Mean-Square Performance Analysis of the Normalized Subband Adaptive Filter

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Abstract—This paper presents the mean-square performance analysis of a class of normalized subband adaptive filters (NSAF) using the concept of energy conservation. The NSAF has a unique weight-control mechanism whereby subband error signals are used to update a fullband tap-weight vector. We show that an energy conservation relation can be established for the weight adaptation. Subsequently, an expression for the steady-state excess mean-square error (MSE) and the necessary condition on the step size for mean-square stability can then be derived by manipulating various error quantities of the energy conservation relation. Simulation results are presented to corroborate the mathematical analysis.

I. INTRODUCTION

Subband and multirate techniques have been employed in designing computationally efficient adaptive filters with improved convergence performance against high eigenvalue disparity [1]-[7]. One application of interest is acoustic echo cancellation, which involves colored excitation and the modeling of a long impulse response. Conventional subband adaptive filter (SAF) [2], [3] decomposes the input signal and desired response into multiple subbands, which are then processed with separate adaptive subfilters. Intuitively, faster convergence is possible because the spectral dynamic range is greatly reduced in each subband. Furthermore, the computational burden can be reduced by decimating both the order and adaptation rate of the subfilters. However, detailed analysis shows that its convergence rate is limited by aliasing and band-edge effects [2], [3], [8]. To remedy these structural problems, a new weight-control mechanism, as depicted in Fig. 1, has been adopted in [4]-[7].

In [6], we derived the recursive equation of the new weight-control mechanism from a deterministic optimization criterion. The devised adaptive algorithm is referred to as the normalized SAF (NSAF) algorithm. The SAFs reported in [4], [5] are different embellishments of the NSAF depicted in Fig. 1. In particular, the modeling filter is implemented in polyphase form in [5], whereas in [4], a closed-loop delayless structure is employed. Nevertheless, the basic concept remains the same: error signals are estimated in subbands, whereas the coefficients that explicitly adapted are the tap weights of a fullband filter.

In this paper, the performance of the NSAF algorithm is evaluated in terms of its convergence stability and the estimation accuracy in the steady state after the algorithm has converged. Such a mathematical analysis of the NSAF algorithm provides a set of working rules that can be used for its design in practical applications. A unique characteristic of the NSAF algorithm is that it uses a set of normalized subband signals to adapt the fullband tap weights of a modeling filter. The modeling filter is placed after the decimators, whereas in [4], a set of subfilters are placed before the decimators. In what follows, the architectural beauty of such a configuration becomes more evident for which it enables a tractable convergence analysis of the subband adaptive filtering algorithm, which is generally a formidable task for the case of conventional SAF.

The remaining of the paper is organized as follows. The NSAF recursion is first reviewed in Section II. Subsequently, Section III introduces the data model and assumptions used in the convergence analysis. The mean-square performance of the NSAF algorithm is then analyzed by means of an energy conservation relation [10], [9] in Section IV, where stability bounds for the step size and an expression for the steady-state MSE are determined. The results obtained from the convergence analysis are confirmed through simulation in Section V. Finally, Section VI concludes the paper.

II. THE NSAF RECURSION

Figure 1 shows the detailed structure of the NSAF adaptive weight-control mechanism [6, 7]. In the subband structure, the diagonalized N-input, N-output system $W(k,z)I_{N\times N}$ is basically a bank of parallel filters with identical transfer function $W(k,z) = \sum_{m=0}^{M-1} w_m(k) z^{-m}$, where $I_{N\times N}$ denotes the $N\times N$ identity matrix and $M$ is the length of the transversal function applied to the subband input signals $d(z)$, $W_k(z)$.
filter. These \( N \) copies of adaptive transversal filter operate on the set of subband signals \( u_i(n) \) at the original sampling rate \( n \), while the tap weights of the filter \( W(k,z) \) being iteratively updated at the decimated rate \( k \). In particular, the weight adjustment applied on the tap-weight vector \( \mathbf{w}(k) = [w_0(k), w_1(k), \ldots, w_{M-1}(k)]^T \) is iteratively calculated via

\[
\mathbf{w}(k+1) = \mathbf{w}(k) + \mu \mathbf{U}(k) \mathbf{A}^{-1}(k) \mathbf{e}_n(k),
\]

