Frequency-domain Method for $\mathcal{H}_2$ Optimization of Time-delayed Sampled-data Systems

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Abstract—A frequency domain method based on the parametric transfer function concept is developed for the design of digital control systems with time-delayed continuous-time plants. The method allows to investigate the behavior of sampled-data systems in continuous-time in terms of input-output relations. An algorithm is presented for the design of an optimal digital controller that minimizes a quadratic cost functional under stochastic disturbances. The method is applicable to the design of digital controllers for arbitrary linear continuous-time plants. © 1997 Elsevier Science Ltd.

1. Introduction

The third approach, which is used in this paper, is based on the parametric transfer function (PTF) concept (Rosenwasser, 1973, 1992a, 1994, 1995a, b; Lampe and Rosenwasser, 1995). It does not call for temporary transmission into auxiliary spaces, and it is applicable for any mathematical description of continuous-time elements of the system. The PTF method allows one to consider systems with distributed parameters, is not based on the separation theorem, and can be used immediately, without auxiliary transformations for systems of arbitrary structure. In contrast to the above-mentioned approaches, the PTF is able to handle signals over the whole time axis $-\infty < t < \infty$ instead of $0 \leq t < \infty$ only. One consequence of this is that only the PTF yields solutions for nonergodic stochastic input signals.

In this paper the PTF method is generalized to digital control systems with time-delayed plants. Previous work on time-delayed sampled-data systems was based on the assumption of the stability of the generalized plant (Chen and Francis, 1991) or the separation theorem (Lennartson, 1989), which leads to restricted applicability (Yould et al., 1976; Åström and Wittenmark, 1984). In Haru et al. (1994) time-delayed sampled-data systems are investigated in a finite-dimensional hybrid state space. The method presented below does not use the separation theorem, and allows one to design $\mathcal{H}_2$-optimal controllers for neutral and unstable plants as well. All this will be done in the frequency domain without the need to change the representation.

The example given in Section 6 shows that it is of practical importance to take pure delay into account in the design procedure, because even a small delay can substantially affect the behaviour of $\mathcal{H}_2$-optimal sampled-data systems.

2. Statement of the problem
Consider a stable closed-loop digital control system with the block diagram shown in Fig. 1, where all of the exogenous inputs, namely the reference signal $r(t)$, the measurement noise $m(t)$ and the disturbance $w(t)$, are centered stationary stochastic processes. The continuous part consists of the time-invariant real rational transfer functions $F(s)$ (plant), $E(s)$ (prefilter) and $G(s)$ (dynamic feedback), as well as two pure time delays. The functions $H(s)$, $G(s)$ and $E(s)$ are assumed to be proper, and $f(s)$ is assumed to be strictly proper. The controlling computer is represented as a series connection of a sampling unit, a digital controller with the transfer function $C(s)$ and a hold circuit with the transfer function $G_D(s)$. The output signal $y(t)$ should follow the reference signal $r(t)$ as closely as possible, with due account of the restrictions on the control action $u(t)$. The principal problems are caused by the fact that the system is nonstationary when it is considered in continuous time, and for stationary stochastic inputs steady-state variances of the error $e = r - y$ and control $u$, denoted by $v_e$ and $v_u$, respectively, are time-dependent and $v_e(t) = v_e(t + T)$, $v_u(t) = v_u(t + T)$, where $T$ is the sampling period. A weighted sum of these variances can be taken as a cost function

$$E(t) = v_e(t) + k^2 v_u(t),$$

(1)

with a real constant $k$ and $E(t) = E(t + T)$. The function (1), theoretically, can be minimized for any $t$, for instance for $t = 0$, but in this case the values $E(t)$ for all other $t$ are ignored. Therefore it is reasonable to minimize a weighted sum of the variances averaged over the period $T$

$$\bar{E} = \bar{v}_e + k^2 \bar{v}_u,$$

(2)

where

$$\bar{v}_e = \frac{1}{T} \int_0^T v_e(t) \, dt, \quad \bar{v}_u = \frac{1}{T} \int_0^T v_u(t) \, dt. $$

(3)
Fig. 1. Sampled-data control system with time delays.

