}$

let w_i be an arbitrary vector perpendicular to ℓ_i (so for each line we can choose the direction of w_i in two ways).

Let M be a maximal clique in G. For $i \in \{1, \ldots, k\}$, let R_i be the last polygon in M found by a sweeping line in the direction of w_i (if there is more than one such a polygon, we choose an arbitrary one). Let r_i be the supporting line of R_i perpendicular to w_i (which is exactly the sweeping line in direction of w_i that found R_i). Define $\widetilde{M} = \{R \in V : \forall i \in \{1, \ldots, k\}\ R \text{ intersects } r_i\}$. We shall prove that $M = \widetilde{M}$.

First let us show that $M \subseteq \widetilde{M}$. Let R be a polygon in M and consider some $i \in \{1, \ldots, k\}$. The polygon R cannot be farther than r_i in the direction w_i, because r_i is the supporting line of R_i, which is the last polygon in M in direction of w_i. On the other hand, the polygon R cannot lie entirely before r_i (again in the direction of w_i), because then r_i would separate two polygons R and R_i in M. Hence R intersects r_i for every $i \in \{1, \ldots, k\}$ and thus it belongs to $\widetilde{M}$.

Now we will show that $\widetilde{M}$ is a clique, which, combined with the fact that $\widetilde{M}$ contains the maximal clique M, implies that $\widetilde{M} = M$. Suppose there exist polygons $Q, R \in \widetilde{M}$ such that $Q \cap R = \emptyset$. So Q and R can be separated by a line ℓ, which is parallel to one of the faces of Q or R and thus to some $\ell_j \in \{\ell_1, \ldots, \ell_k\}$. Since $Q, R \in \widetilde{M}$, we know that both Q and R intersect r_j. Hence we get two parallel lines ℓ and r_j, one separating Q and R and the other intersecting both of them. It is easy to verify that this is not possible, so $\widetilde{M}$ is a maximal clique.

Note that each maximal clique is uniquely determined by the choice of $r_1, r_2, \ldots, r_k$. For each i, every r_i is determined by R_i, which can be chosen in at most n ways. So finally, the number of maximal cliques in G is at most n^k. $\square$

Theorem 6 implies the following corollary.

Corollary 1. *Let P be a convex polygon in $\mathcal{P}(k)$. Then any n-vertex graph in P_{hom} has at most n^k maximal cliques.*

4.2 Lower bounds

In this section we obtain lower bounds for the maximum number of maximal cliques in k_{DIR}-CONV graphs and the subclasses of this class. First we focus on k-DIR graphs (recall from Theorem 1 that k-DIR $\subseteq k_{\text{DIR}}$-CONV).

Theorem 7. *For any $k \geq 2$, the maximum number of maximal cliques over all n-vertex graphs in k-DIR is $\Omega(n^{k(1-\epsilon)})$, for any $\epsilon > 0$. Moreover, if k is a constant, then the bound is $\Omega(n^k)$.*

Proof. Let n be divisible by k. Let $\{\ell_1, .., \ell_k\}$ be a set of pairwise non-parallel lines. For $i \in \{1, .., k\}$ let S_i be a segment parallel to ℓ_i. We make $\frac{n}{k}$ copies of S_i for every $i \in \{1, .., k\}$. Let $S_{i,s}$ be the s-th copy of S_i. We place all segments $S_{i,s}$ for $i \in \{1, .., k\}$, $s \in \{1, .., \frac{n}{k}\}$, in such a way that $S_{i,s}$ and $S_{j,t}$ ($i, j \in \{1, .., k\}$, $s, t \in \{1, .., \frac{n}{k}\}$) intersect if and only if $i \neq j$ (see Figure 13).

Let $G(n, k)$ be the intersection graph of $\{S_{i,s} : i \in \{1, .., k\}, s \in \{1, .., \frac{n}{k}\}\}$. Notice that $G(n, k)$ is a complete k-partite graph $K_{\frac{n}{k},...,\frac{n}{k}}$ and every maximal clique contains exactly one of $S_{i,1}, .., S_{i,\frac{n}{k}}$ for every $i \in \{1, .., k\}$. Hence the number of maximal cliques in G is $(\frac{n}{k})^k = \Omega(n^{k-\log k}) = \Omega(n^{k(1-\epsilon)})$ for any $\epsilon > 0$.

