The Geometry of Minkowski Spaces — A Survey.
Part II

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MSC 2000 Subject Classification: Primary 46-02; secondary 52A21, 46B20.
Abstract. In this second part of a series of surveys on the geometry of finite dimensional Banach spaces (Minkowski spaces) we discuss results that refer to the following three topics: bodies of constant Minkowski width, generalized convexity notions that are important for Minkowski spaces, and bisectors as well as Voronoi diagrams in Minkowski spaces.

Keywords: Minkowski Geometry, Minkowski spaces, finite dimensional normed spaces, bisectors, Voronoi diagrams, bodies of constant width, generalized convexity

1 Introduction

This paper is the second part of a planned series of surveys on Minkowski Geometry, which is the geometry of finite dimensional normed linear spaces (= Minkowski spaces). The origins and basic developments of Minkowski Geometry are connected with names such as Riemann, Minkowski and Busemann, see the Preface of [283] and [196, § 2] for more information. This field can be located at the intersection of Finsler Geometry, Banach Space Theory and Convex Geometry, but it is also closely related to Distance Geometry (in the spirit of Menger and Blumenthal [28]) and was enriched by many results from applied disciplines such as Operations Research, Optimization, Theoretical Computer Science and Location Theory.

The motivation for this series of surveys is given by the facts that there are hundreds of papers in Minkowski Geometry, widespread in very different fields, previous surveys are old, and the recent excellent monograph [283] covers mainly the analytical part of the theory.

In Part I [196] we placed special emphasis on planar results, in some cases with simplified proofs. Thus Part I can be seen as fundamental for the parts following it. In Part II we survey three topics from Minkowski Geometry that are very geometric in nature and show interesting relations to further disciplines, such as Classical Convexity, Abstract Convexity, Computational Geometry and the Foundations of Geometry.

Bodies of constant width are well studied in Convex Geometry. The extensive knowledge on these special convex bodies in Euclidean space is summarized in the surveys [60] and [131]. Although these surveys and the monograph [283] also contain material on bodies of constant width in Minkowski spaces, a complete summary is missing. Our first
section below is the first and, as we believe, complete survey on this subject. It gives an example of how strongly Classical Convexity and Minkowski Geometry are related to each other.

**Generalized convexity notions** in the sense of metric convexity were studied in Distance Geometry and Abstract Convexity. Some of these notions are especially interesting in the context of Minkowski Geometry and applications thereof, e.g. in Location Science and Computational Geometry. We present an overview, particularly showing the state of the art regarding the theory of \(d\)-convex sets. We also take into consideration other types of generalized convexity notions in normed linear spaces.

Geometric properties of **bisectors** in Minkowski spaces yield various deep characterizations of special normed linear spaces, or of Minkowski spaces within more general classes of spaces. This viewpoint is also related to Foundations of Geometry. Furthermore, the study of **Voronoi diagrams** is based in a natural manner on the geometry of bisectors and their extensions. During the last decade, these topics were mainly investigated in Computational Geometry, in many cases even for linear spaces equipped with a nonsymmetric unit ball. Since this more general approach is still based on the tools typical for Minkowski Geometry, it is also considered in the third part of this survey.

Finally we mention here the contents of our planned Part III which is in preparation: smoothness of norms, Chebyshev sets, isometries (Hyers-Ulam and Beckman-Quarles type theorems, as well as Banach-Mazur distance), isoperimetric problems, and various notions of orthogonality and angle measures. A Part IV is also planned and will contain subjects more related to Discrete Mathematics.

We finish this introduction by some notation used in all three sections below. We let \(\mathbb{M}^d\) denote a \(d\)-dimensional Minkowski space, i.e., a \(d\)-dimensional real normed linear space with norm \(\| \cdot \|\) and unit ball \(B := \{ x \in \mathbb{M}^d : \| x \| \leq 1 \}\). As usual we write \(\mathbb{E}^d\) for the \(d\)-dimensional Euclidean space. Points and vectors are in boldface. The origin is denoted by \(o\). A **convex body** \(K \subset \mathbb{M}^d\) is a compact, convex subset of \(\mathbb{M}^d\) with nonempty interior. For two distinct points \(x, y \in \mathbb{M}^d\) we denote the linear segment joining them by \([x, y]\), and the ray emanating from \(x\) and passing through \(y\) by \([x, y)\). (This notation for a segment differs from the notation in Part I. This is so that it corresponds to the usual notation for \(d\)-segments, see the section on generalized convexity notions below). For further general notation and definitions we refer to Part I [196].

## 2 Bodies of constant width in Minkowski spaces

### 2.1 Introduction

If the distance between any pair of parallel supporting hyperplanes of a convex body \(K\) in \(d\)-dimensional Euclidean space \(\mathbb{E}^d, d \geq 2\), is the same, then \(K\) is called a **body of constant width**. Since the time of Euler or even further back it has been known that there are many non-spherical bodies of constant width, the most famous one being the Reuleaux triangle, cf. [232].

Most of the results on bodies of constant width derived up to 1934 are collected in the classical monograph [40]. More recent surveys, showing the pertinent results obtained before 1993, are [60] and [131, Section 5]. Various related concepts (like bodies of constant brightness, etc.) are also discussed in the books [98] and [244]. And there is even a
monograph on bodies of constant width, see [208].

It is natural to extend the notion of bodies of constant width to Minkowski spaces, and in [60] and [131] this more general point of view is also taken into consideration. In addition, we refer to Chapter 4 of Thompson’s monograph [283] and [36, Chapter V], where wider discussions of some geometric properties of bodies of constant width in Minkowski spaces are given. However, with many references not cited in [60], [131] and [283], and since the literature on such bodies in Minkowski spaces is sufficiently grown and widespread, one is motivated enough to write an independent summary on this topic. Nevertheless, for practical reasons our sequence of subsections will follow that from [60] and [131].

2.2 Basic Notions

We start by introducing Minkowski analogues of various notions from (Euclidean) convex geometry. Note that a unit functional \( \varphi \) in the dual \((\mathbb{M}^d)^*\) of \(\mathbb{M}^d\) is associated in a natural way with a directed hyperplane of \(\mathbb{M}^d\). The Minkowski support function of a convex body \(K\) is defined by \(h(K, \varphi) := \sup\{\varphi(x) : x \in K\}\) for any unit functional \(\varphi \in (\mathbb{M}^d)^*\). Thus \(h(K, \varphi)\) is the signed distance from the origin \(o\) to the supporting hyperplane \(H = \{x : \varphi(x) = 1\}\) of \(K\). The Minkowski width (or Minkowski breadth) of \(K\) in direction \(\varphi\) is given by \(w(K, \varphi) = h(K, \varphi) + h(K, -\varphi)\), and \(K\) is said to be of constant Minkowski width \(w(K) \in \mathbb{R}^+\) if \(w(K, \varphi) = w(K)\) for any unit functional \(\varphi\).

For later use, we also introduce the central symmetrical \(\Delta K\) of a convex body \(K \subset \mathbb{M}^d\), defined by

\[
\Delta K := \frac{1}{2}(K + (-K)),
\]

where the (Minkowski or) vector addition is defined by \(K_1 + K_2 := \{x + y : x \in K_1, y \in K_2\}\). A segment \([p, q]\), whose different endpoints \(p, q\) are from the boundary \(bd K\) of \(K\), is called a diametrical chord (or affine diameter) of \(K\) if there exist two parallel supporting hyperplanes \(H_1 \neq H_2\) of \(K\) such that \(p \in H_1\) and \(q \in H_2\); in this situation we say that \([p, q]\) is generated by \(H_1, H_2\). In Euclidean space \(\mathbb{E}^d\), a chord \([p, q]\) of \(K\) is called a normal of \(K\) at \(p \in bd K\) if \(K\) has a supporting hyperplane \(H_1\) with \(p \in H_1\) such that \([p, q]\) is orthogonal to \(H_1\). If, moreover, there is a supporting hyperplane \(H_2 \ni q\) which is parallel to \(H_1\), then \([p, q]\) is called a double normal of \(K\). To transfer these notions to Minkowski spaces, we first need a suitable notion of orthogonality. Let \(H\) be a hyperplane and \(u \neq o\) a vector in \(\mathbb{M}^d\). The vector \(u\) (or a line with \(u\) as its direction vector) is said to be normal to \(H\), denoted by \(u \perp H\), if the two supporting hyperplanes of the unit ball \(B\) which are parallel to \(H\) generate a diametrical chord of \(B\) having direction \(u\) and passing through \(o\). (Thus, if \(B\) is not strictly convex there may be infinitely many directions normal to \(H\).) A chord \([p, q]\) of a convex body \(K \subset \mathbb{M}^d\) is a Minkowski normal of \(K\) at \(p \in bd K\) if \(K\) has a supporting hyperplane \(H_1 \ni p\) such that \([p, q]\) is normal to \(H_1\). Also, \([p, q]\) is a Minkowski double normal of \(K\) if, in addition, there is a supporting hyperplane \(H_2 \ni q\) parallel to \(H_1\).

Finally, a point \(x\) from the interior \(\text{int} K\) of \(K \subset \mathbb{M}^d\) is called an equichordal point of \(K\) if all chords of \(K\) passing through \(x\) have the same Minkowski length.

2.3 Geometric properties and characterizations

There are various well known properties of bodies of constant width in \(\mathbb{E}^d, d \geq 2\), characterizing them within the class of all \(d\)-dimensional convex bodies. A collection of such
characterizations is given in [60, Section 2], and the analogous characterizations of bodies of constant Minkowski width can be summarized by

**Theorem 1.** A convex body $K \subset \mathbb{M}^d$ is of constant Minkowski width $w(K) \in \mathbb{R}^+$ if and only if one of the following statements holds true.

1. The central symmetral $\triangle K$ satisfies $\triangle K = \frac{w(K)}{2} \cdot B$.
2. For each pair $H_1, H_2$ of parallel supporting hyperplanes of $K$ and every direction $u$ normal to $H_1, H_2$ there exists a diametrical chord of $K$ having direction $u$ and generated by $H_1, H_2$.
3. For each pair $H_1, H_2$ of parallel supporting hyperplanes of $K$, every diametrical chord of $K$ generated by $H_1, H_2$ is normal to $H_1, H_2$.
4. All diametrical chords of $K$ have length $w(K)$.

Proofs of the equivalences of (1) and (2) to constant Minkowski width are presented in [85], and the analogues referring to (3) and (4) are verified in [60, Section 2], see also [88] and [127].

Our next statement (I) is also proved in [85], and (II) is an easy consequence of it. In both (I) and (II) there are now restrictions on the unit ball.

**Theorem 2.** In a Minkowski space $\mathbb{M}^d$ with smooth and strictly convex unit ball the following statements hold true.

1. A convex body $K \subset \mathbb{M}^d$ is of constant Minkowski width if and only if any two parallel Minkowski normals of $K$ coincide.
2. A convex body $K \subset \mathbb{M}^d$ is of constant Minkowski width if and only if every chord $[p, q]$, which is a Minkowski normal of $K$ at $p$, is also a Minkowski normal of $K$ at $q$.

It should be noticed that without any restrictions on $B$ the above coincidences of Minkowski normals still imply that $K$ is of constant Minkowski width, but the converse may fail to be true. Vrećica [298] has shown that a convex body $K \subset \mathbb{M}^d$ is of constant Minkowski width if and only if for all $x, y \in \text{int } K$ there is a set $C$ of constant Minkowski width such that $C \subseteq \text{int } K$ and $x, y \in \text{bd } C$. In [189] it is proved that a convex body $K$ in the Euclidean plane is of constant (Euclidean) width $w(K) = \text{diam } K$ if and only if any two mutually perpendicular chords of $K$ with a common point have total length not smaller than $\text{diam } K$. One might ask for a characterization of those norms where the analogous statement holds, if Euclidean perpendicularity of the chords is replaced by normality.

The formulation of the monotonicity lemma in Minkowski spaces involves only Minkowski balls, cf. [196, Section 3.5] or [283, Lemma 4.1.2]. In [132] Heppes proves a characterization theorem which can be treated as a monotonicity lemma for bodies of constant width in $\mathbb{E}^2$. Grünbaum and Kelly [123] extend one of the implications from [132] to strictly convex Minkowski planes, and in [19] the characterization theorem of Heppes is extended to arbitrary Minkowski planes, namely as the two-dimensional part of the following.
Theorem 3. Any hyperplane section $S$ of a convex body $K \subset \mathbb{M}^d$ of constant Minkowski width splits $K$ into two compact, convex sets such that at least one of them has the same diameter as $S$.

For $d = 2$ this property characterizes the bodies of constant Minkowski width within the class of two-dimensional convex bodies.

Similar characterization theorems, referring to double normals and the unimodality of chord parametrizations of planar curves, are obtained in [16] and [17]. A further characterization of bodies of constant Minkowski width is given in [134].

Another type of result is the characterization of special representatives within the class of bodies of constant Minkowski width. For example, Petty and Crotty [220] prove

Theorem 4. If a convex body $K \subset \mathbb{M}^d$ has constant Minkowski width and, in addition, an equichordal point, then $K$ is homothetic to the unit ball $B$.

In a different way, a part of their proof was earlier obtained by Hammer [127], together with further observations on diametrical chords in Minkowski planes.

Hammer and Smith [128] prove that if all binormal chords of a curve of constant width in the Euclidean plane divide its circumference into two equal parts, then it is a circular disc. The Minkowski version of this theorem is announced there without proof.

Let $r$ denote the inradius of a convex body $K \subset \mathbb{M}^d$, i.e., the largest real number such that $K$ contains some translate of $rB$, and $R$ be the circumradius of $K$, that is the smallest real number such that $K$ is contained in some translate of $RB$. The boundaries of these translates of $rB$ and $RB$ are said to be inspheres and circumspheres of $K$, respectively. Chakerian [57] (see also [88]) shows that the following statement holds.

Theorem 5. If a convex body $K \subset \mathbb{M}^d$ has constant Minkowski width $w \in \mathbb{R}^+$, then the equality

$$r + R = w$$

holds. Moreover, the corresponding insphere and circumsphere of $K$ are concentric.

Conversely, these properties do not imply constant width in $\mathbb{M}^d$, but a generalization of Theorem 5 is given in [240]; cf. Section 2.6 below.

Some geometric properties of planar bodies of constant width in the Minkowski plane $(\mathbb{M}^2)^*$, whose unit ball is the isoperimetrix $I$ (i.e., the convex figure of minimal perimeter for given area) of the original plane $\mathbb{M}^2$, are discussed in [150] and [55]. For example, if $C$ is a figure of constant width $w \in \mathbb{R}^+$ relative to $(\mathbb{M}^2)^*$, then $A(C) + A(C, -C) = \frac{w}{2} \cdot A(I)$, where $A(C)$ is the area of $C$ and $A(C, -C)$ denotes the mixed area (cf. [244, § 5.1]) of $C$ and $-C$. They also give further relationships between $\mathbb{M}^2$ and $(\mathbb{M}^2)^*$, e.g.:

1. if $C$ has constant width $w$ relative to $(\mathbb{M}^2)^*$ and $P(C)$ is the perimeter of $C$ relative to $\mathbb{M}^2$, then $P(C) = w \cdot A(I)$,

2. if $C$ is a smooth convex curve of constant width relative to $(\mathbb{M}^2)^*$ such that each of its diametrical chords bisects the Minkowski circumference (or the Minkowski area), then $C$ is homothetic to $I$. 
2.4 Special bodies of constant width

The simplest figure of constant width \( w \in \mathbb{R}^+ \) in \( \mathbb{E}^2 \) apart from a circle is the Reuleaux triangle whose first mechanical usage is ascribed by Reuleaux to Hornblower, the inventor of the compound steam-engine, see [232, § 155]. It is bounded by three circular arcs of radius \( w \) which are centred at the vertices of an equilateral triangle (with side-length \( w \)). There are different ways to define its analogue for Minkowski planes. Replacing the notions “circular” and “equilateral” by their Minkowski analogues, Ohmann [211], Chakerian [56] and Wernicke [300] describe Minkowski Reuleaux triangles as extremal figures with respect to certain metrical problems, see also Section 2.8 below. In addition, the planes with parallelograms or centrally symmetric hexagons as unit circles are the only ones where Reuleaux triangles exist as Minkowski circles, cf. [300]. The paper [231] of Reimann is also geometrically related to Reuleaux triangles in Minkowski planes. Sallee [237] extends these considerations by constructing certain Minkowski Reuleaux polygons.