Thus the optimization problem can be posed as follows. Let the transfer functions of the continuous-time elements and the hold circuit, the sampling period $T$, and the spectral densities of the exogenous inputs be given; then find the transfer function of the digital controller $C(s)$ that stabilizes the closed loop in the sense of Lyapunov, and minimizes the cost functional (2).

For comparison, we consider the problem of minimization of the cost function (1) for $r = 0$, i.e. $E(0)$. In the following, a controller that minimizes $E(0)$ will be called a minimum-variance controller (MV controller), while a controller that minimizes $\tilde{E}$ will be called a continuous minimum-variance controller (CMV controller).

3. Construction of the cost functional

Hereinafter, all input signals are assumed to be stochastically independent. The error and the control variance can be determined using the formulae given by Rosenwasser (1992b):

\[ v(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left[ W_m(s, t)S(s)W_m(-s, t) \right] ds + W_m(s, t)S_m(s)W_m(-s, t) ds, \]  
\[ v(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left[ W_m(s, t)[S_m(s) + S_s(s)]W_m(-s, t) \right] ds + W_m(s, t)S_m(s)W_m(-s, t) ds, \]

where $W_m(s, t)$ stands for the PTF from the input $f(t)$ to the output $g(t)$, and $S_s(s)$ denotes the spectral density of the stationary stochastic process $s(t)$.

Now we introduce for any image $P(s)$ the displaced pulse-frequency response

\[ \varphi_P(T, s, t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} P(s + k\omega) e^{-k\omega t} \]  
and the discrete Laplace transform

\[ \varphi_P(T, s, t) = \varphi_P(T, s, t)e^{\omega t}. \]  

For functions of $s$ or $t = e^{-sT}$ we shall denote

\[ F(s) = F(-s), \quad F(t) = F(-t). \]  

The PTFs of the system shown in Fig. 1 can be written in the form (Rosenwasser, 1992b):

\[ W_m(s, t) = -F \tilde{C} \varphi_{FH,G}(s, t), \]  
\[ W_m(s, t) = -F \tilde{G} \varphi_{FH,G}(s, t), \]  
\[ W_m(s, t) = -F \tilde{G} \varphi_{FH,G}(s, t) - F, \]  
\[ W_m(s, t) = W_m(s, t) - \tilde{H} \varphi_{FH,G}(s, t), \]

with the notation

\[ H(s) = H(s)e^{-\tau s}, \quad G(s) = G(s)e^{-\tau s}, \]  
\[ \tilde{C}(s) = \left[ 1 + CG(s)H(s)H(s) \right]^{-1}. \]

Substituting (8) into (4) and (5), and transforming the integrals to finite integration limits, we can represent (1) in the form

\[ E(t) = v_0 + E_i(t), \]

where

\[ v_0 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left[ S + FF^*S \right] ds, \]
\[ E_i(t) = \frac{T}{2\pi i} \int_{-\infty}^{\infty} \left[ A(s, t)\tilde{C} \tilde{C}^* - B(s, t)\tilde{C} - B(-s, t)\tilde{C}^* \right] ds, \]

with the notation

\[ A(s, t) = \left[ \partial_{\varphi_{FH,G}(s, t)} - \partial_{\varphi_{FH,G}(s, t)}( s, t) \right] \partial_{\varphi_{FH,G}(s, t)}, \]
\[ B(s, t) = \partial_{\varphi_{FH,G}(s, t)}( s, t) \partial_{\varphi_{FH,G}(s, t)}, \]
\[ U(s) = F(s), \quad V(s) = FF^*S. \]

So far, spectral densities of the inputs are assumed such that the functions $U(s)$, $V(s)$ and $S + FF^*S$ are strictly proper.