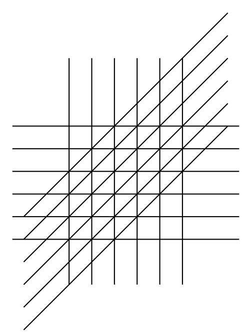

Figure 13: Construction for $k = 3$ and $n = 18$.

□

We can also obtain the same bound as in Theorem 7, but for P_{hom} graphs, where P is a $2k$-gon in $\mathcal{P}(k)$, i.e. P has k pairs of parallel sides.

Theorem 8. *For any $2k$-gon $P \in \mathcal{P}(k)$ there exists an n-vertex graph $G \in P_{hom}$ with $\Omega(n^{k(1-\epsilon)})$ many cliques, for any $\epsilon > 0$. Moreover, if k is a constant, then the bound is $\Omega(n^k)$.*

Proof. Let $L = \{\ell_1, \ell_2, \ldots, \ell_k\}$ be the directions of sides of P, i.e. $P \in \mathcal{P}(L)$. For a copy P' of P, by $s_{1,d}(P')$ and $s_{2,d}(P')$ we shall denote the two sides of P', which are parallel to ℓ_d.

For $d \in \{1, 2, .., k\}$ and $i \in \{1, \ldots, \frac{n}{k}\}$ let $L_{d,i}$ and $R_{d,i}$ be a pair of two disjoint copies of P, such that sides $s_{1,d}(L_{d,i})$ and $s_{2,d}(R_{d,i})$ are parallel and very close to each other in the sense that one can be obtained from the other by a translation by a short vector perpendicular to ℓ_d.

Pairs $(L_{d,1}, R_{d,1}), \ldots, (L_{d,\frac{n}{k}}, R_{d,\frac{n}{k}})$ are placed in such a way that the following conditions are satisfied (see Figure 14, left picture):

- $L_{d,i}$ intersects $R_{d,j}$ for all $i, j \in \{2, \ldots \frac{n}{k}\}$ such that $j < i$,
- $L_{d,i}$ intersects $L_{d,j}$ and $R_{d,i}$ intersects $R_{d,j}$ for all $i, j \in \{2, \ldots \frac{n}{k}\}$.

Observe that in the intersection graph, the vertices $L_{d,1}, \ldots L_{d,\frac{n}{k}}$ (and also $R_{d,1}, \ldots R_{d,\frac{n}{k}}$) form a clique.

Let G_d denote the subgraph induced by $\{L_{d,i}, R_{d,i} \colon i \in \{1, \ldots, \frac{n}{k}\}\}$. In G_d all maximal cliques are of the form: $\{L_{d,i}, L_{d,i+1}, \ldots, L_{d,\frac{n}{k}}, R_{d,1}, \ldots R_{d,i-1}\}$, so there are $\frac{n}{k}$ maximal cliques in G_d for every $d \in \{1, 2, .., k\}$.

Notice that polygons $L_{d,i}$, $R_{d,i}$ can be placed in such a way that $L_{d_1,i}$ and $L_{d_2,j}$ (and, by symmetry $R_{d_1,i}$ and $R_{d_2,j}$) intersect each other for all distinct $d_1, d_2 \in \{1, , .., k\}$ and all $i, j \in \{1, \ldots, \frac{n}{k}\}$ (see Figure 14, right picture).

Hence every maximal clique in G is a disjoint union of k cliques, each in some G_d for $d \in \{1, 2, .., k\}$. Therefore, the number of maximal cliques is $(\frac{n}{k})^k = \Omega(n^{k - \log k}) = \Omega(n^{k(1-\epsilon)})$ for any $\epsilon > 0$. $\square$