Another type of Reuleaux polygon in \( \mathbb{M}^2 \) has been described by Hammer [127]. Its Euclidean variant yields figures of constant width bounded by finitely many circular arcs of possibly different radii (see Rademacher and Toeplitz [228, p. 167] for the Euclidean case).

Petty [216] uses the Minkowski analogue of the notion “evolute” to construct special curves of constant width in Minkowski planes.

2.5 Completeness in Minkowski spaces

A bounded subset \( C \) of \( \mathbb{M}^d \) is called complete (or diametrically complete or diametrically maximal) if it cannot be enlarged without increasing its diameter \( \text{diam} \ C := \sup_{x,y \in C} \|x-y\| \). In other words, \( C \) is complete if it is the intersection of all translates \( x + \lambda B \) with \( x \in C \) and \( \lambda \geq \text{diam} \ C \). Any complete set \( C \) having the same diameter \( \text{diam} \ C = \text{diam} \ K \) as a convex body \( K \subset \mathbb{M}^d \) with \( K \subset C \) is said to be a completion of \( K \). Various results on complete sets in Minkowski spaces are compiled in [36, Chapter V] and [112]. In Euclidean space, the following two statements hold true.

- A convex body \( K \subset \mathbb{E}^d \) is complete if and only if it is of constant width.
- Every convex body \( K \subset \mathbb{E}^d \) has at least one completion.

The first statement is the classical theorem of Meissner, see [204] and [40, §§ 64], and the second one is known as Pál’s theorem, cf. [213] and [40, §§ 64]. It is easy to see that constant width implies completeness also in Minkowski spaces, see, e.g., [88, § 2] for an explicit proof. For \( d = 2, 3 \) and smooth unit balls Meissner [204] investigated the converse, followed by Kelly [149] without restrictions regarding the dimension and smoothness. However, the proofs in [204] and [149] with respect to this converse implication are erroneous for \( d \geq 3 \): there are complete sets that are not of constant width, even if the unit ball is smooth and strictly convex. Such an example was constructed by Eggleston [85], who also gives an example with a polyhedral unit ball. Thus, the extension of Meissner’s classical theorem to Minkowski spaces is true only for \( d = 2 \). However, in higher dimensions the completeness of \( K \subset \mathbb{M}^d \) implies constant width if \( K \) is smooth and strictly convex, cf. [88, Theorem (3.18)]. However, there is a related equivalence theorem holding for all Minkowski spaces. A convex body \( K \subset \mathbb{M}^d \) is said to have the spherical...
intersection property if \( K \) is the intersection of all balls with centre \( x \in K \) and radius \( \text{diam} \ K \), cf. also Section 2.6. Using this notion Eggleston [85] also proves

**Theorem 6.** A compact set \( X \subset \mathbb{M}^d \) is complete if and only if it has the spherical intersection property, or if and only if each boundary point of \( X \) has distance \( \text{diam} \ X \) from at least one other point of \( X \).

An alternative proof of the first statement in this theorem can be found in [88, § 3].

Eggleston [85] compares Minkowski spaces in which the classes of complete sets, sets of constant width and spheres coincide (cf. also [88, § 3]). In [84] he erroneously states that the only Minkowski spaces in which sets of constant width are necessarily balls are those whose unit ball \( B \) is a paralleloplane. In order to obtain the right answer to this question, one has to look at the classification of all irreducible sets, i.e., of all convex bodies \( K \) centred at the origin for which the identity \( K = Q + (-Q) \) is only possible if \( Q \) is centrally symmetric, see Grünbaum [121] and Shephard [249] for first investigations in this direction. A thorough study of irreducible polytopes (i.e., polyhedral unit balls for which sets of constant Minkowski width are necessarily balls) is given by Yost [304], see also [244, § 3.3]. For example, each convex \( d \)-polytope, \( d \geq 3 \), centred at the origin and having less than \( 4d \) vertices is irreducible. Investigating four-dimensional polyhedral unit balls in view of irreducibility, Payá and Yost [215] obtain examples of Minkowski spaces that are interesting regarding the \( n \)-ball property and semi-\( M \)-ideals (notions from Banach space theory).

Eggleston [85] also shows that if the unit ball \( B \) is a \( d \)-dimensional paralleloplane, then every complete set is homothetic to the unit ball. The converse is proved by V. Soltan [261], i.e., we have

**Theorem 7.** If every complete set in \( \mathbb{M}^d \) is homothetic to the unit ball \( B \), then \( B \) has to be a paralleloplane.

The two-dimensional case is contained in Theorem 8.1 of [127]. The infinite dimensional case has been looked at by Dalla and Tamvakis [67] and Franchetti [90]; they also investigate completeness in relation to constant width.

Eggleston [85] also shows the following

**Theorem 8.** Any bounded set \( S \subset \mathbb{M}^d \) can be embedded in a complete set of the same diameter.

Uniqueness of completion in Minkowski planes plays an essential role with respect to Borsuk’s partition conjecture in such planes, see [36, § 33] and Section 2.10 below. Groemer [112] investigates the uniqueness of completion in \( d \)-dimensional Euclidean and Minkowski spaces, e.g. in view of maximal tight covers of a bounded set \( S \subset \mathbb{M}^d \) (which are convex bodies \( K \) of maximal volume satisfying \( S \subset K \) and having the same diameter as \( S \)). He shows that every maximal tight cover of a bounded set \( S \subset \mathbb{M}^d \) is a completion of \( S \) and that any two such completions are translation equivalent; for strictly convex norms there is precisely one completion of this type. Furthermore, symmetry properties of such “maximal completions” are studied, and cardinalities of the set of completions for a given set (without the maximality assumption) are discussed. Various results from [112] are new even for the Euclidean case. We also refer to [21], where some of Groemer’s
theorems are summarized and used to get further results. Vrećica [298] observes that each bounded subset of $M^d$ has a completion within any circumball (i.e., a smallest Minkowski ball containing this set); a sharpening is obtained in [88]. Sallee [242] describes methods for constructing complete sets in $M^d$ containing an arbitrarily given set $X$. He pays special attention to the problem of preassigning boundary parts of such sets and, following ideas from [239], also to questions of preserving symmetries (if this is desired). Some of Sallee’s results are reproved in [21], and it is shown there that these constructions carry over to infinite dimensions, as the result of Vrećica [298] does. On the other hand, it is shown in [21] that, while in finite dimensions there is a completion $C$ satisfying $C \cap B = S \cap B$, the analogue fails in infinite dimensions (here $B$ denotes a circumsphere of the bounded set $S \subset M^d$).

2.6 Intersection properties

Due to Eggleston [85] a convex body $K \subset M^d$ has the spherical intersection property if $K$ is the intersection of all balls with centre $x \in K$ and radius $\text{diam} K$ (see also Theorem 6 above). In Euclidean space, the property of constant width and the spherical intersection property are equivalent, cf. [85] and [60, p. 62]. In Minkowski spaces this is no longer true, but the spherical intersection property is still equivalent to the notion of completeness, see Theorem 6 above and [85]. For a body of constant width $K \subset M^d$ let $t(K)$ denote the smallest number of balls whose intersection equals $K$. V. Soltan [263] proves some results on the possible values of $t(K)$. For example, he derives necessary and sufficient conditions for $t(K) < \infty$. For $d = 2$, he determines unit balls that admit given even values for $t(K)$. Groemer [112] uses the spherical intersection property to study bounded sets with unique completion in $M^d$.

Sallee [240] generalizes Theorem 5 above. He shows that it still holds if one replaces “constant width $w \in \mathbb{R}^+$” by “the spherical intersection property”.

Section 2 of [21] contains also some properties of sets which can be presented as intersections of balls. Analogously, Bavaud [22] introduces the $s$-adjoint transform in $E^2$, which associates to a two-dimensional convex body $K$ its $s$-adjoint $K^s$ as the intersection of all discs of radius $s$ whose centres are from $K$. Explaining the connection of this transform to hyperconvexity (cf. our section on $d$-convexity as well as [25, 52, 198]) and to the concept of constant width, he gives a new construction of the completion of a compact set in the plane. He also presents applications related to stochastic point processes and statistical mechanics. One might ask for extensions of these Euclidean results to Minkowski planes.

Baronti and Papini [21] establish various properties connecting completeness and the spherical intersection property for infinite dimensions.

Kupitz and Martini [163] consider a related notion: a set $S \subset E^2$ of diameter 1, say, has the weak circular intersection property if the intersection of all unit discs whose centres are from $S$ is a set of constant width 1. They determine the least number of points to be added to a finite planar set of given diameter such that the resulting set has the weak circular intersection property. Nothing seems to be known about this notion in higher dimensions or Minkowski spaces.

The dissertation [186] contains Helly-type theorems for sets of constant Minkowski width.
2.7 Curvature and mixed volumes

Let \( K \subset \mathbb{E}^d \) be a body of constant width \( w(K) \in \mathbb{R}^+ \) whose boundary \( \text{bd} \, K \) is twice continuously differentiable, and let \( R_1(u), \ldots, R_{d-1}(u) \) denote the principal radii of curvature at the point \( x \in \text{bd} \, K \) with outward unit normal \( u \). If \( F_i(u) := \sum_{i=1}^{d-1} R_i(u) \), then

\[
F_1(u) + F_1(-u) = (d - 1) \cdot w(K) \tag{1}
\]

for each \( u \in S^{d-1} \), see [40, p. 128]. This can be extended to all bodies of constant width by introducing the surface area function \( S(K, \omega) \), \( \omega \in \mathcal{B} \), where \( \mathcal{B} \) denotes the field of Borel subsets of \( S^d \). The “mixed surface area function” \( S(K_1, \ldots, K_{n-1}; \omega) \) of convex bodies \( K_1, \ldots, K_{n-1} \in \mathbb{E}^d \) is a completely additive set function defined for \( \omega \in \mathcal{B} \). Thus for any convex body \( K_0 \subset \mathbb{E}^d \) the mixed volume (cf. [244, §5.1]) is given by

\[
V(K_0, K_1, \ldots, K_{n-1}) = \frac{1}{n} \int_{S^{d-1}} h(K_0, u)S(K_1, \ldots, K_{n-1}; du). \tag{2}
\]

Here the presented integral is the Radon-Stieltjes integral of \( h(K_0, u) \) with respect to the function \( S(K_1, \ldots, K_{n-1}; \omega) \). Then \( S(K, \omega) := S(K, \ldots, K, \omega) \) is called the surface area function (or first curvature measure of second kind), cf. [244, §4.2]. Since, if \( K \) is of constant width, \( K + (-K) \) is a homothet of the Euclidean ball, the linearity and homogeneity of \( S(K, \omega) \) as a function of \( K \) imply

\[
S(K, \omega) + S(K, -\omega) = w(K) \cdot \mu(\omega), \tag{3}
\]

where \( \mu(\omega) \) is the spherical Lebesgue measure of the Borel set \( \omega \subset S^{d-1} \). By a known integral representation of \( S(K, \omega) \) in terms of \( F_1 \) (see [244, §4.2]), (3) is equivalent to (2) if \( K \) has a sufficiently smooth boundary, and basically due to the Aleksandrov-Fenchel-Jessen uniqueness theorem (cf. [244, §7.2]) it can be concluded that (3) is a characteristic property of all bodies of constant width in \( \mathbb{E}^d \).

It turns out that, in terms of relative differential geometry (see also [40, §38] and [60, §6]), these relations can be extended to Minkowski spaces. Namely, for a smooth convex body \( K \subset \mathbb{E}^d \) let \( L(\text{bd} \, K, x) \) denote the canonical linear mapping of its tangent space \( \text{bd} \, K \) at \( x \in \text{bd} \, K \) into the tangent space of \( S^{d-1} \) at \( u \in S^{d-1} \), where \( x \) and \( u \) are connected by the usual Gauss mapping via parallel normals, with \( u \) as unit outward normal of \( \text{bd} \, K \) at \( x \). If for a smooth unit ball \( B \) in \( \mathbb{M}^d \), \( B_e \) denotes the tangent space of \( \text{bd} \, B \) at \( e \) having the same unit normal as \( \text{bd} \, K \) at \( x \), then the linear mapping \( J : \text{bd} \, K \rightarrow B_e \) can be defined by \( J = L(\text{bd} \, K, x) \cdot L^{-1}(\text{bd} \, B, e) \) and is, as the canonical linear mapping of \( \text{bd} \, K \) into \( \text{bd} \, B \) via parallel normals, invertible with \( J^{-1} = L(\text{bd} \, B, e) \cdot L^{-1}(\text{bd} \, K, x) \).

Thus we may write \( J = J(u) \), where \( u \) is the outward unit normal of \( \text{bd} \, K \) at \( x \), and of \( \text{bd} \, B \) at \( e \). The relative principal radii of curvature \( \overline{R}_1, \ldots, \overline{R}_{d-1} \) of \( \text{bd} \, K \) at \( x \) are the reciprocals of the eigenvalues of \( J(u) \), and the corresponding relative principal directions are the eigenvectors of \( J(u) \). Following [40, p. 64], we write \( \{\overline{R}_1, \ldots, \overline{R}_v\} \) for the \( v \)th elementary symmetric function of the relative principal radii of curvature \( \overline{R}_1, \ldots, \overline{R}_{d-1} \), and we let \( F_v(K, u) = \{\overline{R}_1, \ldots, \overline{R}_v\} \) at \( u \) as above. With this notation, Chakerian [57] proves
Theorem 9. Let $K$ be a smooth convex body of constant Minkowski width $w(K) \in \mathbb{R}^+$ in $\mathbb{M}^d$. Then
\[ \overline{R}_i(u) + \overline{R}_{d-i}(-u) = w(K), \quad u \in S^{d-1}, \quad i = 1, \ldots, d-1, \]
and, in particular, $F_1(K, u) + F_1(K, -u) = (d - 1) \cdot w(K), \quad u \in S^{d-1}$.

We set $S(K, \ldots, K, B, \ldots, B; \omega) =: S(K, r; B; \omega)$ by using the notation given in connection with (2). Here the convex body $K$ appears $r$ times and $B$ appears $n - r - 1$ times.

Chakerian [57] also proves

Theorem 10. If $K$ is a body of constant Minkowski width in $\mathbb{M}^d$, then

\[ S(K, 1; B; \omega) + S(K, 1; B; -\omega) = 2 \cdot S(B, \omega) \tag{4} \]

with $\omega \in B$. Conversely, if the unit ball $B$ is smooth, (4) implies that $K$ is of constant Minkowski width.

Here the smoothness assumption in the second implication cannot be omitted, as an easy construction (with $B$ the convex hull of a Euclidean ball centred at the origin and two points $x, -x$ outside this ball) shows. Hug [139, § 1.4] obtains a further result on surface area measures of bodies of constant Minkowski width, and he also gives an extension of this which is closely related to pairs of constant Minkowski width (for the latter notion see our Section 2.12 below). In [139] one can find also various related characterizations of unit balls.

If a convex body $K \subset \mathbb{E}^2$ has sufficiently smooth boundary, one may denote its radius of curvature at $x \in \text{bd} K$ with outward normal $u = (\cos \alpha, \sin \alpha)$ by $R(\alpha)$. In this notation we can give a necessary criterion on a function $R(\alpha)$ to present the radius of curvature of a body $K \subset \mathbb{E}^2$ of constant width $w(K) \in \mathbb{R}^+$:

\[ R(\alpha) + R(\alpha + \pi) = w(K). \tag{5} \]

In Minkowski planes we have the same result. The sum of the radii of curvature at analogously opposite boundary points of a plane body of constant Minkowski width is constant and equals $w(K)$. In terms of difference bodies (which are homothets of central symmetrals) this was first observed by Vincensini [295, p. 24], and Sz.-Nagy [281, p. 31], and Petty reproved this, see [216, Theorem (6.14)].