It can be verified that the integral (10) can be represented as

\[ E_i(t) = \frac{T}{2\pi i} \int_{-\infty}^{\infty} \left[ A(s, t)\tilde{C} \tilde{C}^* - B(s, t)\tilde{C} - B(-s, t)\tilde{C}^* \right] \frac{d\zeta}{\zeta}, \]

where the contour integral is taken anticlockwise along the unit circle, and $A(s, t)$ and $B(s, t)$ are real rational functions of $\zeta$.

For construction of the cost functional (2), the function (11) has to be averaged over the period $T$, which yields

\[ \tilde{E} = v_0 + \tilde{E}_i, \]

with

\[ \tilde{E}_i = \frac{1}{T} \int_{-\infty}^{\infty} \left[ A(s)\tilde{C} \tilde{C}^* - B(s)\tilde{C} - B(-s)\tilde{C}^* \right] \frac{d\zeta}{\zeta}, \]
\[ \tilde{A}(s) = \frac{1}{T} \int_{-\infty}^{\infty} A(s, t) ds, \quad \tilde{B}(s) = \frac{1}{T} \int_{-\infty}^{\infty} B(s, t) ds. \]

The two last functions are real rational in $\zeta$, because \[ A(s, t) = \frac{1}{T} \left[ \partial_{\varphi_{FH,G}(s, t), T}( s, \zeta) \right] \partial_{\varphi_{FH,G}(s, t), T}, \]
\[ B(s, t) = \frac{1}{T} \partial_{\varphi_{FH,G}(s, t), T}( s, 0), \]
\[ \tilde{A}(s) = \frac{1}{T} \left[ \partial_{\varphi_{FH,G}(s, t), T}( s, 0) \right] \partial_{\varphi_{FH,G}(s, t), T}, \]
\[ \tilde{B}(s) = \frac{1}{T} \partial_{\varphi_{FH,G}(s, t), T}( s, 0). \]

4. System stabilization

The functional (12) can be minimized over stable functions $\tilde{C}(\zeta)$ by the standard Wiener-Hopf technique for discrete-time systems (Chang, 1961). In this case the optimal controller is determined by (9). Nevertheless, this method yields an unstable system if there are unstable plants in the loop. Therefore, in the general case the functional (12) must be minimized over the set of stabilizing controllers. Here the common stability definition in the sense of Lyapunov is used, i.e. for initial-value disturbances. The solution is given below by Theorem 1.

Let us assume the strictly proper function $F_0FHG(s)$ to be irreducible, and its poles may be $s_i (i = 1, \ldots, l)$. In the

\[ s = e^{-sT} \]

\[ \frac{1}{T} \int_{-\infty}^{\infty} \left[ \partial_{\varphi_{FH,G}(s, t), T}( s, \zeta) \right] \partial_{\varphi_{FH,G}(s, t), T}, \]
\[ \frac{1}{T} \left[ \partial_{\varphi_{FH,G}(s, t), T}( s, 0) \right] \partial_{\varphi_{FH,G}(s, t), T}, \]
construction of the set of all stabilizing controllers a major role is played by the function

$$\mathcal{P}_{\mathcal{M}_{1}}([T, \zeta, 0]) = \frac{n(\zeta)}{d(\zeta)},$$  \hspace{1cm} (13)

where \(n(\zeta)\) and \(d(\zeta)\) are polynomials, and \(n(0) = 0\).

**Theorem 1.** Let the following nonpathological conditions hold:

$$e^{T \zeta} \neq e^{T \gamma}, \quad \gamma \neq kT, \quad i \neq k, \quad i, \ k = 1, \ldots, l, \quad G_{0}(\zeta) \neq 0, \quad i = 1, \ldots, l.$$  

Then

(i) the function (13) is irreducible;

(ii) the set of all stabilizing controllers can be represented in the form

$$C(\zeta) = (a_{0} + d\zeta)(b_{0} – n\zeta)^{-1},$$  \hspace{1cm} (14)

where \(Q(\zeta)\) is an arbitrary stable real rational function, and the polynomials \(a_{0}(\zeta)\) and \(b_{0}(\zeta)\) are from an arbitrary solution of the Diophantine equation

$$a_{0}n + bd = 1;$$  \hspace{1cm} (15)

(iii) all stabilizing controllers are causal (physically realizable);

(iv) all stabilizing controllers are insensitive, i.e. the system remains stable under sufficiently small continuous-time process parameter variations.