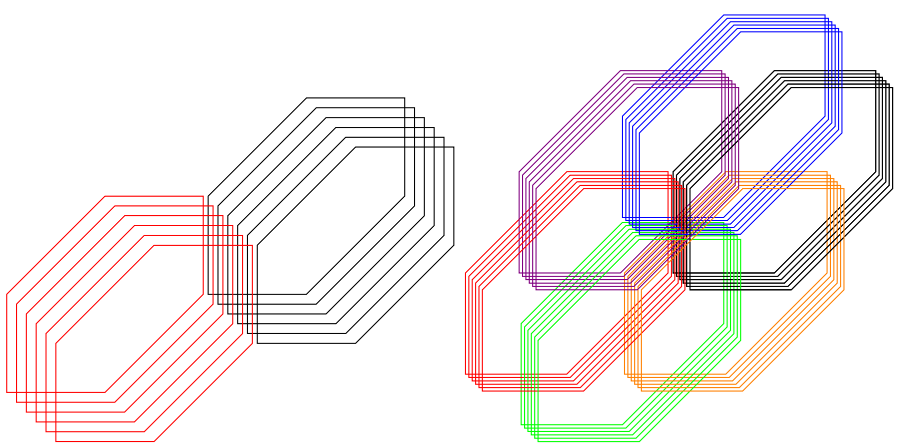

Figure 14: *Left:* A placement of polygons $L_{d,i}$ and $R_{d,i}$ in the construction from Theorem 8. *Right:* The representation of G.

Observe that the construction in fact works for $P_{translate}$ graphs, as we do not use any scaling. Also note that Theorem 8 gives a tight bound for P being a parallelogram.

We can provide a very similar construction for all regular polygons, even if the number of sides is odd. However, in this case the number of maximal cliques is much lower (although still increasing with k).

Theorem 9. *For any k and any regular k-gon P there exists an n-vertex graph $G \in P_{hom}$ with $\Omega(n^{\lfloor \frac{k}{2} \rfloor (1-\epsilon)})$ maximal cliques, for any constant $\epsilon > 0$. Moreover, if k is a constant, then a bound $\Omega(n^{\lfloor \frac{k}{2} \rfloor})$ holds.*

Proof. The case where k is even is covered by Theorem 8. For the case where k is odd, a construction very similar to the proof of Theorem 8 works.

For simplicity, set $q := \lfloor k/2 \rfloor$. Let $F_1, F_2, .., F_q$ be q consecutive sides of P. For each $d \in \{1, 2, .., q\}$, let c_d denote the corner of P which lies opposite the side F_d. For a copy P' of P, by $f_d(P')$ (resp., $c_d(P')$) we shall denote the appropriate side (resp., corner) of P'.

For every $d \in \{1, 2, .., q\}$ and $i \in \{1, \dots, \frac{n}{q}\}$ we take a pair $L_{d,i}$, $R_{d,i}$ of copies of P and place them in such a way that they are disjoint, but $F_d(L_{d,i})$ is very close to $c_d(R_{d,i})$ (see Figure 15, left picture).

Pairs $(L_{d,1}, R_{d,1}), \dots, (L_{d,n/q}, R_{d,n/q})$ are placed is such a way that the following conditions are satisfied (again, refer to Figure 15, right picture):

- $L_{d,i}$ intersects $R_{d,j}$ for all $i, j \in \{2, \dots n/q\}$ such that $j < i$,
- $L_{d,i}$ intersects $L_{d,j}$ and $R_{d,i}$ intersects $R_{d,j}$ for all $i, j \in \{2, \dots n/q\}$.

By the same reasoning as in the previous proof one can verify that the intersection graph for this configuration of polygons has $\Omega\left((n/q)^q\right)$ maximal cliques. □

Again, observe that the construction above in fact works for $P_{translate}$ graphs. We strongly believe that a similar construction can be conducted for any polygon P with at least 4 directions of sides.

Conjecture 1. For every $k \geq 4$ and every convex k-gon P there exists an infinite family of P_{hom} graphs with $\Omega(n^{\lfloor k/2 \rfloor}(1-\epsilon))$ maximal cliques, for any $\epsilon > 0$.