Chakerian [59] shows that for a figure $K$ of constant width in $\mathbb{M}^2$ the integral

\[ I(K) = \frac{1}{2} \int \overline{R}^2 dS, \]

where $\overline{R}$ is the relative radius of curvature and $dS$ denotes the relative arc length element, can also be expressed by the functional

\[ I(K) = \iint n(x_1, x_2)dx_1dx_2, \]

where $n(x_1, x_2)$ denotes the number of diameters of $K$ passing through $x = (x_1, x_2) \in K$.

Let $C$ be a twice continuously differentiable curve in $\mathbb{E}^2$. The Four Vertex Theorem says that $C$ has at least four vertices which are the points of $C$ where the curvature of
this curve has a stationary value. Heil \[129\] derives a generalized version of the analogous theorem for Minkowski planes. His considerations imply that any curve of constant Minkowski width has at least six vertices.

Defining the $k$-th quermassintegral of a convex body $K \subset \mathbb{E}^d$ in terms of mixed volumes by $W_k(K) := V(K, \ldots, K, B, \ldots, B)$ (here $K$ appears $n-k$ times, and $B$ appears $k$ times, see [40, p. 49]), the equality

$$W_{n-k}(K) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} W_{n-i}(K)[w(K)]^{k-i}, \quad k = 0, 1, \ldots, n,$$

holds if $K$ is a body of constant width $w(K) \in \mathbb{R}^+$, cf. [77]. Chakerian [58] generalizes (6) as follows: For convex bodies $K, L, M$ with $K + L = M$ let $V(K, k; *)$ denote the mixed volume with $K$ occurring $k$ times and * representing $n-k$ fixed entries, and $V(L, i; M, k-i; *)$ be the mixed volume of $i$ times $L$ and $k-i$ times $M$ with the same $n-k$ fixed entries as above. Then

$$V(K, k; *) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} V(L, i; M, k-i; *).$$

Assuming $L = -K$, $M = w_B(K) \cdot B$ and setting the remaining entries equal to the unit ball $B$ of $\mathbb{M}^d$, an analogue of (6) for bodies of constant Minkowski width is obtained, see also [60].

Guggenheimer [124, p. 327] announces related results on bodies of constant width relative to a unit ball which is no longer centrally symmetric, i.e., only a gauge body.

2.8 Inequalities

The Blaschke-Lebesgue Theorem states that among all figures of constant width $w \in \mathbb{R}^+$ in $\mathbb{E}^2$ only the Reuleaux triangle has minimum area (see [60, § 7] and, for a short proof, e.g. [56]). The analogous theorem for Minkowski planes was obtained by D. Ohmann and, independently, by K. Günther in their dissertations (both in Marburg, 1948); Ohmann published his approach in [211], see also Petty [216, pp. 15-16]. Since Reuleaux triangles of fixed width in Minkowski planes can have different areas in general, we describe the construction of the particular type which this theorem refers to. Let the Minkowski unit vectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \in \mathbb{M}^2$ have the property $\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 = \mathbf{0}$. Then, by suitable translates of the segments $[\mathbf{0}, \mathbf{r}_1]$, one can form a triangle $\mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3$ with $\mathbf{a}_1 = \mathbf{0}$ and $\mathbf{a}_2, \mathbf{a}_3 \in \text{bd } B$, cf. Figure 1. Connecting now the vertices $\mathbf{a}_1$ and $\mathbf{a}_2$ as well as $\mathbf{a}_2$ and $\mathbf{a}_3$ by corresponding boundary arcs of $B$ (see again the figure), we get a figure of constant Minkowski width 1 whose boundary contains the points $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3$. Due to Ohmann any homothetical copy of such a figure is called a general Reuleaux triangle in $\mathbb{M}^2$. With this notation we can formulate

**Theorem 11.** Among all figures of constant width $w \in \mathbb{R}^+$ in a Minkowski plane, a general Reuleaux triangle has minimum area.

Chakerian [56] presents another proof of this theorem, and also Kubota and Hemmi [161] give an independent approach by investigating various inequalities for convex figures in $\mathbb{E}^2$. From this they deduce that precisely the above mentioned analogues of Reuleaux triangles are extremal with respect to an inequality in terms of diameter and minimal

The Firey-Sallee Theorem says that among all Euclidean Reuleaux polygons having \( n \geq 3 \) vertices and width \( w \in \mathbb{R}^+ \) the regular one has maximal area. In [238] Sallee announces without proof that his methods for getting this result in \( \mathbb{E}^2 \) can be modified to obtain analogous results in Minkowski planes.

Lenz [183] proves that in a Minkowski plane whose unit circle \( B \) is a Radon curve the ball has largest area among all bodies of fixed constant width. However, by the Rogers-Shephard inequality [236] in any Minkowski space the ball has the largest volume among all bodies of fixed constant width.

Wernicke [300] shows that the area of a Reuleaux triangle \( \Delta \) of width 1 in a Minkowski plane satisfies \( \frac{1}{6} \leq \frac{\text{Area}(\Delta)}{\text{Area}(B)} \leq \frac{1}{4} \), where equality holds iff the unit ball \( B \) is an affine regular hexagon (left hand side) or a parallelogram (right hand side). Furthermore, he proves that only in planes with \( B \) a parallelogram or a centrally symmetric hexagon there exist Reuleaux triangles that are Minkowski circles.

Castro Feitosa [88] uses extremal properties of bodies of constant width in \( \mathbb{M}^2 \) to extend various inequalities of Scott [246] for convex figures in \( \mathbb{E}^2 \) to convex figures in Minkowski planes. E.g., he uses the facts that among all convex figures in \( \mathbb{M}^2 \) with given diameter and circumradius the sets with largest thickness (= minimal width), perimeter, inradius or area are of constant width.

Further inequalities for the area of figures of constant Minkowski width and related results are derived in [54] and [151]. In [151] an upper bound on the area of a figure \( C \subset \mathbb{M}^2 \) of constant width in terms of the Minkowski arc length of its pedal curve and other quantities is given; this bound is attained iff \( C \) is homothetic to a pedal curve of the isoperimetric of \( \mathbb{M}^2 \).

Investigating products of dual cross section measures of convex bodies and their polar reciprocals, Ghandehari [104] proves that the \( d \)-dimensional volume of the polar reciprocal \( K^* \) of a body \( K \subset \mathbb{M}^d \) of constant width 2 satisfies \( V(K^*) \geq V(B^*) \), in which \( B^* \) is the polar reciprocal of the unit ball \( B \). Here equality holds if and only if \( K = B \).

Another type of inequality is considered in [190]. For \( X \) a finite point set in \( \mathbb{E}^d \) with \( d \)-dimensional convex hull \( P \), the points \( x_i, x_j \in X \) are called antipodal if there are different
parallel supporting hyperplanes $H', H''$ of the polytope $P$ with $x_i \in H'$, $x_j \in H''$. If $\mathbb{E}^d$ is endowed with a Minkowski metric, one might ask for the number of pairs in $X$ whose Minkowski distance is maximal. This number is not larger than the number $a(X)$ of antipodal pairs in $X$, and if $P$ is of constant Minkowski width, then equality holds. In [190] several upper bounds on $a(X)$ are given.

2.9 Inscribed and circumscribed bodies

As is well known, a hexagon (regular in the considered norm) can be inscribed in any Minkowski circle. We refer to [283, Chapter 4] for a broad and nice representation of related results and how this can be applied to construct curves of constant Minkowski width, in particular Reuleaux triangles. As a special case of a result of Doliwa [79] (conjectured by Lassak [172]) we have that any planar body of constant Minkowski width 1, say, has an inscribed pentagon whose vertices are in at least unit distance to each other. A generalization to arbitrary equilateral inscribed polygons is announced by Lassak (personal communication).

2.10 Packings, coverings, lattice points

Loomis [186] considers so-called three-coverings within the family of bodies of constant width $w$ in Minkowski spaces. A set $K$ three-covers the set $L$ if $L \subseteq \{a_1, a_2, a_3\} + K$ for some three points $a_1, a_2, a_3$. Among other results, Loomis shows that a Reuleaux triangle of width $w$ three-covers any figure of constant width $w$. Further on, if $B$ is a centrally symmetric octagon, then each figure of constant width three-covers every other set of the same constant width.

Inspired by a question of P. C. Hammer, Sallee [237] considers bodies of constant width in association with lattices. In analogy to the Euclidean situation, he defines a Minkowski Reuleaux polygon to be a set of constant width $w$ in $\mathbb{M}^2$ which is the intersection of a finite number of (properly chosen) translates of $wB$. Saying that a set $S$ avoids another set $X$ if $\text{int} S \cap X = \emptyset$, he proves the following statements for any Minkowski plane with strictly convex unit ball $B$: Every set of maximal constant width avoiding a square unit lattice $L$ is a Minkowski Reuleaux triangle $P$ where each of the three open “edges” of $P$ contains at least one point of $L$. If the lattice $L$ is replaced by a locally finite family $X$ of convex sets in an arbitrary Minkowski plane, then the corresponding maximal sets are Reuleaux polygons in $\mathbb{M}^2$ all of whose open “edges” contain points from $X$.

Surveys on the famous partition problem of Borsuk are presented by Grünbaum [122], [36, Chapter V] and Raigorodskii [229], see also [1, Chapter 15]. This problem is closely related to bodies of constant width, cf. [60], [131], [33] and also [36, Chapter VIII]. A first investigation of Borsuk’s problem in Minkowski planes is due to Grünbaum [120]. He shows that if $B$ is not a parallelogram, then any set of diameter 1 can be covered by three balls each of diameter less than 1. Let $F \subset \mathbb{M}^d$ be a bounded set of diameter $h$. What is the smallest integer $k$ such that $F$ is the union of $k$ sets each of which has a diameter strictly smaller than $h$? Denoting this smallest number by $a_B(F)$, Boltyanski and V. Soltan [39] prove that for $d = 2$ one has $a_B(F) \in \{2, 3, 4\}$, where $a_B(F) = 4$ occurs if and only if $B$ is a parallelogram and the convex hull of $F$ is a homothet of $B$. And $a_B(F) > 2$ holds if and only if one of the following two conditions is satisfied: (i) There is a unique completion of $F$ to a figure $C$ of constant width $h$. (ii) For any two parallel supporting hyperplanes of $C$ at least one has nonempty intersection with $F$. 
2.11 Rotors in polytopes

It is obvious that bodies of constant width in $\mathbb{E}^d$ are rotors in cubes and in some other polytopes. On the other hand, there are also polytopes with rotors not of constant width (but still having various similar properties). With this (a little extended) point of view, a remarkable list of related references can be taken from [40, pp. 139-140], [60, p. 80] and [131, p. 367].

Regarding Minkowski geometry, Ghandehari and O’Neill [105] derive inequalities for the self-circumference $U$ of rotors in equilateral triangles and figures of constant width. Here $U$ is measured by taking these rotors or figures of constant width themselves as gauge figures or unit circles of a Minkowski plane. The inequalities compare their areas and mixed areas (taking also the polar reciprocal) with $U$. Some higher dimensional results are also given in [105].

Weakly related, Sorokin [275] studies certain classes of convex bodies which can roll in Minkowski spheres.

2.12 Concepts related to constant Minkowski width and further results

Heil’s concept of reducedness (cf. [130]) is in a sense dual to completeness. A convex body $K \subset \mathbb{E}^d$ which does not properly contain a convex body of the same minimal width is called a reduced body. For a discussion of results on reduced bodies see [131, §§ 5.4]. The most striking open questions on reduced bodies in $\mathbb{E}^d$ are:

(a) Do there exist reduced polytopes for $d \geq 3$?

(b) Is a strictly convex reduced body in $\mathbb{E}^d, d \geq 3$, necessarily of constant width?

Recently Lassak and Martini [173] extended this notion to Minkowski spaces. It is easy to see that for certain norms (with the Manhattan norm as most simple case) and $d \geq 3$ reduced polytopes exist, whereas in [173] the extension of (b) is again only verified for $d = 2$. Furthermore, it is shown that there exist reduced bodies in Minkowski spaces of dimensions $d \geq 3$ having minimal width 1, say, but arbitrarily large diameter.

Let $\delta(K, \mathbf{u})$ denote the $(d-1)$-dimensional volume of the orthogonal projection of a convex body $K \subset \mathbb{E}^d$ onto the subspace orthogonal to $\mathbf{u} \in S^{d-1}$. Usually $\delta(K, \mathbf{u})$ is called the brightness of $K$ at $\mathbf{u}$, cf. Chapters 3, 4, 8, and 9 of [98] for related results. A convex body $K \subset \mathbb{M}^d$ is said to be of constant brightness with respect to $B$ if $\delta(K, \mathbf{u})$ is proportional to $\delta(B, \mathbf{u})$. Chakerian [57] shows that if both $B$ and $K$ in $\mathbb{M}^3$ have $C^2$ boundary with everywhere positive curvature, then $K$ is a homothet of $B$. More generally, Petty defines the Minkowski brightness of $K \subset \mathbb{M}^d$ at $\mathbf{u} \in S^{d-1}$ as the minimal Minkowski cross section area of the cylinder $K + L$, where $L$ is the 1-subspace of direction $\mathbf{u}$. In [217] and [218] he derives results on bodies of constant Minkowski brightness. The $k$-girth of a convex body $K \in \mathbb{E}^d$ in direction $\mathbf{u}$ is given by the mixed volume $dV(K, \ldots, K, B, \ldots, B, [\mathbf{u}])$, where $K$ appears $k$ times, the Euclidean ball $B$ occurs $n - k - 1$ times, and $[\mathbf{u}]$ denotes the unit line segment parallel to $\mathbf{u}$, see also Section 2.7 above. Chakerian [57] considers sets of constant $k$-girth in Minkowski spaces, and Petty [218] studies bodies of constant Minkowski curvature.

Due to Maehara [188], two convex bodies $K_1, K_2 \subset \mathbb{E}^d$ are said to be a pair of constant width if $K_1 + (-K_2)$ is a ball. Analogously, Sallee [241] defines a pair $K_1, K_2$ of convex
bodies to be a pair of constant width in $\mathbb{M}^d$ if $h(K_1, u) + h(K_2, -u) = \lambda \cdot h(B, u)$ for some $\lambda > 0$ and all directions $u \in S^{d-1}$. Among other results, he proves that $K_1$ is a summand of the unit ball $B$ iff there is a $K_2$ such that $\{K_1, K_2\}$ is a pair of constant width in $\mathbb{M}^d$. Also Sallee’s generalization of Theorem 5 with respect to sets having the spherical intersection property in $\mathbb{M}^d$ (cf. Section 2.6 above) can be formulated in terms of pairs of constant width in $\mathbb{M}^d$, cf. [240].

Petty and Crotty [220] show that there are $d$-dimensional Minkowski spaces with convex bodies having exactly two equichordal points.

Rodriguez Palacios [235] points out that results on summands of Banach spaces may be interpreted in terms of sets having constant width in Banach spaces.

In view of multiplication with scalars, Minkowski addition and suitable combinations thereof, the family of all convex bodies in $\mathbb{E}^d$ forms an abelian semigroup with scalar operators. Having such an algebraic structure and the Hausdorff metric in mind, Ewald and Shephard [87] introduce an equivalence class structure for the subclass of bodies of constant width, which yields an incomplete normed linear space. They remark that (due to a hint of Grünbaum) their respective results may be easily extended to bodies of constant Minkowski width. Such extensions were given by Sorokin [275], even for non-symmetric unit balls which, on the other hand, have to be smooth and strictly convex. Taking the minimum width of certain representatives as a metric (in the above mentioned space), Lewis [185] shows that then a conjugate Banach space with complete norm is obtained. For related considerations we also refer to [86].