The proof of Theorem 1 is similar to that given by Rosenwasser (1992b) for systems without pure time delay, and is omitted here.

5. **Optimal controller design**

The aim of this section is the minimization of the functional (12) over the set of functions \(C(\zeta)\) satisfying (9) and (13)-(15).

Let us introduce some notations used below. Any real rational function \(F(\zeta)\) that is free of poles on the unit disk can be expanded in a Laurent series as

$$F(\zeta) = (F(\zeta))^{+} + (F(\zeta))^{-},$$

where \((F(\zeta))^{+}\) is a polynomial, while the strictly proper functions \((F(\zeta))^{+}\) and \((F(\zeta))^{-}\) incorporate only stable (outside the unit disk in the \(\zeta\) plane), or only unstable poles respectively. We denote by \((F(\zeta))^{0}\) the sum of all the terms of \((F(\zeta))^{+}\) that have common poles with the polynomial \(d(\zeta)\).

As shown by Rosenwasser et al. (1996), under the given assumptions (even if \(F(s)\) has poles on the imaginary axis, but \(w(t) \neq 0\)), there exists a factorization

$$d_{\zeta} = K(\zeta)A_{\zeta},$$

where the polynomial \(g(\zeta)\) is defined above. Substituting (26) and (27) in (25) with due account of (20) gives

$$Q(\zeta) = K^{-1} - L_{d},$$  \hspace{1cm} (25)

Since the poles of \(L_{d}(\zeta)\) are not poles of \(L(\zeta)\), the second term in (18) may be written in the form

$$\frac{a_{0}K}{d} = L_{o} + L_{3},$$

where \(L_{3}(\zeta)\) is a real rational function. Then

$$L_{d}(\zeta) = L_{0} + L_{1} + L_{2} + L_{3},$$

Solving (15) with respect to \(a_{0}(\zeta)\) and taking into account the stability of \(K(\zeta)\), we obtain

$$L_{3}(\zeta) = \left(\frac{K}{d_{\zeta}}\right)^{-1},$$

where the polynomial \(g(\zeta)\) is defined above. Substituting (26) and (27) in (25) with due account of (20) gives

$$Q(\zeta) = L_{d} - \frac{a_{0}K}{d},$$

which, on substitution into (14), leads to (19) and (20).

Now let us determine the minimal value of the functional (16). It is immediately seen that, for (25),

$$J(\zeta) = \frac{1}{2\pi} \int \left(\mathcal{M}(\zeta) – B\zeta^{-1} \cdot A_{\zeta}^{-1} \cdot \frac{d\zeta}{\zeta}\right),$$

where

$$M(\zeta) = \left(L_{d}(\zeta) – \frac{a_{0}K}{d}\right)^{-1},$$

hence (21) follows. □

It was shown by Rosenwasser et al. (1996) that for \(w(t) \neq 0\)
the poles of \( F(s) \) are not poles of the function \( L(\xi) \) in (18), and therefore they have to be incorporated into \( L_0(\xi) \). Thus Theorem 2 allows one to design the optimal controller also for neutral plants and leads to a closed-loop system with nonzero stability margin. This is of great importance for the synthesis of optimal digital controllers for real dynamic plants with poles at \( s = 0 \).

**Remark 3.** The method presented above is also applicable to the minimization of the functional (11) for any given \( i \), for instance for \( i = 0 \), which yields the ordinary minimum-variance controller.

6. **Numerical example**

Consider the system of Fig. 1 with

\[
F(s) = \frac{s + 0.5}{s - 0.2}, \quad T = 0.2, \quad C(s) = \frac{1}{s}, \quad S(s) = \frac{4}{s^2 + 4}.
\]

The transfer function of the plant \( F(s) \) has an unstable and a neutral pole, and a zero-order hold is used as the hold circuit.