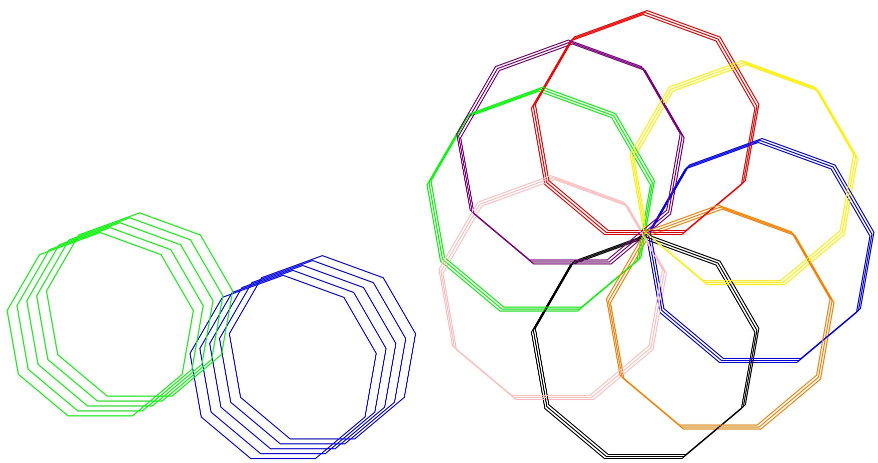

Figure 15: *Left:* A placement of polygons $L_{d,i}$ and $R_{d,i}$ in the construction from Theorem 9. *Right:* The representation of G.

As a last result in this section, we give a general bound for P_{hom} graphs, where P is any convex polygon but a parallelogram. It is a simplified and generalized version of the construction for homothetic triangles, presented by Kaufmann *et al.* [11].

Theorem 10. *If P is not a parallelogram then the maximum number of maximal cliques in an n-vertex graph in P_{hom} is $\Omega(n^3)$.*

Proof. Let F be a face of P and ℓ be a line containing F. Consider the set of vertices of P, which are at the largest distance from ℓ. If there is only one such vertex, denote it by $D(F)$. It there are two such vertices let $D(F)$ denote the face spanned by these two vertices.

Choose a side of P and call it F_1 and let $P_1 = D(F_1)$. Let F_2, F_3 be sides of P adjacent to P_1 and let $P_2 = D(F_2)$ and $P_3 = D(F_3)$.

Let h, r, t, v be four copies of P. By $F_i^h, F_i^r, F_i^t, F_i^v, P_i^h, P_i^r, P_i^t, P_i^v$ we denote the sides and corners in polygons h, r, t, v corresponding to F_i, P_i in polygon P for $i \in \{1, 2, 3\}$, respectively (see Figure 17). We can adjust the sizes and positions of h, r, t, v in such a way that:

1. t and h are touching and F_1^t intersects P_1^h,
2. t and v are touching and F_1^t intersects P_1^v,

3. h and v are touching and F_2^h intersects P_2^v,

4. r and t are touching and F_3^r intersects P_3^t,

5. r and h intersect.

6. r and v intersect.

For every polygon $x \in \{h, r, t, v\}$ we make $\frac{n}{4}$ copies $x_1, ..x_{\frac{n}{4}}$ and move them slightly with respect to the position of h, r, t, v in such a way that:

1. t_i and v_j intersect iff $i \geq j$,

2. h_i and v_j intersect iff $i \leq j$,

3. h_j and t_j intersect iff $i \leq j$,

4. r_i and t_j intersect iff $i \geq j$,

5. r_i and v_j intersect for all $i, j \in \{1, .., \frac{n}{4}\}$,

6. r_i and h_j intersect for all $i, j \in \{1, .., \frac{n}{4}\}$.

For any α, β, γ such that $1 \leq \alpha \leq \beta \leq \gamma \leq \frac{n}{4}$ the set $\{h_1, .., h_\alpha, v_\alpha, .., v_\beta, t_\beta, .., t_\gamma, r_\gamma, .., r_{\frac{n}{4}}\}$ is a maximal clique in G. Hence there are at least $\binom{\frac{n}{4}}{3} = \Omega(n^3)$ maximal cliques in total.

$\square$

5 Towards higher dimensions

In this section we generalize the concept of intersection graphs of convex polygons to arbitrary dimensions. The definitions we use here are straightforward generalizations of the definitions for the 2-dimensional case.

For a polytope P, let $\dim(P)$ denote its dimension. Let ^{d} be the set of all $(d-1)$-dimensional hyperplanes in $\mathbb{R}^d$, containing the origin point. For any $L \in \binom{d}{k}$, by $\mathcal{P}^d(L)$ we denote the set of all polytopes in $\mathbb{R}^d$, whose every facet (i.e., a $(d-1)$-dimensional face) is parallel to one of the hyperplanes in L. By $\mathcal{P}^d(k)$ we denote the set $\bigcup_{L \in \binom{d}{k}} \mathcal{P}^d(L)$.