Finally we shortly mention the concept of bodies of constant affine width in the sense of affine differential geometry, see Süss [280] for an early contribution. For further references to this subject we refer to the final paragraphs of the surveys [60] and [131] and, in particular, to the investigations in [135] and [24], relating the concept of affine width to that of Minkowski width.

3 Generalized convexity notions in Minkowski geometry

3.1 Introduction

In Section 3 we deal with modifications of the usual convexity notion, in most cases yielding natural extensions of basic theorems on convex sets. The main part will refer to a type of metric convexity which is usually called $d$-convexity, but also other kinds of convexity will be discussed. The letter ‘$d$’ is used in two different meanings: for the dimension of the space, and for the historically fixed notions of $d$-segment and $d$-convexity. Since the distinction will always be clear from the context, we let it as it is.

The notion of $d$-segment is based on so-called metric betweenness points and the metric betweenness relation which were first considered by Menger [206, Part I] and Blumenthal [27, Chapter II] in the context of complete, convex metric spaces. See also [207], [28, Chapter II], [114], [234], [5], [253], [29, Part 3], and [256]. Also based on betweenness points, Menger and Busemann proposed to complete Fréchet’s axioms for a metric space to ensure the existence of geodesics, cf. [45, Chapters I and II], [47, Chapter I], and its continuations [48] and [51]. Replacing usual straight line segments in the common definition of convex sets by $d$-segments, the concept of $d$-convex sets is obtained, see [216], [114], [5], [53], [253], and [256] for the definition and first investigations. (We note that
Petty [216] speaks about the concept of “Minkowski convexity”. Wider presentations of this concept can be found in the monographs [37], [270] and [36] (see the chapters with the headline “d-convexity”), and it is also mentioned in the Handbook of Convex Geometry, cf. [193, § 4]. For further generalized convexity notions we refer to [70, § 9], [293] and [250].

3.2 d-segments in Minkowski spaces

Let $\gamma$ be a simple curve in a $d$-dimensional Minkowski space $M^d$ which is parametrized by $[t_0, t_n]$ and whose length is defined, in an elementary way, by

$$|\gamma| := \sup \left\{ \sum_{i=1}^{n} \| a_{i-1} - a_i \| : n \in \mathbb{N}, a_i = \gamma(t_i), t_0 < t_1 < \cdots < t_n \right\},$$

with the endpoints $a_0 = \gamma(t_0), a_n = \gamma(t_n)$. A metric segment is a curve isometric to a closed segment of the real line, a metric line is a curve isometric to the real line, and a geodesic (cf. [45, § 3] and [47, p. 32]) is a curve that is locally a metric segment, i.e., each point of the curve has a closed neighbourhood which is a metric segment.

For a metric $d(x, y) = \|x - y\|$ in a Minkowski space, the set $[a, b]_d := \{z \in M^d : d(a, z) + d(z, b)\}$ is called the $d$-segment with endpoints $a, b \in M^d$.

It is easy to show that any metric segment with endpoints $a, b$ is contained in $[a, b]_d$ and, on the other hand, that $[a, b]_d$ is the union of all metric segments with endpoints $a, b$. Each point $z \in M^d$ satisfying $a \neq z \neq b$ and $d(a, b) = d(a, z) + d(z, b)$ is said to be a betweenness point of the distinct points $a$ and $b$. See Menger [206, p. 77], who introduced betweenness points for arbitrary metric spaces and studied basic properties of the related betweenness relation (cf. also [27, Chapter II], and [28, Chapter II]). It is clear that the betweenness relation is closely related to the triangle inequality and therefore, in Minkowski spaces, to the (strict) convexity of the unit ball [109], cf. our first survey [196, § 3]. In particular, any metric segment is a straight line segment iff the unit ball $B$ of the Minkowski space is strictly convex, see [96], [73, p. 144], [42], [43], [257], and [76], also for further equivalent properties. We recall that strict convexity is equivalent to many notions such as the monotone property of the distance function and Chebyshev sets; see [196, § 3] as well as the further references [9] and [92]. Extensions of this characteristic property to more general spaces and related observations are given in [252], [209], [42], [230], [94], [75], [95], [303], and [292]. See also [9] and [29, Chapters 6 and 7]. For detailed discussion of the following observations we refer to [196, § 4]. Gölb and Härten [109] and, independently, Toranzos [290] have shown that the extreme points of the unit ball $B$ of a Minkowski space coincide with the directions of unique metric segments, i.e., of curves $\gamma$ from $a$ to $b$ such that $|\gamma| = \|a - b\|$. Nitka and Wiatrowska [210] observe that the origin $o$ is a betweenness point of $p, q \in \partial B$ iff the straight line segment $[p, \varphi_o(q)] \subset \partial B$, where $\varphi_o(q)$ denotes the reflection of $q$ at $o$. From this it follows that, given three arbitrary non-collinear points $a, b, c \in M^d$, one can always find a norm such that $b$ is betweenness point of $a$ and $c$. To see this, it suffices to choose a Minkowski ball centred at $b$ whose boundary contains $[a, \varphi_b(c)]$.

Verheul [294] says that a metric space $X$ is modular if $[a, b]_d \cap [b, c]_d \cap [c, a]_d$ is nonempty for any triple $a, b, c \in X$, and $X$ is called median if this intersection is always exactly one point. He shows that a Minkowski space is median iff it is isometric to $\ell_1^d$. 
the $d$-dimensional Minkowski space with norm $\|x\| = \sum_{i=1}^{d} |x_i|$, i.e., iff the unit ball is a cross-polytope. Verheul uses lattice-theoretic techniques to study modular metrics and modular Banach spaces.

Finally we want to give a geometric description of $d$-segments in terms of the boundary structure of the unit ball, see [257] and [36, § 9]: For $x, y \in M^d$, denote by $B_x, B_y$ the Minkowski balls of radius $\|x - y\|$ with centres $x$ and $y$, respectively. Furthermore, we write $F_x$ for the face of $B_x$ in $bd B_x$ that contains $y$ and, analogously, $F_y$ for the face of $B_y$ in $bd B_y$ containing $x$, see Fig. 2. We denote by $C_x$ the cone with apex $x$ consisting of all points $x + \lambda(a + x)$, where $a \in F_x$, $\lambda \geq 0$, and by $C_y$ the cone with apex $y$ and the representation $y + \lambda(b - y)$, $b \in F_y$ and $\lambda \geq 0$. Then the $d$-segment with endpoints $x, y$ is the intersection of the cones $C_x$ and $C_y$ (which are symmetric with respect to the midpoint of the straight line segment $[x, y]$), i.e., we have $[x, y]_d = C_x \cap C_y$. Soltan [270, Theorem 11.22] extends this observation to the infinite dimensional case.

![Figure 2. Construction of a $d$-segment](image)

### 3.3 Fundamentals of $d$-convexity

As Menger emphasizes in [206, Part I], the usual definition of convexity, using straight line segments, cannot be used for general metric spaces, and he defines a metric space $X$ to be **metrically convex** if for any two distinct points $x, y \in X$ there exists a betweenness point $z$, i.e., $d(x, z) + d(z, y) = d(x, y)$ has to be satisfied. With this concept, sometimes also called “Menger convexity” or $M$-convexity (see, e.g., Busemann [47, § 6]), convex metric spaces were studied, cf. the basic reference [28, § 14] (or Section 3.8 below). A slight modification yields the notion of $d$-convexity which is more interesting for Minkowski spaces and seems to have been first defined by Petty [216] and, independently, by de Groot [114]. Using the term “$d$-segment”, one can formulate this definition (referring to a Minkowski space $M^d$) as follows.

A set $A \subset M^d$ is **$d$-convex** if for any two distinct points $a, b \in A$ the $d$-segment $[a, b]_d$ is contained in $A$. Equivalently, $A \subset M^d$ is $d$-convex provided for any three points $a, b \in A, x \in M^d$, $d(x, a) + d(a, b) = d(x, b)$ has to be satisfied. With this concept, sometimes also called “Menger convexity” or $M$-convexity (see, e.g., Busemann [47, § 6]), convex metric spaces were studied, cf. the basic reference [28, § 14] (or Section 3.8 below). A slight modification yields the notion of $d$-convexity which is more interesting for Minkowski spaces and seems to have been first defined by Petty [216] and, independently, by de Groot [114]. Using the term “$d$-segment”, one can formulate this definition (referring to a Minkowski space $M^d$) as follows.

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the equality \( \|a - b\| = \|a - x\| + \|x - b\| \) implies that \( x \in A \). Since \([a, b] \subset [a, b]_d\) for any \( a, b \in \mathbb{M}^d \), each \( d \)-convex set is also convex in the usual sense (i.e., linearly convex). Obviously the converse is not necessarily true.

A broad presentation of the theory of \( d \)-convex sets in Minkowski spaces and linear normed spaces of any dimension is given in [270, Chapter II], see also Chapter II, § 24 and the first six research problems in Chapter VIII of [36]. Forerunners of these presentations are [38] and [37]. In the following, we will be concerned with the most important results in this direction, and we will try to give a historically correct representation how these results came to be.

3.4 \( d \)-convex sets in metric (and Minkowski) spaces

This section refers to results about \( d \)-convex sets which even hold in metric spaces, but are also important for Minkowski geometry. Since it is easy to prove that the intersection of an arbitrary family of \( d \)-convex sets is itself \( d \)-convex, the following notion makes sense. The smallest \( d \)-convex set containing a set \( A \subset \mathbb{M}^d \) (i.e., the intersection of all \( d \)-convex sets containing \( A \)) is called the \( d \)-convex hull of \( A \) and denoted by \( \text{conv}_d A \). (Note that \( \text{conv} A \) is used for the usual convex hull.)

A well known procedure to obtain \( \text{conv} A \) from \( A \) is that of \textit{segment joining}: Considering the union of all straight line segments \([a, b] \) with \( a, b \in A \), a finite iteration of segment joining yields \( \text{conv} A \) (by Carathéodory’s theorem). The following theorem, which is due to [289] and [102], refers to the analogous procedure with \( d \)-segments. For this we introduce the following notation. For an arbitrary set \( A \subset \mathbb{M}^d \) we write \( P_0(A) = A \), and then define

\[
P_i(A) = \bigcup \{[x, y]_d : x, y \in P_{i-1}(A)\} \text{ for each } i \geq 1.
\]

\textbf{Theorem 12.} For an arbitrary set \( A \subset \mathbb{M}^d \), the set \( \bigcup_{i=1}^{\infty} P_i(A) \) is the \( d \)-convex hull of \( A \).

Using the inequality \( P_1(A) \leq 2 \cdot \text{diam} A \) for the diameter of \( P_1(A) \), the authors of [174] show the following: Let \( \alpha(A) := \frac{\text{diam}(\text{conv}_d A)}{\text{diam} A} \), and let \( \beta(A) \) denote the smallest number \( k \) for which \( \text{conv}_d A = \bigcup_{k \geq 0} P_k(A) \) (if such a \( k \) does not exist, then \( \beta(A) = \infty \)). Then \( \alpha(A) \leq 2^{\beta(A)} \).

A metric space \( X \) (in particular, a Minkowski space) is called an \( \alpha \)-space (?-space) if \( \alpha(A) < \infty \) (?-space) for any bounded set \( A \subset X \), and it is called an \( \alpha^* \)-space (?-space) if \( \alpha^*(X) = \sup \alpha(A) \) (?-space) is finite, where the supremum is taken over all bounded subsets \( A \) of \( X \). This notation is from [174], where examples are given as well. Spaces where \( \alpha(A) = \infty \) or \( \beta(A) = \infty \) are also considered there. See also [270, pp. 107-111], [36, § 9 and § 11], and below.

From \( \alpha(A) \leq 2^{\beta(A)} \) we have that these four classes of spaces satisfy the following diagram of inclusions:

\[
\text{class of } \beta^*-\text{spaces} \subset \text{class of } \beta\text{-spaces} \\
\cap \\
\text{class of } \alpha^*-\text{spaces} \subset \text{class of } \alpha\text{-spaces}
\]

The following results are derived in [270, § 14]: For any set \( A \subset \mathbb{M}^2 \)

\[
\text{conv}_d A = \text{conv} A \cup P_1(A) = P_1(\text{conv} A) = \text{conv} P_1(A) = P_2(A)
\]

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and for bounded $A \subset \mathbb{M}^d$

$$\text{diam}(\text{conv}_d A) \leq 2 \cdot \text{diam} A \quad \text{and} \quad \text{conv}_d(\text{cl} A) = \text{cl}(\text{conv}_d A).$$

Besides various further observations on (special) metric spaces, in [101] and [258] the following results are obtained:

- For any bounded set $A \subset \mathbb{M}^d$ and the unit ball $B$ of $\mathbb{M}^d$ the inequality $\alpha(A) \leq \alpha(B)$ holds, yielding also $\alpha^*(\mathbb{M}^d) = \alpha(B)$ and the property that for any $\alpha'$ with $1 \leq \alpha' \leq \alpha(B)$ a set $Q \subset \mathbb{M}^d$ exists such that $\alpha(Q) = \alpha'$.  

- We have $1 \leq \alpha^*(\mathbb{M}^2) \leq 2$ with equality on the left-hand side iff $B$ is $d$-convex, and on the right-hand side iff $B$ is a parallelogram.

- We have $\beta^*(\mathbb{M}^2) \in \{1, 2\}$ with $\beta^*(\mathbb{M}^2) = 1$ iff $B$ is a parallelogram.

The following definition is taken from [267]. A point $z \in A$ is said to be $d$-extreme with respect to a set $A$ in a metric space if $z \notin [x, y]_d$ for any $x, y \in A \setminus \{z\}$. We denote by $\text{ext}_d A$ the family of all $d$-extreme points of a set $A$. It follows that for a $d$-convex set $A$ the point $z \in A$ is $d$-extreme iff $A \setminus \{z\}$ is $d$-convex. Furthermore we have $\text{ext}_d(\text{conv}_d A) \subset \text{ext}_d A$.

In [182] the following notion is given. A point $z \in A$ is called a $d$-boundary point of a set $A$ in a metric space if there exists a point $y \in A$ such that $z \notin [y, x]_d$ for any $x \in A \setminus \{z\}$. We write $\text{bd}_d A$ for the family of all $d$-boundary points of a set $A$.

Based on $\text{ext}_d A \subset \text{bd}_d A$ it is shown in [267] that for compact subsets $A$ of a metric space $\text{conv}_d A = \text{conv}_d(\text{bd}_d A)$ holds.

Furthermore, we mention here that an overview to results on $d$-convex sets in Cartesian products of metric spaces is given in [270, pp. 117–121]; various observations from there are weakly related to Minkowski spaces, see also [254], [164], [174] and [274]. In particular, the paper [165] of Lassak has to be mentioned here. Among other interesting observations, it contains the result that the Helly number of such Cartesian products equals the maximal Helly number of the considered metric spaces (for the notion of Helly number see below after Theorem 20). Thus [165] contains results more general than those for Minkowski spaces discussed at the end of Section 3.5 below, and these results of Lassak have consequences even for the study of Cartesian products of abstract convexities, cf. again [270, pp. 117–121].

### 3.5 $d$-convexity in Minkowski spaces

Now we turn to results and observations that no longer hold for all metric spaces but for Minkowski spaces. Our first theorem in this group of results gives a characterization of $d$-convex sets in a Minkowski space $\mathbb{M}^d$ and is taken from [255].

**Theorem 13.** A set $A \subset \mathbb{M}^d$ is $d$-convex iff for arbitrary points $a, b \in A$ any simple arc of length $\|a - b\|$ and with endpoints $a, b$ (i.e. a metric segment) from $a$ to $b$ is contained in $A$.