where \( \mu \) and \( \alpha \) are the step-size and regularization parameters, respectively. The \( i \)th subband regressor \( \mathbf{u}_i(k) \), the \( M \times N \) subband data matrix \( \mathbf{U}(k) \), and the \( N \times 1 \) desired response vector \( \mathbf{d}_n(k) \) are defined, respectively, as follows

\[
\mathbf{u}_i(k) = [u_i(kN), u_i(kN-1), \ldots, u_i(kN-M+1)]^T,
\]

\[
\mathbf{U}(k) = [\mathbf{u}_0(k), \mathbf{u}_1(k), \ldots, \mathbf{u}_{N-1}(k)],
\]

\[
\mathbf{d}_n(k) = [d_{0,n}(k), d_{1,n}(k), \ldots, d_{N-1,n}(k)]^T.
\]

The NSAF algorithm collectively adapts multiple spectral bands of a fullband filter by means of subband data \( \{\mathbf{U}(k), \mathbf{e}_n(k)\} \). The weights of the transversal filter are considered as fullband because the subband signals actuating the tap-weight adaptation collectively cover contiguous spectral bands of the original frequency bandwidth. It should also be noted that, by decimating the tap-weight adaptation rate, responses of the modeling filter \( W(k,z) \) to the \( N \) subband input signals, \( u_i(n) \) for \( i = 0,1,\ldots,N-1 \), are evaluated at \( 1/N \)th of the input sampling rate. The computational complexity remains unchanged even though there are \( N \) copies of \( W(k,z) \) in the subband structure of Fig. 1.

III. DATA MODEL AND ASSUMPTIONS

In this section, we introduce a linear data model for the desired response, a paraunitary assumption for the filter banks, and a novel multiband MSE function as the performance measure for the NSAF algorithm. These assumptions allow the convergence behavior to be analyzed in a tractable manner.

A. Linear Data Model

A linear data model assumes that the physical mechanism for generating the desired response \( d(n) \) is described by

\[
d(n) = \mathbf{w}_n^T \mathbf{u}(n) + \eta(n),
\]

where \( \mathbf{w}_n = [w_{n,0}, w_{n,1}, \ldots, w_{n,N-1}]^T \) denotes an \( M \times 1 \) unknown parameter vector of the model, \( \mathbf{u}(n) = [u(n), u(n-1), \ldots, u(n-M+1)]^T \) is the fullband input vector, and \( \eta(n) \) accounts for the additive white noise that is statistically independent of the input signal \( u(n) \). Furthermore, it is also assumed that the length \( M \) of the adaptive transversal filter \( W(k,z) \) is equal to that of the data model.

By virtue of the linearity of the analysis filters and the decimators, the linear data model (7) can be equivalently represented as

\[
\mathbf{d}_n(k) = \mathbf{U}^T(k) \mathbf{w}_n + \mathbf{e}_n(k),
\]

where \( \mathbf{e}_n(k) = [\eta_{0,n}(k), \eta_{1,n}(k), \ldots, \eta_{N-1,n}(k)]^T \) denotes a noise vector comprised of the critically-decimated subband noises.

\[
\eta_{i,n}(k) = \eta_i(kN) = \sum_{l=0}^{k-1} h(l) \eta(kN-l),
\]

where \( h(n) \) is the impulse response of the \( i \)th analysis filter of length \( L \). Using (8) in (2), the estimation error under the linear data model assumption can then be expressed as

\[
\mathbf{e}_n(k) = \mathbf{U}^T(k) \mathbf{e}(k) + \mathbf{e}_n(k),
\]

where \( \mathbf{e}(k) = w_n - \mathbf{w}(k) \) denotes the weight-error vector. Notice that the estimation error \( \mathbf{e}_n(k) \) consists of two components, with \( \mathbf{U}^T(k) \mathbf{e}(k) \) gives the modeling error, while \( \mathbf{e}_n(k) \) represents the measurement noise.

B. Paraunitary Assumption

The filter banks for band-partitioning the input signal \( u(n) \) and desired response \( d(n) \) are assumed to be identical and paraunitary. The paraunitary property leads to the following three-fold consequence:

(i) The filter banks are power complementary.