By $k^d_{\mathrm{DIR}(L)}$-CONV we define the class of intersection graphs of polygons in $\mathcal{P}^d(L)$, while k^d_{DIR}-CONV is defined as $\bigcup_{L \in \binom{d}{k}} k^d_{\mathrm{DIR}(L)}$-CONV.

Now we recall the following separation theorem for polytopes.

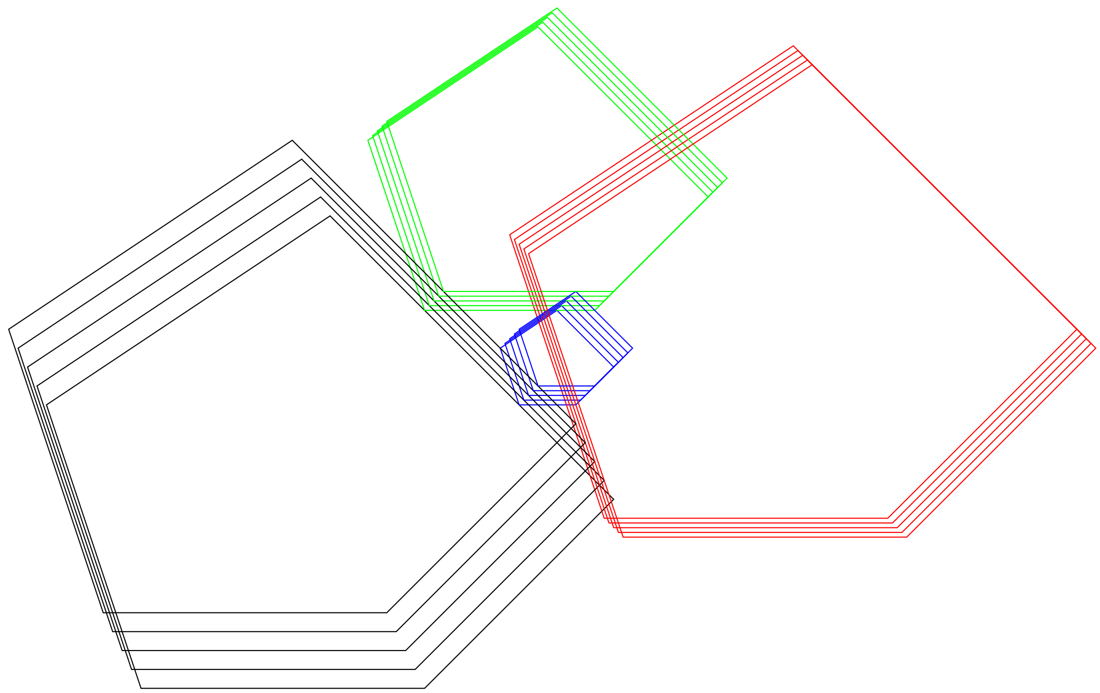

Figure 16: Construction for the lower bound of $\Omega(n^3)$.

Theorem 11 (Wright, 2010 [25])**.** *Consider nonempty convex polytopes P_1 and P_2 in a Euclidean space and suppose that P_1 and P_2 can be properly separated*[1]*. Then there exist parallel hyperplanes H_1 and H_2 properly separating P_1 and P_2, for which* $\dim(H_1 \cap P_1) + \dim(H_2 \cap P_2) \geq \dim(P_1 \cup P_2) - 1$.

From this theorem we can easily obtain the following corollary, generalizing Lemma 1.

Corollary 2. *Let P_1 and P_2 be disjoint convex d-dimensional polytopes. Then they can be separated by a hyperplane H, which is parallel to some d_1-dimensional face of P_1 and to some d_2-dimensional face of P_2, such that $d_1 + d_2 = d - 1$ (either d_1 or d_2 may be equal to 0).*

We shall bound the maximum number of maximal cliques in a similar way as we did in the proof of Theorem 6. Let G be an n-vertex graph in

[1] We say that a hyperplane H *properly separates* convex sets A and B if at least one of those sets does not lie entirely within H.