An immediate consequence is that each $d$-convex set is convex in the usual sense. On the other hand, there are many examples of convex sets $A \subset \mathbb{M}^d$ that are not $d$-convex,
and such sets might be unit balls or even $d$-segments of $M^d$, see [270, § 11] or [36, § 9]. E.g., the unit ball of the $\ell_1$ norm (i.e., the cross-polytope $C$ whose vertices are presented by the unit vectors of a Cartesian coordinate system) is not $d$-convex since its $d$-convex hull is the cube having the vertices of $C$ as midpoints of its facets. For $d$-segments that are not convex see the discussion of $d$-convex hulls below. However, the following theorem holds.

**Theorem 14.** The following conditions are equivalent for any Minkowski space $M^d$.

1. The unit ball $B$ of $M^d$ is strictly convex.

2. The families of $d$-convex and linearly convex sets in $M^d$ coincide.

3. For any $a, b \in M^d$ one has $[a, b] = [a, b]_d$.

These equivalences were observed by many authors, see [43, 290, 13, 257, 76] and our discussion of $d$-segments above. (Similar equivalences are shown to hold for star-shaped and $d$-star-shaped sets in [274], see also below.) In [270, § 11] it is shown that $M^d$ is strictly convex iff the set of extreme points of $A$ coincides with $\text{ext}_d A$ for any convex set $A \subset M^d$, and connections between strictly convex spaces and convex and $d$-convex functions are discussed in [273], cf. also Section 3.7 below. Related open questions are given in [36, Chapter VIII] (see Problem 1 there).

As usual, we will use the abbreviations $\text{ext} A$, $\text{bd} A$, $\text{rbd} A$ for the set of extreme, boundary and relative boundary points of a set $A$. The following structure of inclusions is known (see [270, § 11]), where $A \subset M^d$ denotes a convex set:

$$
\begin{align*}
\text{ext} A & \subset \text{rbd} A \\
\text{ext}_d A & \subset \text{bd}_d A
\end{align*}
$$

We continue with results on $d$-extreme and $d$-boundary points in Minkowski spaces derived in [267], see also [268]. For example, $\text{ext}_d A = \emptyset$ (or $\text{ext} A = \text{bd}_d A$) for any $d$-convex set $A$ in a Minkowski plane $M^2$ iff its unit circle $B$ is a parallelogram. Also $\text{bd}_d A = \text{rbd} A$ for any convex (or $d$-convex) set $A \subset M^2$ iff $B$ is not a parallelogram, and $\text{ext}_d A \neq \emptyset$ for all compact, convex (or $d$-convex) sets $A \subset M^2$ iff $B$ is not a polygon. A Minkowski plane $M^2$ is called *angular* if there are three extreme points $a, b, c$ of its unit circle $B$ such that the segments $[a, b], [b, c]$ are from the boundary of $B$. In [267] it is shown that a normed plane $M^2$ is not angular iff $A = \text{conv}_d(\text{ext}_d A)$ for any compact $d$-convex set $A \subset M^2$. Sharpening the structure of inclusions (7) it is shown in [268] that for any $d$-convex set $A \subset M^d$ with nonempty interior the inclusions $\text{ext}_d A \subset \text{ext} A \subset \text{bd}_d A \subset \text{bd} A$ hold.

Our next observations are mainly taken from [268] and give still more information related to this structure of inclusions. Namely, for any convex set $A \subset M^d$ the equality $\text{rbd} A = \text{bd}_d A$ holds iff $o \in \text{bd}_d K$ for any proper convex cone $K$ with apex $o$, and $\text{ext} A \subset \text{bd}_d A$ iff $o \in \text{bd}_d K$ for any acute convex cone $K$ with apex $o$. If, in particular, the set $A$ is diametrically complete (cf. Section 2.5), then also $\text{rbd} A = \text{bd}_d A$. If $M^d$ is a $\beta^*$-space, then $\text{ext}_d A = \emptyset$ for any $d$-convex set $A$ with $\dim A \geq 2$ iff the unit ball of $M^d$ is
a cross-polytope. If $M^d$ is an $\alpha$-space, then $\text{ext}_d A = \text{bd}_d A$ for any $d$-convex set $A \subset M^d$ iff the unit ball is a cross-polytope.

Lassak [171] studies certain extensions of the notion of $d$-extreme points in Minkowski spaces. Here we also mention results of Lassak (cf. [164] and [167], but also [270, § 13]) on concepts related to affinely or convexly independent sets of points, yielding also results on strictly convex normed spaces, $\beta$-spaces and further types of spaces.

The following statements can be found in [102] and [100] (see also [101] for a summary):

- The supporting (or inscribed) cone with apex at any point of a $d$-convex set $A \subset M^d$ is itself $d$-convex.
- For any $d$-convex set $A \subset M^d$, its affine hull, relative interior or closure is $d$-convex as well.
- Each face of a $d$-convex set $A \subset M^d$ is $d$-convex ($\Rightarrow$ a half-space is $d$-convex iff its bounding hyperplane is $d$-convex).
- Any $d$-convex body $A$ is the intersection of all closed $d$-convex half-spaces supporting $A$.

In particular, it is proved in [270, § 11] that a set $A \subset M^d$ is $d$-convex iff each $x \in \text{bd} A$ belongs to at least one $d$-convex supporting hyperplane of $A$. Also, if a set $A \subset M^d$ is open, then $\text{conv}_d A$ is open, and $\text{conv}_d(\text{cl} A) \subset \text{cl}(\text{conv}_d A)$, cf. [101] and [174]. On the other hand, if $A \subset M^d$ is itself $d$-convex, then $\text{cl} A$ and retint $A$ are also $d$-convex, cf. [102].

Related results for unbounded $d$-convex sets were obtained in [102] and [100], see also [36, § 11]. E.g., the maximal inscribed cone of an unbounded $d$-convex set in $M^d$ is also $d$-convex.

A Minkowski space is $d$-strictly convex if for any two points $x, y$ on the unit sphere, $[x, y]_d$ intersects the interior of the unit ball. Calder [53] studies an infinite dimensional analogue called weak uniform convexity, and proves that $\ell_\infty^d$ has this property. It is easily seen that the finite-dimensional $\ell_\infty^d$, hence any Minkowski space with a parallelepiped as unit ball, is $d$-strictly convex. Calder also shows what amounts for Minkowski spaces to the following: Each closed $d$-convex set $A$ in a $d$-strictly convex Minkowski space is Chebyshev, i.e., for any point $a$ there is a unique point $b \in A$ nearest to the point $a$. (In particular, $d$-convex subsets of $\ell_\infty^d$ are Chebyshev.)

We continue now with properties of $d$-convex hulls. For instance, with the help of $d$-convex hulls (of two points) it can be shown that in certain Minkowski spaces (which are not $\alpha$-spaces) even $d$-segments might not be $d$-convex. The following example is given in [101] for $d = 3$, and it is extended to any $d \geq 3$ in [270, Example 12.1], see also [36, § 9]. Let $M^d$, $d \geq 3$, be a Minkowski space whose unit ball $B$ is a $d$-cube (maximum norm), and let $x, y$ be any two different points with the property that the segment $[x, y]$ is not parallel to any spatial diagonal (connecting two opposite vertices) of $B$. Then the $d$-convex hull of $x, y$ is the whole space $M^d$. (On the other hand, $[x, y]_d$ is obviously a bounded set.) We also note that for $d = 2$ any $d$-segment is a $d$-convex set and refer to the related Problem 1 in [36, Chapter VIII], see also [257]. An analogue of the example above for the infinite dimensional situation is also given in [270, Example 11.2].
We continue with properties of Minkowski spaces with \( d \)-convex unit balls. In [259] it is proved that if the unit ball \( B \) of \( M_d \), \( d \geq 3 \), is \( d \)-convex, then it cannot be a polytope. An alternative proof is given in [36, § 11]. In [270, § 12] it is shown that if \( B \) is \( d \)-convex, then for any \( k \) with \( 1 \leq k < d \) there are infinitely many \( d \)-convex linear \( k \)-subspaces. V. Soltan [259] adds the observations that if a \( d \)-convex unit circle in a Minkowski plane is a polygon, then it has \( 4k + 2 \) \((k \geq 1)\) sides, and that for any of these numbers Minkowski planes exist whose unit circles are \( d \)-convex \((4k + 2)\)-gons.

The next theorem is proved in [258].

**Theorem 15.** The following conditions are equivalent:

1. The unit ball \( B \) of a Minkowski space \( M_d \) is \( d \)-convex.
2. \( \text{diam}(\text{conv}_d A) = \text{diam} A \) for any bounded set \( A \subset M_d \).

Strongly related is the equivalence of the conditions \( \alpha(A) < \infty \) (for any bounded set \( A \)) and \( \alpha(B) < \infty \), see again [258]. Here we also mention the observation of Petty [216] that the isoperimetrix of a Minkowski plane has to be \( d \)-convex.

Various results on \( d \)-convex flats are also related to this. Since all the properties under consideration are translation invariant, it suffices to look at subspaces. A key result in this connection is the following statement from [101], which again implies the coincidence of linear convexity and \( d \)-convexity in the case of strictly convex unit balls.

**Theorem 16.** A linear subspace \( L \) of \( M_d \) is \( d \)-convex iff for any point \( x \in \text{bd} B \cap L \) the face \( \Phi_x \) of \( x \) in \( \text{bd} B \) is contained in \( L \).

In [258] it is shown that for any Minkowski space \( M_d \) there exist one-dimensional \( d \)-convex subspaces \( l_1, \ldots, l_d \) whose direct vector sum yields the whole space, implying also that every Minkowski plane is an \( \alpha \)-space. On the other hand, a Minkowski space is an \( \alpha \)-space iff it has \( d \)-convex \((d - 1)\)-subspaces \( L_1, \ldots, L_d \) whose intersection is the origin, see again [258]. For arbitrary nontrivial linear subspaces \( L_1, L_2 \) of \( M_d \), the expression

\[
\ominus(L_1, L_2) = \varrho(L_1 \cap B, L_2 \cap B)
\]

with \( \varrho \) as Hausdorff distance defines a metric on the family \( G \) of all nontrivial subspaces of \( M^d \), and the topology determined by this metric in \( G \) does not depend on the choice of the corresponding norm. The set \( G_k \) of all \( k \)-dimensional linear subspaces is contained in \( G \), and so \( G_k \) is a metric space with the metric (8). If \( E_k \) is used to denote the family of all \( d \)-convex \( k \)-dimensional linear subspaces, we have \( E_k \subset G_k \), i.e., \( E_k \) is a subspace of the metric space \( G_k \). For the following result we refer to [102], see also [36, § 11].

**Theorem 17.** The set \( E_{n-1} \) (with the metric (8)) is compact.

One can easily show that there exist \( d \)-dimensional Minkowski spaces and positive integers \( k < d - 1 \) such that \( E_k \) is not compact, see e.g. [36, Example 11.11].

Our next results refer once more to \( \alpha \)-, \( \beta \)-, \( \alpha^* \)- and \( \beta^* \)-spaces. In [174] it is shown that, if \( M^d \) has the representation \( M^d = M_1 \oplus \cdots \oplus M_m \) with norm \( \| \cdot \| = \sum_{i=1}^m \| \cdot \|_i \), then \( \alpha^*(M^d) = \).
\[ \sum_{i=1}^{m} \alpha^*(M_i). \] For any bounded set \( A \) in a \( \beta \)-space we have \( \text{cl}(\text{conv}_d A) = \text{conv}_d (\text{cl} A) \), and if the \( d \)-convex hull of a set \( A \) is a polytope, then \( \beta(A) \) is finite (cf. [174]). It is not hard to find Minkowski spaces that are \( \alpha \)-spaces but not \( \alpha^* \)-spaces, see [270, Example 9.3]. Furthermore it is easy to construct Minkowski spaces for which \( \text{conv}_d B \) is bounded but not closed, i.e., which are \( \alpha^* \)-spaces but not \( \beta \)-spaces, and also one can easily describe \( \beta \)-spaces that are not \( \beta^* \)-spaces (see [270, Example 9.2], [36, Example 10.4], and [270, Examples 12.4 and 12.5]). In addition, for any Minkowski space \( \mathbb{M}^d \), \( d \geq 2 \), one has \( d \leq 2^{\beta(M^d)} \), which is exact for \( \ell^1_1 \), cf. [174]. Using the following notions, one can describe unit balls of \( \beta^* \)-spaces geometrically. A one-dimensional linear subspace \( l \) of \( \mathbb{M}^d \) is said to be special if it is the intersection of \( d - 1 \) \( d \)-convex linear \((d - 1)\)-subspaces. A point \( x \in \text{bd} B \) is special if it is the intersection of \( \text{bd} B \) and a special subspace of \( \mathbb{M}^d \). The following theorem (cf. [262]) gives a geometric picture of unit balls of \( \beta^* \)-spaces.

**Theorem 18.** If \( \mathbb{M}^d \) is a \( \beta^* \)-space, then its unit ball is the convex hull of the closure of all its special points.

Let \( \mathbb{R}^{k+l} \) be the direct sum of its subspaces \( \mathbb{R}^k \) and \( \mathbb{R}^l \), and let \( B_1 \subset \mathbb{R}^k, B_2 \subset \mathbb{R}^l \) be unit balls of Minkowski spaces \( \mathbb{M}^k \) and \( \mathbb{M}^l \) obtained from \( \mathbb{R}^k \) and \( \mathbb{R}^l \), respectively. If the normed space \( \mathbb{M}^{k+l} \), obtained from \( \mathbb{R}^{k+l} \), has the unit ball \( B = \text{conv}\{B_1 \cup B_2\} \), then \( \mathbb{M}^{k+l} \) is said to be the join of \( \mathbb{M}^k \) and \( \mathbb{M}^l \). Then the norm of \( \mathbb{M}^{k+l} \) is \( \| \cdot \| = \| \cdot \|_1 + \| \cdot \|_2 \) where \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) are the norms of \( \mathbb{M}^k \) and \( \mathbb{M}^l \), respectively. The following results are obtained by P. Soltan, see [254] and [255]: Let \( \mathbb{M}^{k+l} \) be the join of \( \mathbb{M}^k \) and \( \mathbb{M}^l \), and \( a, b, x \) be points from \( \mathbb{M}^{k+l} \) such that \( (a - x) \in \mathbb{M}^k, (b - x) \in \mathbb{M}^l \). Then \( x \in [a, b]_d \). From this one can deduce the equality

\[ \text{conv}_d A = \text{conv}_d \pi_1(A) + \text{conv}_d \pi_2(A) \]

for an arbitrary set \( A \subset \mathbb{M}^{k+l} \), where \( \pi_1 \) denotes the parallel projection of \( \mathbb{M}^{k+l} \) along \( \mathbb{R}^l \) onto \( \mathbb{R}^k \), and \( \pi_2 \) vice versa. In addition, we have

\[ \text{diam}(\text{conv}_d B) = \text{diam}(\text{conv}_d B_1) + \text{diam}(\text{conv}_d B_2). \]

We say that the family of all \( d \)-convex sets in \( \mathbb{M}^d \) has the \( d \)-cone property if for any \( d \)-convex set \( A \subset \mathbb{M}^d \) and any point \( z \in \mathbb{M}^d \)

\[ \text{conv}_d (A \cup z) = \bigcup \{ [z, a]_d, a \in A \} \]

holds.

Furthermore, the family of all \( d \)-convex sets in \( \mathbb{M}^d \) has the \( d \)-jointness property if for any two \( d \)-convex sets \( A, B \subset \mathbb{M}^d \) the set \( \text{conv}_d (A \cup B) \) is the union of all \( d \)-segments \( [a, b]_d \) with \( a \in A, b \in B \). Also \( \mathbb{M}^d \) is said to be the join of linear subspaces \( L_1, \ldots, L_k \) if it is their direct vector sum and the unit ball \( B \) of \( \mathbb{M}^d \) satisfies \( B = \text{conv}(B \cap \bigcup_{i=1}^k L_i) \). In [260] the following theorem is established.