(ii) The mean-square error (MSE) produced by the NSAF algorithm can be written as the average of the mean-square value of the subband estimation errors in the following form

\[
J(\mathbf{w}) = \frac{1}{N} \mathbb{E}\left[ \| \mathbf{d}_n(k) \|^2 \right].
\]

(iii) The noise vector \( \mathbf{e}_n(k) \) in (8), at any particular time instant \( k \), is uncorrelated with the current and all the previous weight estimates, \( \mathbf{w}(k), \mathbf{w}(k-1), \mathbf{w}(k-2), \ldots \).

The first point is well understood in the literature, for examples, in [11, pp. 124]. The second point is the direct consequence of the power complementary properties, as we shall see in the next subsection. The justifications for the third point are given in the appendix. The major motivation of the paraunitary assumption is to avoid the parameters of the filter banks in the mathematical formulation. More precisely, filter banks that complicate the mathematical manipulation are not explicitly included in the convergence analysis. Instead, the properties imposed by the filter banks, which we assume to be paraunitary, on various subband quantities are identified (as listed above) and taken into consideration.

C. Multiband MSE function

The error function \( J(\mathbf{w}) \) in (11) defines a multiband MSE function in terms of subband estimation errors \( e_{i,n}(k) \), for \( i = 0, 1, \ldots, N-1 \). A power complementary filter bank preserves the second-order moments of its input signal, in which case the multiband MSE function can be shown to reduce to

\[
J(\mathbf{w}) = \sigma^2 - 2\mathbf{p}^T \mathbf{w} + \mathbf{w}^T \mathbf{R} \mathbf{w},
\]

where

\[
\sigma^2 = \mathbb{E}\left[ d^2(n) \right] = \mathbb{E}\left[ \| \mathbf{d}_n(k) \|^2 \right]/N,
\]

\[
\mathbf{p} = \mathbb{E}\left[ \mathbf{u}(n) d(n) \right] = \mathbb{E}\left[ \mathbf{U}(k) \mathbf{d}_n(k) \right]/N,
\]

\[
\mathbf{R} = \mathbb{E}\left[ \mathbf{u}(n) \mathbf{u}^T(n) \right] = \mathbb{E}\left[ \mathbf{U}(k) \mathbf{U}^T(k) \right]/N
\]

are the second-order moments characterizing the MSE function.
Using (10) in (11), the multiband MSE function under the linear data model assumption can be expressed in the following form
\[ J(k) = \frac{1}{N} E \left[ \| \mathbf{b}_0 (k) \|^2 \right] + \frac{1}{N} E \left[ \| \mathbf{e}_0 (k) \|^2 \right] \]
\[ + \frac{1}{N} E \left[ \| \mathbf{e}_s (k) \|^2 \right] , \]
where \( \mathbf{e}_s (k) = \mathbf{U}^T (k) \mathbf{e} (k) \) denotes the undisturbed estimation error. The cross-term \( E \{ \mathbf{b}_0 (k) \mathbf{e}_s (k) \} \) can be eliminated by exploiting the fact that the noise vector \( \mathbf{b}_0 (k) \) is uncorrelated with the current tap weights \( \mathbf{w}(k) \) due to the paraunitary condition imposed on the filter banks. By so doing, the multiband MSE function reduces to the following form
\[ J(k) = \sigma_0^2 + \frac{1}{N} E \left[ \| \mathbf{e}_s (k) \|^2 \right] , \]
where
\[ \sigma_0^2 = E \{ \eta^2 (n) \} = \frac{1}{N} E \left[ \| \mathbf{b}_0 (k) \|^2 \right] \]
is the variance of the additive white noise \( \eta(n) \), considering that the analysis filter bank is power complementary.