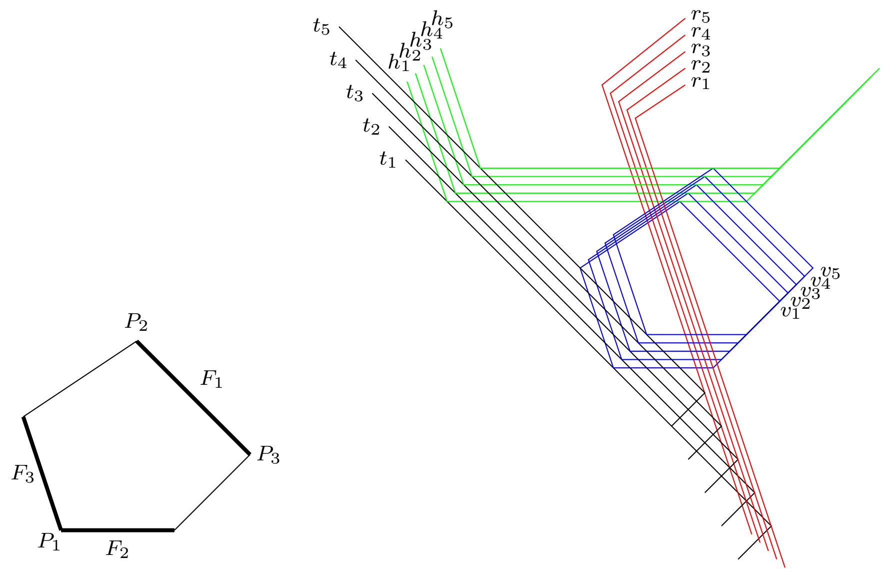

Figure 17: Idea of the construction (for $n = 20$).

k^d_{DIR}-CONV. Fix some representation of G with d-dimensional polytopes, whose every facet is parallel to one hyperplane from $L = \{\ell_1, .., \ell_k\} \in \binom{d}{k}$.

Let $P_1, P_2 \in \mathcal{P}^d(L)$. Let $\mathcal{H}$ be the set of all $(d-1)$-dimensional hyperplanes H containing the origin point, such that H is parallel to some i-dimensional face of P_1 and to some $(d-1-i)$-dimensional face of P_2 (for $i \in \{0, 1, .., d-1-i\}$). Define $h := |\mathcal{H}|$ and let $\mathcal{H} = \{H_1, H_2, .., H_h\}$. Notice that each i-dimensional face is defined as an intersection of some $d-i$ facets. Thus $h \le \sum_{i=0}^{d-1} \binom{k}{d-i}\binom{k}{i+1}$.

For every $j \in \{1, \ldots, h\}$ let w_j be an arbitrary normal vector of H_j. Now we can proceed in the same way as in the proof of Theorem 6, considering sweeping $(d-1)$-dimensional hyperplanes in each direction w_j (for $j \in \{1, 2, .., h\}$), instead of sweeping lines.

This leads us to the conclusion, that the maximum number of maximal cliques in G is at most $n^h \le n^{\sum_{i=0}^{d-1} \binom{k}{d-i}\binom{k}{i+1}}$.

6 Parametrized complexity of the Clique problem in P_{hom} graphs

In this paper we have shown that the number of maximal cliques in any k_{DIR}-CONV graph (and therefore any P_{hom} graph for $P \in \mathcal{P}(k)$) is at most n^k. Tsukiyama *et al.* [23] presented an algorithm enumerating all the maximal cliques in an n-vertex graph in time $O(n^3 \cdot C)$, where C is the number of maximal cliques. Thus the Clique problem can be solved in time $O(n^{k+3})$ for any $G \in k_{\mathrm{DIR}}$-CONV (even if the geometric representation is not known), and therefore is in XP when parameterized by k. One can observe that the proof of Theorem 6 yields a slightly better, $O(k \cdot n^{k+2})$ algorithm for this problem. However, it requires that the geometric representation of the input graph is given.

It is interesting to know if the problem is in FPT, or more generally, to answer the following question.

Problem 1. What is the parametrized complexity of the Clique problem for P_{hom} graphs (parametrized by the number of directions of sides of P)?

Acknowledgements

We thank Michael Kaufmann for valuable discussions on the topic.

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