**Theorem 19.** For a Minkowski space \( \mathbb{M}^d \) the following conditions are equivalent.

1. The family of all \( d \)-convex sets in \( \mathbb{M}^d \) has the \( d \)-cone property.
2. The family of all $d$-convex sets in $M^d$ has the $d$-jointness property.

3. $M^d$ is the join of some linear subspaces $L_1, \ldots, L_k$ each of which has a strictly convex unit ball or is two-dimensional.

We say that the family of all $d$-convex sets in $M^d$ has the normality property if any two disjoint $d$-convex sets $A, B$ can be separated by some complementary $d$-convex half-spaces (or if there are $d$-convex sets $C, D$ with $A \subset C, B \subset D$ and $C \cup D = M^d$). Using this notion, V. Soltan [260] shows the following.

**Theorem 20.** The following properties of a Minkowski space $M^d$ are equivalent.

1. The family of all $d$-convex sets in $M^d$ has the normality property, i.e., any two $d$-convex sets $A, B$ with relint $A \cap \text{relint } B = \emptyset$ are strictly separable by some $d$-convex hyperplane.

2. The vector sum $A + B$ of any two $d$-convex sets $A, B$ is itself a $d$-convex set.

3. The vector sum of any two $d$-convex linear subspaces $L, M$ of $M^d$ is a $d$-convex subspace of $M^d$.

4. The space $M^d$ is the join of some of its linear subspaces $L_1, \ldots, L_k$ each of which has a strictly convex unit ball or is two-dimensional.

In [101] and [100] it is shown that if $d$-convex sets $A_1, A_2 \subset M^d$ satisfy relint $A_1 \cap \text{relint } A_2 = \emptyset$, then there are such maximal (under inclusion) closed $d$-convex cones $K_1, K_2$ such that $A_1 \subset K_1, A_2 \subset K_2$ and relint $K_1 \cap \text{relint } K_2 = \emptyset$.

At the end of this section we will discuss some combinatorial aspects of $d$-convexity, in particular related to generalizations of Helly’s theorem. The Helly dimension $\text{him } F$ of an (infinite) family $F$ of sets in $\mathbb{R}^d$ was introduced in [254], see also [38], [37, Chapter IV] and [36, Chapter II]. It is the smallest integer $m > 0$ such that for every collection $\{M_1, \ldots, M_s\} \subset F$ with $s > m + 1$ the following holds: if each $(m+1)$-subfamily from this collection has a common point, then $M_1 \cap \cdots \cap M_s \neq \emptyset$. If $F$ does not have this property, then $\text{him } F = \infty$. Thus if $0 < \text{him } F < \infty$ is assumed, then $\text{him } F$ is the largest integer $m$ for which there are sets $M_1, \ldots, M_{m+1} \in F$ such that each $m$ of them have nonempty intersection and $M_1 \cap \cdots \cap M_{m+1} = \emptyset$.

In the literature one can also find the Helly number of $F$, which in general differs from him $F$ only by one, cf. [70], [270] and [83].) For a Minkowski space $M^d$ one can now consider the family $\mathcal{C}_d$ of all $d$-convex sets and the family $\mathcal{K}_d$ of all $d$-convex bodies (i.e. with non-empty interior) with the conventions him $\mathcal{C}_d = \text{him } M^d$ and him $\mathcal{K}_d = \text{him } (b) M^d$, calling the first of these numbers the Helly dimension of the space $M^d$. It easily follows that, due to $\mathcal{C}_d \supset \mathcal{K}_d$, $\text{him } M^d \geq \text{him } (b) M^d$, and from [165] (even for metric spaces), [37, § 18] and [36, § 14] one can read off various further results on these numbers. E.g., if $M^d$ is representable as a join of the spaces $L_1, \ldots, L_k$, then

$$\text{him } M^d = \max_{i=1,\ldots,k} \text{him } (b) L_i,$$
a result obtained by Lassak [165] even for metric spaces. Further results on the Helly dimensions of special set families (see e.g. Chapter IV in [36]) can be carried over to Minkowski spaces if such a set family is exactly the family of d-convex sets.

Analogously one can extend further classical theorems, such as those of Radon and Carathéodory, in terms of d-convexity, see [270, Chapter II], [164], and [167].

3.6 On d-star-shaped sets in Minkowski spaces

Again we start with some results holding even in metric spaces.

A set $A \subset X$ is called d-star-shaped with respect to $z \in A$ if $[z, x]_d \subset A$ for any $x \in A$. The family of all points $z \in A$ with respect to which $A$ is d-star-shaped is said to be the d-kernel of $A$ and denoted by $\text{kern}_d A = \{ z \in A : x \in A \Rightarrow [z, x]_d \subset A \}$.

The authors of [274] prove that for any $x, y \in X$ and all $\delta \geq d(x, y)$ the “metrical ellipsoid” $U(x, y, \delta) = \{ z \in X : d(x, z) + d(z, y) \leq \delta \}$ is d-star-shaped, where its “foci” $x, y$ belong to $\text{kern}_d U(x, y, \delta)$. From this it follows that any d-segment $[x, y]_d$ and any ball $B_\delta(x)$ of radius $\delta$ around $x$ are d-star-shaped.

We continue with results about d-star-shaped sets restricted to Minkowski spaces and define (besides the d-kernel $\text{kern}_d A$ of a set $A \subset \mathbb{M}^d$) also the set $K(A)$ as the intersection of all maximal (under inclusion) d-convex sets contained in $A$. Obviously, one has $K(A) \subset \text{kern}_d A \subset \text{kern} A$, with $\text{kern} A$ as the usual kernel of $A$, defined by $\text{kern} A = \{ z \in A : x \in A \Rightarrow [z, x] \subset A \}$. In addition, it is clear from the considerations above that any d-star-shaped set is star-shaped in the usual sense. Again from [274] we take

**Theorem 21.** For a Minkowski space $\mathbb{M}^d$ the following conditions are equivalent.

1. The families of d-star-shaped and star-shaped sets coincide.

2. $K(A) = \text{kern} A$ for any set $A \subset \mathbb{M}^d$.

3. $\text{kern}_d A = \text{kern} A$ for any set $A \subset \mathbb{M}^d$.

4. The unit ball of $\mathbb{M}^d$ is strictly convex.

For any set $A$ and every $x \in A$ let $A_x := \{ z \in A : [x, z]_d \subset A \}$, and for $N \subset A$ we write $A_N = \bigcap \{ A_x : x \in N \}$. A point $x \in A$ is a k-extremal point of $A$ if $x \notin \text{relint} S$ for any $(k + 1)$-dimensional simplex $S \subset A$. It is shown in [285] that for a compact set $A \subset \mathbb{M}^d$, where $\mathbb{M}^d \setminus A$ is connected, the following holds: If $N$ is the union of all $(d - 2)$-extremal points of $A$, then $A$ is d-star-shaped iff $A_N \neq \emptyset$; in this case we even have $A_N = \text{kern}_d A$. Furthermore, a point $x \in A$ is said to be locally d-convex in $A$ if there is a neighbourhood $N(x)$ of $x$ such that $N(x) \cap A$ is d-convex; otherwise $x$ is locally d-nonconvex in $A$. Besides other related results, the paper [284] contains the following statement: For a compact, connected set $A \subset \mathbb{M}^d$ we have $x \in \text{kern}_d A$ iff each $y \in A$ which is locally d-nonconvex in $A$ has a neighbourhood $N(y)$ such that $[x, z]_d \subset A$ for $z \in A \cap N(y)$. Moreover, a closed connected set $A \subset \mathbb{M}^d$ is d-convex iff any of its points is locally d-convex. Further related results on d-star-shaped sets are obtained in [274], [264], [265], [286], [287], [288], and [35], see also [270, § 14]. In particular we want to mention results that are related to Krasnosel’ski’s theorem: if every $d + 1$ points of a star-shaped set $A \subset \mathbb{E}^d$ are visible in $A$ from a point $x \in A$, then all points of $A$ are visible in $A$ from
a point of $A$ (note that $y \in A$ is visible in $A$ from $x \in A$ if $[x, y] \subset A$). E.g., in [35] an analogue of Krasnosel’ski’s theorem by means of $d$-visibility (i.e., visibility defined with the help of $d$-segments instead of usual segments) is given, which has a forerunner already in [38], see also [36, § 15]. A further extension of the usual visibility notion is proposed in Problem 5 of [36, Chapter VIII].

3.7 Some applications of $d$-convexity

First we mention that the notion of a convex function can be successfully extended to $d$-convex functions, cf. [273, 266, 269]. Continuing [81], the authors of [197] investigate the Fermat-Torricelli problem from Location Science in Minkowski spaces, i.e., the search of those points having the minimal sum of distances to arbitrarily given points. For solving certain cases of this problem they use the concept of concurrent $d$-segments. Verheul [294] uses the concept of modular Banach spaces for getting related results on Steiner trees. Brown [41] considers Menger’s betweenness relation in the context of Approximation Theory in Minkowski spaces.

In Computational Geometry $d$-segments are also studied, see [153]. In particular we observe that any Voronoi region of a Voronoi diagram in $M^d$ is in fact $d$-star-shaped. Again with the background of computational geometry, partitions of compact point sets into special (convex) parts by cuts in prescribed directions (with various practical applications) can be successfully studied in terms of $d$-convexity, see [227], [271] and [195]. Also one should mention that there are many results on $d$-convexity in ordinary (including infinite) graphs, see the respective parts of [270]. The only survey we could find in this field is [272]. In the context of combinatorial geometry, fixing systems of $d$-convex bodies are studied in [34].

3.8 Further convexity notions in Minkowski geometry

First we will discuss the notion of $B$-convexity (ball convexity) which is introduced by Lassak in [166]. (To avoid confusion, we note that another notion of $B$-convexity is considered in [106], also for normed linear spaces, but quite different in nature.) A set $A$ in a metric space $X$ is said to be $B$-convex if, for any finite number of points, $A$ contains the intersection of all closed balls containing these points. In [166] the relation $\text{diam} \, \text{conv}_B A = \text{diam} A$ for any set $A \subset X$ is shown, where $\text{conv}_B A$ denotes the intersection of all $B$-convex sets containing $A$, also called the $B$-convex hull of $A$. That paper contains further results for Minkowski spaces. E.g., if $A$ is $B$-convex, then also the affine hull, the interior and the relative interior of $A$ are $B$-convex. For each boundary point $x$ of a $B$-convex body $K$ there exists a $B$-convex hyperplane supporting $K$ at $x$, and any $B$-convex body $K$ is representable as an intersection of $B$-convex closed half-spaces or (in the bounded case) of balls. In Euclidean space, usual convexity and ball convexity coincide, and in [168] this statement is shown to hold also for smooth unit balls. This paper contains further generalizations of the above statements, such as the observation that a set $A$ is $B$-convex and compact iff it is an intersection of $B$-convex closed half-spaces or (in the bounded case) of balls. In Euclidean space, usual convexity and ball convexity coincide, and in [168] this statement is shown to hold also for smooth unit balls. This paper contains further generalizations of the above statements, such as the observation that a set $A$ is $B$-convex and compact iff it is an intersection of $B$-convex closed half-spaces or (in the bounded case) of balls. Also various results on $B$-convex hulls, $B$-convex cones and related separation properties are derived. It is interesting to compare $B$-convexity with the notion of $d$-convexity, as it is done in [169]. For instance, let us consider two Minkowski spaces, one with unit ball $B$ and the other with metric $d$ such that the $d$-convex hulls of finite sets are closed. Then the families of $B$-convex sets and $d$-convex sets are identical iff the families of $B$-convex subspaces and
A mapping of a Minkowski plane onto itself iff it is \( d \)-contractive with respect to \( \| \cdot \| \). A mapping \( f \) of \( \mathbb{M}^d \) is called \( \| \cdot \| \)-contractive if for any \( x, y \in \mathbb{M}^d \) the relation \( \| f(x) - f(y) \| \leq \| x - y \| \) holds. Besides various related results, Gruber [118] shows that a set \( F \) is the fixed point set of some \( \| \cdot \| \)-contractive mapping of a Minkowski plane onto itself iff \( F \) is closed and metrically convex with respect to \( \| \cdot \| \); see also the references there for related investigations. In another paper Gruber [119] gives a characterization of Chebyshev sets in Minkowski planes by means of a sharpening of metric convexity.

Another kind of convexity was introduced by Boltyanski, see, e.g., [36, Chapter III] for a summary. Namely, let \( H \) denote an arbitrary subset of the unit sphere \( S^{d-1} \) in Euclidean \( d \)-space. Each half-space of the form \( \{ x : \langle x, y \rangle \leq \lambda \} \) with \( y \in H \) and \( \lambda \in \mathbb{R} \) is said to be an \( H \)-convex half-space, and any set representable as intersection of a family of \( H \)-convex half-spaces is called an \( H \)-convex set. There are relations between \( H \)-convexity and \( d \)-convexity in a Minkowski space \( \mathbb{M}^d \), cf. [36, § 24]. E.g., one can identify \( H \) with the set of all Euclidean unit vectors orthogonal to \( d \)-convex hyperplanes. With this convention, a closed convex set \( A \subset \mathbb{M}^d \) with interior points is \( d \)-convex iff it is \( H \)-convex, and so also Helly-type statements can be generalized for Minkowski spaces with the help of \( H \)-convexity.
There are more types of generalized convexity notions that might be successfully combined with Minkowski geometry. However, we did not find further substantial references related to this.

4 Bisectors and Voronoi diagrams

4.1 Introduction

Various unsatisfactory definitions of planes in Euclidean space led Leibniz to define a plane as the locus of points having equal distances from two different points \( p \) and \( q \). Since this definition can be carried over to various types of metric spaces, it makes sense to introduce, in this general setting, the analogous point sets as bisectors (or equidistant sets) with respect to \( p \) and \( q \), see [194], [44], [46], [148], [9], and [214] for early contributions to this notion, the oldest being the theorem of Mann [194] that a Minkowski space is Euclidean if its bisectors are convex. A really deeper study of geometric properties of bisectors in Minkowski spaces started only with the development of Computational Geometry. In the following we will survey results on bisectors in Minkowski Geometry. The main applications of bisectors in this field are

- constructions of (generalized) Voronoi diagrams and
- motion planning with respect to translations,

but also interesting characterizations of special types of Minkowski spaces can be obtained. There are various other definitions of bisectors, and also the given geometric configuration \{p, q\} can be modified. Therefore we will also present results on natural extensions and related concepts. E.g., many recent results about Voronoi diagrams in the spirit of Computational Geometry were obtained without the assumption that the unit ball is centrally symmetric (i.e., it is often assumed to be a gauge body), and also the notion of angular bisector is interesting in these spaces. In addition, bisectors were used to characterize Minkowski spaces within classes of more general types of spaces. We will also survey results on Voronoi diagrams closely related to Minkowski spaces. Basic references containing material about this viewpoint are [153], [14], and [15], but see also the monograph [212]. Furthermore, we present applications of Voronoi diagrams (such as translational motion planning) and similar concepts. Finally some further related topics, namely besides angular bisectors also sets equidistant to only one given site, are discussed.

4.2 Basic geometric properties of bisectors

The bisector of two points \( p \neq q \) in a Minkowski space \( \mathbb{M}^d \) is the set

\[
B(p, q) := \{x \in \mathbb{M}^d : \|x - p\| = \|x - q\|\}.
\]

The tools needed to study bisectors mainly belong to the field of classical convexity. In particular, one has to consider intersections of homothets of a convex surface, cf. our first survey [196, §§ 3.3] for an extensive discussion of related results. In addition one has to underline that applications of bisectors in computational geometry (mainly referring to Voronoi diagrams) often allow an immediate consideration of underlying gauges (i.e. the unit ball does not necessarily have a centre of symmetry), where computational geometors
mainly use the term “convex distance function”. In other words, the unit ball is a compact, convex body with the origin \( o \) in its interior. So in this section the underlying real linear space has a gauge, and whenever we consider the particular case of a Minkowski space, this will be said explicitly or will be clear from the context.