Perfect adaptation is achieved when the adaptive tap-weight vector \( \mathbf{w}(k) \) takes on the optimum value \( \mathbf{w}_* \), in which case \( \mathbf{e}_s (k) = \mathbf{0} \). Using this perfect adaptation condition in (15), it follows that the minimum MSE, \( J_{\text{min}} \), under the linear data model assumption is given by the variance of the disturbance \( \eta(n) \). Nevertheless, due to the stochastic nature of adaptive algorithms, the tap-weight vector will never terminate at \( \{ \mathbf{w}_*, \mathbf{w}_0 \} \). Instead, it executes a random motion around the optimum solution, giving rise to the so-called steady-state excess MSE expressed as
\[ J_{\text{ss}} (\infty) \equiv J(\infty) - J_{\text{min}} = \frac{1}{N} E \left[ \| \mathbf{e}_s (\infty) \|^2 \right] . \]
The steady-state excess MSE indicates the difference between the MSE \( J(\infty) \) produced by an adaptive algorithm operating in steady state and the minimum MSE \( J_{\text{min}} \). In the next section, we shall use (17) to evaluate the steady-state performance of the NSAF algorithm.

IV. MEAN-SQUARE ANALYSIS

A. Energy Conservation Relation

Let \( \mathbf{e}_s (k) = \mathbf{U}^T (k) \mathbf{e} (k) \) and \( \mathbf{e}_p (k) = \mathbf{U}^T (k) \mathbf{e} (k+1) \) denote the \textit{a priori} and the \textit{a posteriori} errors, respectively. Recall that \( \mathbf{e}_s (k) \) represents the undisturbed estimation error produced by the current tap-weight vector, whereas \( \mathbf{e}_p (k) \) indicates the undisturbed estimation error produced by the updated tap-weight vector. Subtracting (1) from \( \mathbf{w}_* \) and multiplying both sides of the resulting equation from the left by \( \mathbf{U}^T (k) \), the error quantities \( \mathbf{e}_s (k) \) and \( \mathbf{e}_p (k) \) can be related via
\[ \mathbf{e}_p (k) = \mathbf{e}_s (k) - \mu \left[ \mathbf{U}^T (k) \mathbf{U}(k) \right] \mathbf{A}^{-1} (k) \mathbf{e}_0 (k) . \]
Assuming that the subband data matrix \( \mathbf{U}(k) \) has full column rank, and thus the matrix \( \mathbf{U}^T (k) \mathbf{U}(k) \) is invertible, (18) can be solved for \( \mu \mathbf{A}^{-1} (k) \mathbf{e}_0 (k) \). Using this result in (1), rearranging terms, the NSAF recursion can be written in the following equivalent form
\[ \mathbf{e}(k+1) + \mathbf{U}(k) \left[ \mathbf{U}^T (k) \mathbf{U}(k) \right] \mathbf{A}^{-1} (k) \mathbf{e}_0 (k) \]
\[ = \mathbf{e}(k) + \mathbf{U}(k) \left[ \mathbf{U}^T (k) \mathbf{U}(k) \right]^{-1} \mathbf{e}_s (k) \]
\[ = \mathbf{e}(k) + \mathbf{U}(k) \left[ \mathbf{U}^T (k) \mathbf{U}(k) \right]^{-1} \mathbf{e}_s (k) \]
Evaluating the energy (i.e., taking the squared Euclidean norm) of expression on both sides of (19), we arrive at the following energy conservation relation:
\[ \| \mathbf{e}(k+1) \|^2 + \mathbf{e}_s^T (k) \left[ \mathbf{U}^T (k) \mathbf{U}(k) \right]^{-1} \mathbf{e}_s (k) \]
\[ = \| \mathbf{e}(k) \|^2 + \mathbf{e}_s^T (k) \left[ \mathbf{U}^T (k) \mathbf{U}(k) \right]^{-1} \mathbf{e}_s (k) \]
\[ = \| \mathbf{e}(k) \|^2 + \mathbf{e}_p^T (k) \left[ \mathbf{U}^T (k) \mathbf{U}(k) \right]^{-1} \mathbf{e}_p (k) . \]
The energy conservation relation shows how the energies of the weight-error vectors at two successive iterations are related to the weighted energies of the \textit{a priori} and \textit{a posteriori} errors. It has been shown in [10], [9] that similar energy conservation relation can be established for a handful of adaptive filtering algorithms, for examples, least-mean-square (LMS), normalized LMS (NLMS), affine projection (AP), and recursive least-squares (RLS) algorithms.