The poles are \( \xi = 1 \) and \( \xi = 0.819 \) of \( B^* \), caused by the poles \( \tau = 0 \) and \( s = 1 \) of the transfer function \( F(s) \), are not poles of \( L(\xi) \) (Rosenwasser et al., 1996). Therefore the separation of \( L_0(\xi) \) in (17) gives

\[
L_0 + (L_2)^2 + (L_3)^2 = 0.1826(\xi - 1.094)(\xi - 1.160)
\]

In this case \( g(\xi) = 1 \); therefore

\[
\frac{K^*}{n^0} = 0, \quad \xi = L_0 + (L_2)^2 + (L_3)^2.
\]

Equations (20) yield the discrete transfer function of the CMV controller:

\[
C(\xi) = \frac{1.924 - 25.304\xi + 7.038\xi^2}{1 - 0.569\xi - 0.978\xi^2 + 0.610\xi^3}
\]

The average output variance, calculated by (12), is \( \bar{v}(\xi) = 0.0348 \). The output variance at the sampling instants, calculated by (11) for \( i = 0 \), is \( r_i(0) = 0.0256 \).

For comparison, let us design the MV controller under the same assumptions, providing the minimal value \( E(t) \) for \( i = 0 \). Using Remark 3, we obtain

\[
C(\xi) = \frac{14.198 - 19.551\xi + 6.008\xi^2}{1 - 0.701\xi - 0.760\xi^2 + 0.521\xi^3}
\]

The average output variance, calculated by (12), is \( \bar{v}(\xi) = 0.0348 \). The output variance at the sampling instants, calculated by (11) for \( i = 0 \), is \( r_i(0) = 0.0256 \).

Figures 3 and 4 present the transients of the systems with the controllers (30) and (31) respectively when a unit step acts on the input \( u(t) \).

With the MV controller (31), the transient (Fig. 3) is essentially oscillatory, and fades out very slowly. At the same time, the transient in the system with the CMV controller (Fig. 4) is much faster; this system has a greater stability margin. Thus the digital controller must be optimized, in the general case, with respect to the cost functional (2), that penalizes the continuous-time behaviour.

Let us investigate the effect of the pure time delay \( 0 < \tau_1 \leq T \) on the performance of \( \xi \)-optimal systems. Figure 5 shows a plot of \( \bar{v}(\xi) \) versus the delay \( \tau_1 \) for MV and CMV control. With some \( \tau_1 \), the MV-controlled system has a very large average output variance, and is practically inoperable.

This is caused by the fact that there is a zero in the function (13) that is close to the oscillatory stability bound \( \xi = 1 \) (Hara et al., 1989). This zero (or if it is not stable, the corresponding stable zero) becomes a pole of the MV-controller and this causes large intersample oscillations. This effect is demonstrated for the MV controller (31) designed for \( \tau_1 = 0.093 \).

7. **Conclusions**

The paper has presented a frequency-domain method for the design of an \( \xi \)-optimal digital controller for time-delayed continuous-time plants under stationary, possibly nonergodic stochastic disturbances. The method is based on the parametric transfer function concept, which allows one to investigate the continuous-time behaviour of sampled-data systems in terms of input–output relations. A weighted sum of the average variances of the continuous-time signals is taken as a performance criterion; thus the continuous-time behaviour is penalized. A procedure has been presented that allows one to design optimal digital controllers also for unstable and neutral plants.

The performance of MV- and CMV-optimal systems has been compared. It has been shown that in many cases MV control can cause a large increase in the average variance and highly oscillatory transient responses. Therefore it is recommended, in the general case, to use CMV controllers.

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**Fig. 2. Special structure of Fig. 1.**
Fig. 3. Transient using CMV control.

Fig. 4. Transient using MV control.

Fig. 5. Average variance using MV and CMV control.
which provide for a satisfactory performance of the system in continuous time.

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