Geometric properties of bisectors in (special) Minkowski planes were first studied by [160], [180] and [181] in the \( \ell_1 \) metric and, more generally, in \( \ell_p \) norms. Independently, [233] found an interesting duality between the maximum and Manhattan norm in the plane by using geometric properties of bisectors, \( d \)-segments and circles in these norms; a discrete version of this duality and the needed geometry of bisectors is presented in [71].

For two-dimensional gauges, the first explicit proofs of the properties presented below, and related ones, were given in [200, 66, 142, 187], see also [63]. For unit balls in the plane and centred at the origin, an earlier contribution is due to Holub [136], and some if his results were presented in [196]. For special norms, investigations were done by Lee [180] and in [301]. A broad presentation can be found in the second chapter of [187].

In a Minkowski plane consider two different points \( p \) and \( q \). Let the line segment \([p, q]\) be horizontal, with \( p \) to the left of \( q \), and let \( B_p, B_q \) be translates of unit balls having the centres \( p \) and \( q \), respectively. The top point of \( B_p \), denoted by \( t_p \), is the leftmost point of \( T \cap B_p \), and the top point of \( B_q \), denoted by \( t_q \), is the right most point of \( T \cap B_q \), where \( T \) is the top supporting line of \( B_p \) and \( B_q \) parallel to \([p, q]\). Analogously, the bottom points \( b_p, b_q \) are defined with the help of the bottom supporting line \( D \) of \( B_p \) and \( B_q \) which is parallel to \([p, q]\), see Figure 3.

![Figure 3. Constructing a bisector](image)

**Proposition 22.** In a Minkowski plane the bisector \( B(p, q) \) is fully contained in the bent strip bounded by the rays \([p, t_p]\), \([q, t_q]\), \([p, b_p]\), and \([q, b_q]\). It is homeomorphic to a line iff \([p, q]\) is not parallel to a non-degenerate segment in \( \text{bd } B_p \), and it contains two two-dimensional regions iff \([p, q]\) is parallel to some non-degenerate line segment in \( \text{bd } B_p \).

This statement can be easily verified by considering \( B(p, q) \) as the union of all intersections of equally sized homothetical copies of \( \text{bd } B_p \) and \( \text{bd } B_q \) with centres \( p \) and \( q \), respectively. Thus in Figure 3 we have \( x \in B(p, q) \). If the Minkowski plane is strictly convex, the bent strip in the above proposition is in fact only a strip bounded by two
parallel lines. It is furthermore easy to see from the above proposition that it is only in strictly convex planes that the bisector is always contained in a strip bounded by two parallel lines. So one can conclude that, e.g., for strictly convex unit balls all bisectors are homeomorphic to lines (Holub [136] found this characterization for Minkowski planes, see also [196, § 3.3]). This can be analogously transferred to higher dimensions, see [143] for $d = 3$. Independently, this statement was verified for $d \geq 3$ and Minkowski unit balls by Horváth [137], i.e., together with Example 3 from [137] referring to the converse implication we have

**Theorem 23.** If the unit ball of a Minkowski space $M^d$ is strictly convex, then all bisectors are homeomorphic to a hyperplane. On the other hand, there exists a space $M^d$ for every $d \geq 3$ such that each bisector is a topological hyperplane, but the unit ball of $M^d$ is not strictly convex.

The authors of [143] give a local formulation for unit balls $B$ in 3-space that are not necessarily centrally symmetric: if each supporting line of $B$ parallel to the line through the two sites $p, q$ meets $B$ in exactly one point, then $B(p, q)$ is homeomorphic to a plane; see also [187, § 3]. Closely related is a paper of J. E. Valentine [291]: A Minkowski plane is said to satisfy Euclid’s Proposition 7 provided for four points $a, b, c, d$, with $c, d$ on the same side of $aff\{a, b\}$ and $|a - c| = |a - d|$ as well as $|b - c| = |b - d|$, the coincidence $c = d$ holds. It is proved in [291] that a real Banach space is strictly convex iff each of its 2-subspaces satisfies Euclid’s Proposition 7.

In addition, the paper [65] contains some further geometric properties of bisectors in Minkowski planes that are somehow unexpected. They can be summarized by

**Theorem 24.** In a Minkowski plane with strictly convex unit ball, bisectors do not necessarily have asymptotic lines, and pairs of bisectors can exist that intersect infinitely many times.

In the terminology of [156] (see also [153, § 1.2]) this means that strictly convex norms are not always nice metrics. However, the authors of [65] derive a necessary and sufficient condition for a given bisector to have an asymptotic line. They prove that if the boundary of the strictly convex unit ball is a semialgebraic curve, then the intersection of any two bisectors has a finite number of connected components (yielding therefore a nice metric).

Based on Blaschke’s notion of shadow boundary (of the unit ball, cf. the surveys [219] and [131] for related results), Hetzelt [133] geometrically describes the asymptotic behaviour of bisectors in Minkowski spaces, which is then applied to Approximation Theory. Horváth [138] proves that in a Minkowski 3-space all bisectors are homeomorphic to a plane iff all shadow boundaries of the unit ball are topological circles.

Refining the methods suggested by Proposition 22 and Figure 3 above, the authors of [145] (cf. also [144] and [299]) construct bisectors $B(p, q)$ in the plane where the distances from $p$ and $q$ are measured with respect to different gauges. Here the bisector can contain bounded or unbounded two-dimensional regions, and pieces of the bisector may even appear inside the region of all points closer to $p$ than to $q$. In particular, polyhedral gauges are taken into consideration, and applications in Location Science are discussed.

Finally we mention here a paper of Guggenheimer [125], in which properties of bisectors where the unit ball is a triangle are discussed.
4.3 Characterization theorems based on bisectors

We start with characterizations of inner product spaces within the family of normed linear spaces by bisector properties. An old result of this type was proved by Mann [194]: A Minkowski space is Euclidean iff all its Leibnizian half-spaces (consisting of all those points which are nearer to the origin than to another point) are convex. It is clear that, considering the bounding hyperplanes of Leibnizian half-spaces, this means that all bisectors are hyperplanes. Thus, strictly speaking Mann’s theorem is a forerunner of a known characterization theorem of Day [72] that was obtained as a corollary of a result of James [146], but see also [202]. Namely, we have

**Theorem 25.** All bisectors in a Minkowski space $\mathbb{M}^d$ are hyperplanes iff the unit ball is an ellipsoid, i.e., $\mathbb{M}^d$ is a Euclidean space.

Strongly related characterizations are collected in §3 of Amir’s book [8]. For example, Kalisch and Straus [148] define a subset $A$ of a Minkowski space $\mathbb{M}^d$ to be a determining set if $\|p - a\| = \|q - a\|$ for all $a \in A$ implies that $p = q$. They show that $\mathbb{M}^d$ is Euclidean iff every $A \subset \mathbb{M}^d$ not contained in a hyperplane is a determining subset of $\mathbb{M}^d$. Not cited in [8], but strongly related to Mann’s observation is a result of Panda and Kapoor [214]. They show that replacing “hyperplanes” in Theorem 4 by “convex” gives the same characterization, and they continue with similar results, taking special care for $\ell_p$ spaces. Considering tilings in infinite dimensional Banach spaces, Klee [152] points out that the results in [148] and [214] are near to his investigations.

There are several sharpenings and extensions of Theorem 4 that should be mentioned here. First we mention that the famous Blaschke-Marchaud characterization of ellipsoids by plane shadow boundaries (cf. the surveys [219] and [131] for related references) implies that for $d \geq 3$ a $d$-dimensional Minkowski space is Euclidean iff each bisector is contained in a region bounded by two parallel hyperplanes. (For $d = 2$ this property characterizes strict convexity, see Proposition 22 and the remark following it.) Woods [302] extends Mann’s result to distance functions whose unit ball is not necessarily centrally symmetric or convex. In a series of papers on ellipsoid characterizations (see [115, 116, 117]) Gruber gives further generalizations, such as the following: The statement of Mann still holds if the unit ball is star-shaped with respect to the origin, but not necessarily bounded, convex, or centrally symmetric. Also, Satz 5 in [115] says that a bounded distance function yields the Euclidean norm iff there is a subset $Q$ of the Euclidean unit sphere $S^{d-1}$ having interior points with respect to $S^{d-1}$ and the property that for each $x \in Q$ the Leibnizian half-space of $\{o, x\}$ is convex. Theorem 25 could also be deduced from another theorem of Gruber (see [116, Satz 3]): If $K$ is a convex body in $E^d (d \geq 3)$ and the intersection of the boundaries of the bodies $K'$ and $K$ is contained in a hyperplane for all translates $K'$ of $K$ ($K' \neq K$), then $K$ has to be an ellipsoid. Goodey [110] gives a further extension: If $K_1, K_2$ are convex bodies in $E^d$, $d \geq 3$, and the intersection of the boundaries of the bodies $K_1$ and $K_2$ is contained in a hyperplane for all translates $K'_2$ of $K_2$ ($K'_2 \neq K_1$), then $K_1, K_2$ are homothetic ellipsoids, see also [111] and [113]. For special (convex) surfaces this result was earlier obtained by Shaidenko [247].

Beem [23] extends Theorem 25 to indefinite inner product spaces considering the flatness of bisectors in real Hausdorff topological vector spaces.
Guijarro and Tomás [126] define a perpendicular bisector in a real normed space to be a hyperplane containing the midpoint of the segment $[p, q]$ and satisfying an orthogonality condition with respect to $[p, q]$ which is defined by a certain generalization of the inner product. They show that the space is Euclidean iff for all pairs $p \neq q$ the perpendicular bisector and $B(p, q)$ coincide.

Next we mention an interesting characterizing property of Hilbert spaces. It is easy to see that for $p \neq q$ in Euclidean space the bisector $B(p, q)$ is a usual sphere if one considers differently weighted distances from $p$ and $q$, respectively. More precisely, such a bisector is an Apollonius sphere, i.e., the geometric locus of those points for which the ratio of the distances from $p$ and $q$ (defined by the different weights) is constant. Answering a question of Stechkin, Danelich [68] proves that a normed linear space whose Apollonius spheres (or weighted bisectors) are spheres in the norm is a Hilbert space.

Here two characterization theorems should be mentioned that are more related to reflection (inversion) than to bisectors. Namely, in a Minkowski space $M^d$ the inversion at its unit sphere is the mapping $x \mapsto x/\|x\|^2$ for all $x \in M^d \setminus \{0\}$. Stiles [278] proves the following two statements.

**Theorem 26.** If there exists a line in a Minkowski plane whose inversive image is a Minkowski circle, then the plane is Euclidean. Also, if in a $d$-dimensional Minkowski space the inverse image of any supporting hyperplane of the unit ball is centrally symmetric, then this space is Euclidean.

For special norms this mapping was also investigated in [108].

We also mention that in [7] a type of angle bisectors of triangles is used to characterize inner product spaces, and similar results are given in [6].

Busemann [44] proposes an extension of Leibniz’s definition of a plane (namely to be the bisector $B(p, q)$ of two points $p \neq q$ in Euclidean space) to certain metric spaces. More precisely, he assumes that such a metric space $S$ is finitely compact (i.e., every bounded sequence of points has a convergent subsequence) and that any two distinct points from $S$ lie on a metric line. He shows that $S$ is a finite dimensional Euclidean or hyperbolic space iff all its bisectors are linear, i.e., for any bisector $B(p, q)$ with $x, y \in B(p, q)$, $x \neq y$, also the metric line through $x$ and $y$ is in $B(p, q)$.

Motivated by Fréchet’s characterization of normed linear spaces [91] among general metric spaces and continuing Busemann’s work [44] on bisectors, Andalafte and Blumenthal [9] characterize Minkowski spaces and also Euclidean spaces among all finitely compact, metric spaces by properties that are closely related to bisectors, see also [29, Chapter 7]. In [45, Chapter IV], [46, § 16], [47, § 46.1 and § 47.4], and [51, Chapter IV] one can find further characterizations of finite-dimensional normed linear spaces, Euclidean, hyperbolic and spherical spaces via bisectors similar to the results from [44]. In [51, Chapter IV] the flatness of bisectors is used, see also [223] for a stronger statement in the planar case; Guggenheimer [125] gives related results for gauges.

A similar characterization of Minkowski planes, namely by the flatness of bisectors with respect to a given point and a given line, is derived in [49], and for a generalization we refer to [50]. Note that the concurrency of bisectors in triangles is discussed in [49], see also [125] for gauges. Also in the spirit of [44], Andalafte and Freese [12] give a characterization of inner product spaces within a large class of metric spaces.
Danzer [69] introduces convex sets in a metric space $S$ as intersections of Leibnizian half-spaces. Then from Busemann’s investigations in [44] the following can be concluded if $S$ is a Minkowski space $M^d$ and $A \subset M^d$ is an arbitrary closed convex set in the above sense: $M^d$ is an inner product space iff any two different points from $A$ have a betweenness point in the sense of Menger, cf. Section 3.2 above.

4.4 Sets equidistant to more than two points

In view of vertices in Voronoi diagrams it is essential to investigate sets that are equidistant to three or more given sites. Although the term “$k$-sectors” would be more appropriate, we will follow many authors and use the notion of $k$-bisectors for sets equidistant to $k > 2$ given points. It is clear that the union of the midpoints of all spheres (defined by the given metric or norm) that pass through all these $k$ points is the $k$-bisector.

We refer to [196, §§7.1] for a discussion of the following result of Kramer and Németh [159], see also [158].

**Proposition 27.** A Minkowski plane is smooth iff through any three non-collinear points there passes at least one Minkowski circle.

The analogous statement for strictly convex gauges was derived in [200], [199] and [142], related further observations are collected in §2 of Ma’s dissertation [187].

The proof of Kramer and Németh is generalizable to give the forward implication in $d$-dimensional spaces with gauges. According to Kramer and Németh this was a conjecture of Turán. However, the corresponding statement was proven earlier by Gromov [113].

**Theorem 28.** Let $\mathbb{R}^d$ be equipped with a gauge $\gamma_B$. If the unit ball $B$ is smooth then at least one sphere passes through any $d + 1$ non-collinear points.

Using complicated topological methods, Makeev [191] reproves that theorem; see also [192], where a local version is shown.

Based on results from the theory of additive complexity, Lê [175, 176] proves that for $p$ an even integer there exists an upper bound on the number of $l_p$-spheres in $d$-dimensional space that can pass through $d + 1$ points in general position, this upper bound depending only on $d$ but not on $p$. In [178] and [179] he proves further related results for the $d$-dimensional situation. E.g. the following observation, based on Goodey’s [110] characterization of homothetical ellipsoids, is established in [179]: If the gauge in $d$-space, $d \geq 3$, is not determined by an ellipsoid, then there exist $d + 1$ affinely independent points such that their ($d + 1$)-bisector consists of more than one point.

There are drastic differences between the geometric properties of $k$-bisectors in two- and three-dimensional spaces. However, a deeper study in 3-space is only very recent. First we consider $3$-bisectors for $d = 3$. For smooth and strictly convex gauges it is shown in [142] that all 3-bisectors of three non-collinear points are homeomorphic to a line. On the other hand, for non-smooth gauges in 3-space there is even no general upper bound on the number of their connected components, cf. [177]. This is sharpened in [143] and [187, §3]: Using the concept of *silhouettes* (also called sharp shadow boundaries in Convexity Theory), they give conditions for the connectedness of 3-bisectors, or for the precise number of their connected components. They assume that each supporting line of the unit ball $B$, which is parallel to any segment connecting two of the three
non-collinear given sites, meets $B$ in exactly one point. The union of these points for one of the three directions is called the silhouette of $B$ in the respective direction. It is homeomorphic to a simple closed curve partitioning the boundary of $B$ into two open half-spheres. The results mentioned above are obtained by studying suitable intersection properties regarding pairs of such half-spheres, namely with respect to the connectedness of these intersections. It is pointed out in [143] that these results might be the starting point for computing 3-bisectors, e.g. for polyhedral unit balls.

Lastly we consider 4-bisectors in 3-space. In [142] the following statement is established.