B. Variance Relation

The practical virtue of the energy conservation relation is that it allows the mean-square performance of the NSAF algorithm to be analyzed by evaluating the variance of the error quantities that appear in the relation. By taking the expectation of both sides of (20), the energy relation translates into the following variance relation:
\[ c(k+1) + E \left[ \mathbf{e}_s^T (k) \left[ \mathbf{U}^T (k) \mathbf{U}(k) \right]^{-1} \mathbf{e}_s (k) \right] \]
\[ = c(k) + E \left[ \mathbf{e}_s^T (k) \left[ \mathbf{U}^T (k) \mathbf{U}(k) \right]^{-1} \mathbf{e}_s (k) \right] \]
\[ = c(k) + E \left[ \mathbf{e}_s^T (k) \left[ \mathbf{U}^T (k) \mathbf{U}(k) \right]^{-1} \mathbf{e}_s (k) \right] \]
\[ = c(k) + E \left[ \mathbf{e}_s^T (k) \left[ \mathbf{U}^T (k) \mathbf{U}(k) \right]^{-1} \mathbf{e}_s (k) \right] \]
where \( c(k) \) denotes the mean-square deviation, defined by
\[ c(k) = E \left[ \| \mathbf{e}(k) \|^2 \right] . \]
Knowing that \( \mathbf{e}_s (k) \) is related to \( \mathbf{e}_a (k) \) via (18), the variance relation can be simplified to
\[ c(k+1) = c(k) - 2 \mu E \left[ \mathbf{e}_s^T (k) \mathbf{A}^{-1} (k) \mathbf{e}_p (k) \right] \]
\[ + \mu^2 E \left[ \mathbf{e}_s^T (k) \mathbf{A}^{-1} (k) \left[ \mathbf{U}^T (k) \mathbf{U}(k) \right] \mathbf{A}^{-1} (k) \mathbf{e}_0 (k) \right] . \]
To this end, it should be noted that no approximation has been used to establish the energy conservation relation (20) and the variance relation (23) as well. Hence, reliable results for the mean-square performance can be obtained by manipulating the variance relation.

C. Stability of the NSAF Algorithm

Assuming that the diagonal assumption [6, 7] is valid (i.e., the off-diagonal elements of the matrix \( \mathbf{U}^T (k) \mathbf{U}(k) \) are negligible), and the regularization parameter \( \alpha \) has been set to zero, such that \( \mathbf{U}^T (k) \mathbf{U}(k) = \mathbf{A}(k) \), the variance relation (23) can be further simplified to
\[ c(k+1) - c(k) = \mu^2 E \left[ \mathbf{e}_s^T (k) \mathbf{A}^{-1} (k) \mathbf{e}_0 (k) \right] \]
\[ - 2 \mu E \left[ \mathbf{e}_s^T (k) \mathbf{A}^{-1} (k) \mathbf{e}_0 (k) \right] . \]
For the algorithm to be stable, the mean-square deviation \( c(k) \) must decrease monotonically with increasing \( k \), i.e., \( c(k+1) < c(k) \) from one iteration to the next. From (24), for \( c(k+1) \) to be less than \( c(k) \), the step size \( \mu \) has to fulfill the following condition

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Under noisless situation, where the disturbance \( \eta_0(k) \) in (10) is negligible, it is obvious that the estimation error \( e_0(k) \) is equal to the undisturbed error \( e(k) = U^T(k)e^c(k) \). Hence, in the absence of disturbance, the necessary and sufficient condition for the convergence in the mean-square sense is that the step-size parameter must satisfy the double inequality

\[
0 < \mu < 2. \tag{26}
\]

### D. Steady-state Excess MSE

Again, we assume that assume that the desired response arises from a linear data model, the diagonal assumption is valid, and the regularization parameter has been set to zero. These assumptions allow us to substitute (10) into (24) to obtain

\[
c(k+1) - c(k) = (\mu^2 - 2\mu) E\{e^c(k)\Lambda^{-1}(k)\eta_0(k)\} + \mu^2 E\{\eta_0^2(k)\Lambda^{-1}(k)\eta_0(k)\}. \tag{27}
\]

To arrive at the above equation, we have exploited the fact that the noise vector \( \eta_0(k) \) is uncorrelated with the weight-error vector \( e^c(k) \) by means of the pararnitary assumption.