**Theorem 29.** For each $n \geq 0$ there exist a smooth, strictly convex Minkowski 3-space and four points in it such that the corresponding 4-bisector contains exactly $2n + 1$ points. If the four points are moved independently in small three-dimensional neighbourhoods, the 4-bisector still contains at least $2n + 1$ points.

There are further related results in [175, 176] and [187], for example referring to special $l_p$ norms, polyhedral gauges and algorithmical approaches to corresponding bisectors in 3-space. E.g., in [187, §3.2] the following selected results for polyhedral gauges with $k$ facets are established, in every case assuming that the occurring silhouettes are simple closed polygons.

- A bisector which has $n$ vertices can be computed in optimal time $\Theta(n + k)$.
- Any 3-bisector is the union of segments and rays, and it has $O(k^2)$ vertices.
- For $k \geq 6$ there exist 4-bisectors consisting of at least two points.

### 4.5 Voronoi diagrams

The interdisciplinary concept of Voronoi diagrams can be roughly described as follows: In a space $D$ let there be given a set $S$ of fixed sites, together with a well-defined quantity of influence that a site $s \in S$ can exert on a point $x \in D$. The Voronoi region of $s$ then consists of all points $x \in D$ that are stronger influenced by $s$ than by any other member of $S$. The survey [14] presents some different names for Voronoi regions in various scientific fields, such as Thiessen polygons in Meteorology and Geography, or Wigner-Seitz zones in Chemistry and Physics. However, the first scientists who gave a formal introduction to this concept were the mathematicians Dirichlet [78] and Voronoi [296], [297], and so the structure created by Voronoi regions of $n \geq 3$ point sites in the Euclidean plane (taking the Euclidean distance as quantity of influence) has been called a Dirichlet tessellation or Voronoi diagram. Voronoi [297] was also the first who considered the dual structure, in which any two point sites are connected by a segment if their Voronoi regions have common boundary. Delaunay [74] observed that the same structure is obtained in the following way: two point sites are linearly connected if they lie on a circle whose interior has empty intersection with $S$. Hence this dual concept is usually called Delaunay triangulation.

From the geometric point of view, Voronoi diagrams and their generalizations (in higher dimensions and non-Euclidean geometries, or with sites that are no longer points) were mainly studied in Computational Geometry and Stochastic Point Processes, or as special tilings. Basic references showing the large variety of existing literature in these and
related directions are Aurenhammer [14], Klein [153], Okabe, Boots and Sugihara [212], Fortune [89], and Aurenhammer and Klein [15], and various textbooks and monographs in Computational Geometry contain chapters on Voronoi diagrams. Basic references for stochastic geometers are the two monographs [212] and [279].

Among the various ways of generalizing the original concept, we are interested in geometric properties of Voronoi diagrams in Minkowski spaces and their direct generalizations with respect to gauges.

Perhaps the oldest result on Voronoi diagrams in Minkowski spaces is due to Mann [194]. He proved that if for all lattices of the underlying space the closed Voronoi region of a lattice point is convex then the norm is Euclidean. (It should be noticed that Gruber [116] generalized this to distance functions with bounded star-shaped unit balls, and for Minkowski spaces Horváth [137] continued Mann’s investigations on lattice-like Voronoi diagrams.)

Chew and Drysdale [63] note that Voronoi regions in Minkowski planes are star-shaped (the proof holding in Minkowski spaces as well and in fact gives that the regions are d-starshaped). On the other hand, convexity fails to hold already in the $\ell_1$-plane, see also [153, p. 23]. This star-shapedness and the convexity of the unit ball yield the possibility to apply divide & conquer algorithms for planar Voronoi diagrams. Thus Hwang [140], Lee and Wong [181] investigate the $\ell_1$-plane, and Lee [180] the $\ell_p$-planes; they succeed with algorithms to construct Voronoi diagrams of $n$ sites in $O(n \log n)$ time. In [301] an optimal algorithm for the computation of a Voronoi diagram is given, where the distance function is based on polygonal unit balls, and Chew and Drysdale [63] present a construction in $O(n \log n)$ time for arbitrary gauges in the plane. Kühn [162] gives a randomized parallel algorithm for the computation of Voronoi diagrams in Minkowski planes, and Skyum [251] constructs the dual of the Voronoi diagram for arbitrary gauges in the same time complexity as Chew and Drysdale did. The paper [66] deals with the question when two planar Voronoi diagrams for the same sites, but for different distance functions, have the same combinatorial structure. If $C, D$ denote two smooth, strictly convex unit balls and the corresponding Voronoi diagrams $V_C(S), V_D(f(S))$ have the same combinatorial structure for each set $S$ of at most four points (for some bijection $f$ of the plane), then $f$ is linear and $f(C) = D$, up to scaling. Even stronger (in the case where $D$ is an ellipse) [66] contains

**Theorem 30.** Let $C$ be the unit ball of a Minkowski plane that is not Euclidean. Then there exists a set of 9 points whose Voronoi diagram with respect to $C$ has a combinatorial structure that no Euclidean Voronoi diagram can achieve.

Related investigations regarding the $\ell_1$ and $\ell_\infty$ metric are presented by Chew [62].

Further particular results on Voronoi diagrams with respect to gauges were obtained for three dimensions, and they are based in a natural way on the results on 4-bisectors given above. For example, Theorem 29 implies that there exists no upper bound on the number of vertices of a Voronoi diagram of four points in a smooth, strictly convex Minkowski 3-space. However, for special cases more can be said. For example in [32] it is shown that for $n$ given points in general position (the precise definition depending on the norm) in 3-space the complexity of a Voronoi diagram is $\Theta(n^2)$ if the unit ball is an octahedron, a cube or a tetrahedron (with any interior point of the tetrahedron as the
More generally, Tagansky [282] obtains $O(k^3 \alpha(k)n^2 \log n)$ for polyhedral unit balls with $k$ facets in 3-space. This was improved in [141] to $O(n^2 k^3 \alpha(\min(k, n)))$ if the unit ball again has $k$ facets, which simplifies to $\Theta(n^2)$ for $k$ fixed. Here $\alpha(n)$ is the (extremely slowly growing) inverse of the Ackermann function, see [2, §§4.1]. See also [187] for related geometric reflections.

Also in $d$-space some special cases were investigated. From a result of Sharir [248] on lower envelopes it follows that the complexity of a Voronoi diagram of $n$ points in $d$-space with the $l_p$-norm is in $O(n^{d+\varepsilon})$, the constant, however, tending to $\infty$ with $p$. Also, the above result on octahedral and tetrahedral unit balls from [32] is extended there to arbitrary $d$ for $d$-cubes and $d$-simplices as unit balls, with complexity $\Theta(n^{\lceil d/2 \rceil})$.

### 4.6 More general Voronoi diagrams and applications

Some more general Voronoi diagrams were already discussed above, such as Gruber’s result on star-shaped unit balls, see Section 4.3. Another concept was successfully invented by Klein in 1988, see [154], [156], and the main reference [153]. This is the concept of abstract Voronoi diagrams, where assumptions on the boundaries of Voronoi regions are mainly topological in nature (e.g., their pathwise connectedness is assumed, and bisecting curves must have only finitely many intersection components). It turns out that this approach is still sufficiently strong to solve the corresponding algorithmical problems, see [15, § 4.6]. A related axiomatic approach was proposed by Stifter [277]. We will not follow that line and only mention that the family of nice metrics [153], important in this framework, is also a generalization of gauges.

Another extension is obtained by generalizing the geometric configuration under consideration, e.g., by assuming that the sites are no longer points but segments, lines, or polygons. Then the Voronoi diagrams are not necessarily piecewise linear anymore, but consist of semi-algebraic pieces if the norm is for example Euclidean or polyhedral. Their complexity is still defined to be the total number of cells, a notion which is made precise in semi-algebraic geometry, see [30]. There are also results in this direction which are combined with gauges. For example, the complexity of the Voronoi diagram with respect to a family of lines in Euclidean 3-space is $O(n^{3+\varepsilon})$ (cf. [248]). It is shown in [64] that for polyhedral unit balls with a constant number of edges in 3-space the Voronoi diagram of $n$ lines is of complexity $O(n^2 \alpha(n) \log n)$, where again $\alpha(n)$ is the above-mentioned inverse of the Ackermann function. In addition, there exist line families satisfying the lower bound $\Theta(n^2 \alpha(n))$.

Having translational motion planning (cf. [245]) in mind, Leven and Sharir [184] investigate planar Voronoi diagrams with polygonal sites with respect to gauges with unit ball of a simple shape, i.e., simple enough to guarantee that certain steps of constructing the Voronoi diagram can be done in constant time. For investigating the translational motion of the unit ball amidst polygonal barriers (which are now the given sites) their main tool is the following statement: With respect to a sufficiently simple convex distance function, the Voronoi diagram of $N$ polygonal convex sites, having $n$ sides altogether, can be computed in time $O(n \log N)$. Analogously, the authors of [203] consider translational motion planning with respect to $k$ polygonal sites having a total of $n$ vertices and a convex $m$-gon $P$ as moved object. Using the respective Voronoi diagram generated by the gauge
with unit ball \( P \), they give a \( O(k \log n \log m) \) algorithm. A further discussion of motion planning and its relations to Voronoi diagrams is given in [15, § 5.4].

Voronoi diagrams are also used in Location Science, see [80].

Another application field for Voronoi diagrams is that of Minimum Spanning Trees, see [15, § 5.2], and also [140] and [155].

4.7 Subjects related to Voronoi diagrams

The first subject to look at is the dual concept of Delaunay triangulations. There seem to be only two papers combining these triangulations with (special) gauges, namely [251] and [243]. Considering partitions of finite point sets by Euclidean spheres, one gets a natural relation between Delaunay triangulations in Euclidean space and oriented matroids. The author of [243] explores Delaunay triangulations and the corresponding oriented matroids for gauges. One of the results is that in smooth, strictly convex Minkowski planes which are not Euclidean there exist eight points such that their Minkowski Delaunay triangulation is not the projection of the lower envelope of a 3-polytope, and the corresponding oriented matroid is not realizable. In [251] an \( O(n \log n) \) algorithm for Delaunay triangulations with respect to gauges is obtained.

Farthest-point Voronoi diagrams with respect to the maximum norm are considered in [205].

4.8 Angular bisectors

For unit vectors \( a \neq b \) in a Minkowski plane \( \mathbb{M}^2 \) we call the convex set bounded by the rays \([o, a], [o, b]\) an angle \( \langle aob \rangle \) with the origin \( o \) as its apex.

Glogovskii [107] defines the angular bisector of \( \langle aob \rangle \) in a Minkowski plane to be the ray all points of which have the same Minkowski distances to \([o, a]\) and \([o, b]\), respectively. With this definition the three angular bisectors of a triangle intersect at the centre of the (unique) inscribed circle, see also our survey [196, § 7.1] and Sowell [276] for a special norm. Using homothets of the isoperimetrix instead of circles, Guggenheimer [125] proves an analogue for planes with gauges. Averkov [18] uses a higher-dimensional analogue of Glogovskii’s definition of angular bisectors to obtain results on Minkowski balls touched by all facet hyperplanes of a simplex in \( \mathbb{M}^d \).

For suitable \( x, y \neq o \) in a Minkowski space the authors of [93] define the measure of \( \langle xoy \rangle \) by

\[
A(x, y) = \cos^{-1} \left[ \frac{1}{2} \left( 2 - \|\hat{x} - \hat{y}\|^2 \right) \right],
\]

where \( \hat{x} := \frac{x}{\|x\|} \). For independent \( x \) and \( y \), a point \( z = \lambda_1 x + \lambda_2 y \) \( (\lambda_1, \lambda_2 > 0) \) is called the angular bisector of \( \langle xoy \rangle \) provided \( A(x, z) = A(z, y) \). This is equivalent to the property that there exists a point \( z \) in the plane through \( o, x \) and \( y \) such that \( \|z - \hat{x}\| = \|z - \hat{y}\| \). It is easy to see that such a bisector always exists. A Minkowski space is said to have the angle bisector property if for all independent \( x, y \) with \( \|x\| = \|y\| = 1 \) the element \( z = \frac{x + y}{\|x + y\|} \) satisfies \( A(x, z) = A(z, y) = \frac{1}{2} A(x, y) \). It is proved in [93] that any Minkowski space having the angle bisector property is Euclidean, and it is asked whether this implication still holds for a weakening of the angle bisector property. In [11] the angle bisector property from [93] is considered in more general spaces.
The following definition of an angular bisector is considered by Busemann [49]: A ray \((o, c)\) is said to be an angular bisector of \(<aob\) in \(M^2\) if \(c\) is the midpoint of the chord \([a, b]\) of the unit circle \(B\) of \(M^2\). Using only distances and hence avoiding angular measures, the bisector of an angle in the Euclidean plane can then be described by the following property:

\(\text{(P)} \) Inside a convex angle with sides \(N_1, N_2\) and apex \(a\), there is a ray \(M\) with apex \(a\) such that any segment \([a_1, a_2]\) with \(a_i \in N_i\) and \(a_i \neq a\) \((i = 1, 2)\) intersects \(M\) in a point \(b\) for which

\[
\frac{\|a - a_1\|}{\|a - a_2\|} = \frac{\|b - a_1\|}{\|b - a_2\|}.
\]

In [49] Busemann characterizes the Minkowski planes as those Desarguesian planes (which themselves are planes whose geodesics fall on ordinary affine lines) which satisfy \((P)\). Assuming that the circles, whose convexity follows from \((P)\), are differentiable, he gives the same characterization of Minkowski planes among all two-dimensional straight \(G\)-spaces, cf. [47, Chapter 1] for a definition. Phadke [225] shows that the differentiability hypothesis in the second characterization is not needed.

Düvelmeyer [82] proves that a Minkowski plane is Radon (see [196, § 6.1.2]) iff Glogovskii’s and Busemann’s definitions of angular bisectors coincide.

A concept from computational geometry is closely related to angular bisectors, see [3] and [4]. Namely, for the case of a given polygon \(P\), a straight skeleton is obtained as the interference pattern of certain wavefronts propagated from the edges of \(P\). Until now, this concept waits its extension to normed planes and spaces, although there are relations to other types of distance functions, cf. [20].

4.9 Sets equidistant to only one site

Although point sets equidistant to only one given site (such as a hyperplane) are no longer bisectors, some results on such sets are very close to those on bisectors presented here. For example, some interesting characterizations of Minkowski spaces, or their extensions with gauges, can be obtained by using such equidistant loci.

The authors of [99] show that for a closed subset \(A\) of a Minkowski space with strictly convex or differentiable norm and almost every \(r > 0\) the \(r\)-level set (= union of all points whose distance from \(A\) is \(r\)) contains a relatively open subset which is a \((d - 1)\)-dimensional Lipschitz manifold and whose complement relative to the level set has \((d - 1)\)-dimensional Hausdorff measure zero (for Minkowski planes a sharper result is obtained).

Within a large class of spaces (containing the hyperbolic ones and also spaces which were studied in connection with Hilbert’s fourth problem), Phadke [221] gives a characterization of the linear spaces equipped with gauges. Namely, these spaces are characterized by the property that the equidistant loci on both sides of any hyperplane are themselves hyperplanes. In [222] he gives a sharpening for the planar case and continues with related results in [224]. For \(d = 2\) and under strong differentiability and regularity assumptions, his characterization theorem is a special case of a theorem of Funk [97].

Acknowledgements. This paper was written under a grant from the DFG-NRF agreement. It was partially written during a Research in Pairs visit of both authors to Oberwolfach, September/October 2001, and during a visit of Martini to Unisa, January 2002.

We thank Rolf Klein, Marek Lassak, Valeriu Soltan and Tony (A. C.) Thompson for valuable remarks and suggestions, and for providing certain literature.
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[Abbreviations]

Jbuch. Jahrbuch über die Fortschritte der Mathematik  
Zbl. Zentralblatt MATH  
MR Mathematical Reviews


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