An adaptive algorithm is said to operate in steady state [10], [9] if its mean-squared deviation \( c(k) \) tends to a finite constant value \( c \) as \( k \) increases without bound, as follows

\[
c(k+1) = c(k) = c < \infty, \text{ as } k \to \infty. \tag{28}
\]

Introducing this steady-state condition into (27), we obtain the following equation, which describes the steady-state relation between the subband undisturbed error signals \( e^c_{ia}(k) \) and the subband disturbances \( \eta_{ia}(k) \), as follows

\[
\sum_{i=0}^{N-1} E\left\{ \frac{e^c_{ia}(k)}{\|u_i(k)\|^2} \right\} = \frac{\mu}{2 - \mu} \sum_{i=0}^{N-1} E\left\{ \frac{\eta_{ia}^2(k)}{\|u_i(k)\|^2} \right\}, \tag{29}
\]

as \( k \to \infty \). If we further assume that the subband signals are uncorrelated, then we may deduce that the \( i \)th terms of the summations on both sides of (29) correspond to each other, as follows

\[
E\left\{ \frac{e^c_{ia}(k)}{\|u_i(k)\|^2} \right\} = \frac{\mu}{2 - \mu} E\left\{ \frac{\eta_{ia}^2(k)}{\|u_i(k)\|^2} \right\}, \tag{30}
\]

for \( i = 0,1,\ldots,N-1 \), as \( k \to \infty \). For high-order filter, where \( M \gg 1 \), the fluctuation in the subband energy \( \|u_i(k)\|^2 \) from one iteration to the next can be assumed to be small enough, thereby, justifying the following assumptions

\[
E\left\{ \frac{e^c_{ia}(k)}{\|u_i(k)\|^2} \right\} = E\left\{ e^c_{ia}(k) \right\} \quad E\left\{ \frac{\eta_{ia}^2(k)}{\|u_i(k)\|^2} \right\} = E\left\{ \eta_{ia}^2(k) \right\}. \tag{31a}
\]

and

\[
E\left\{ \frac{\eta_{ia}^2(k)}{\|u_i(k)\|^2} \right\} = E\left\{ \eta_{ia}^2(k) \right\}, \tag{31b}
\]

Inserting this approximation into (30), we arrive at

\[
E\{e^c_{ia}(k)\} = \frac{\mu}{2 - \mu} E\{\eta_{ia}^2(k)\}, \tag{32}
\]

for \( i = 0,1,\ldots,N-1 \), as \( k \to \infty \). Using (32) in (17), the steady-state excess MSE for the NSAF algorithm can be expressed as

\[
J_{\eta}(\infty) = \frac{\mu}{2 - \mu} \left\{ \frac{1}{N} \sum_{i=0}^{N-1} E\{\eta_{ia}^2(k)\} \right\} = \frac{\mu \sigma^2_\eta}{2 - \mu}, \tag{33}
\]

where \( \sigma^2_\eta \) denotes the variance of the additive white noise, as defined in (16). The expression holds as long as the frequency responses of the analysis filters do not overlap significantly and their stopband attenuation is sufficiently high. Otherwise, the excess MSE would generally be higher than that given in (33), because the variance of the undisturbed error signal \( E\{e^c_{ia}(k)\} \) would receive contribution from cross-channel disturbances.

The expression for steady-state excess MSE given by (33) is identical to that of the NLMS algorithm reported in [10], [9]. This is not surprising as NLMS algorithm can be seen as special case of the NSAF algorithm with \( N = 1 \) [7]. It is also obvious that the excess MSE \( J_{\eta}(\infty) \) is independent of the number of subbands \( N \). Hence, the convergence of the NSAF algorithm can be accelerated by increasing the number of subbands \( N \), while maintaining the same level of steady-state excess MSE with the step size remains unchanged. The drawback is that additional delay is introduced into the signal path, since higher order analysis filters are required in order to achieve sufficient stopband attenuation.

### V. Simulations

We consider a system identification problem in the following simulations. The system to be identified \( w_a \) is an acoustic response of a room, truncated to 1024 taps. The adaptive weight vector has identical length of \( N = 1024 \). The excitation signal to the adaptive identification system is an AR(2) random process with coefficients \((1.0, -0.1, -0.8)\). White noise is added to the output of the unknown system giving a 40 dB SNR. Paraunitary cosine-modulated filter banks [11] with filter length of \( L = 8N \) are used for the analysis and synthesis sections, since we impose the paraunitary condition on the analysis filters for mathematical simplicity.

Figure 2 shows the steady-state MSE \( J(\infty) \) of the NSAF algorithm (with \( N = 4, 8, 16 \), and 32 subbands) plotted as a function of the step-size \( \mu \). At each step-size \( \mu \) that varies from 0.01 to 1.99, the ensemble average learning curve is first obtained by averaging the instantaneous squared error curve over 200 independent trials. For each trial, the NSAF algorithm is iterated until a steady-state is reached. The time-average of the last 15,000 samples of the ensemble average learning curve is then used to calculate the steady-state MSE. The steady-state MSE curve, treated as a function of the step-size, is bounded from below and from the right by two asymptotes, namely, \( J(\infty) = J_{\min} = -40 \text{ dB} \) and \( \mu = 2.0 \), as illustrated in Fig. 2. As the step-size approaches the stability bound at \( \mu = 2.0 \), the steady-state MSE increases tremendously. The algorithm diverges when the step-size \( \mu \) is outside the stability bounds defined in (26). The minimum MSE, \( J_{\min} \), is given by the variance of the disturbance, \( \sigma^2_\eta = 1 \times 10^{-3} \). Smaller step-size leads to a lower steady-state MSE towards the vicinity of the horizontal asymptote \( J(\infty) = J_{\min} \). Nevertheless, the attainable steady-state MSE is always higher than \( J_{\min} \), which reflects the stochastic nature of the algorithm.
Figure 3 shows the experimental and theoretical values of the steady-state excess MSE for the NSAF algorithm at various step-sizes. The theoretical values are calculated by using the expression in (33). Observe that the expression leads to a good fit between the theory and practice over the range of step-size from 0.01 to 1.00. Both the theoretical and experimental results also agree that the steady-state excess MSE, $J_{ex}$ ($\infty$), is independent of the number of subbands, $N$.

VI. CONCLUSION

Mean-square performance of the NSAF algorithm was analyzed based on an energy conservation relation. In particular, stability bounds for the step-size and an expression for the steady-state MSE were derived. A good match between theoretical and experimental results is confirmed through simulations. Such a mathematical analysis of the NSAF algorithm provides a set of working rules that can be used for its design in practical applications. Furthermore, it facilitates the comparison of the NSAF algorithm with various adaptive filtering algorithms (e.g., NLMS and AP algorithms) that can also be analyzed under the same energy conservation framework.

APPENDIX: NOISE VECTOR

The disturbance $\eta(n)$ in the linear data model (7) is assumed to be a white noise process such that its successive samples are uncorrelated, as shown by

$$E[\eta(n)\eta(n-l)] = \sigma_\eta^2 \delta(l),$$

(34)

where $\sigma_\eta^2$ represents the variance of $\eta(n)$ and $\delta(l)$ denotes the unit sample sequence. Imposing the paraunitary condition on the analysis filter bank, it can be shown that

$$E[\eta_{p,D}(k)\eta_{p,D}(k-l)] = \left\{
\begin{array}{ll}
\sigma_\eta^2 \delta(l), & \text{for } i = p, \\
0, & \text{for } i \neq p,
\end{array}
\right. \quad (35)$$

which implies that the critically-decimated subband noises, $\eta_{p,D}(k)$ for $i = 0, 1, \ldots, N-1$, are white and are uncorrelated to each other. Now, by using (35), the fact in point (iii) of Subsection IIIIB can be easily observed from the update equation of the NSAF algorithm. Substituting (10) into (1) and setting $k = k-1$, the current weight estimate $w(k)$ can be expressed as

$$w(k) = w(k-1) + \mu U(k-1)A(k-1)$$

$$\times [U^T(k-1)e(k-1) + \eta_0(k-1)].$$

(36)

Clearly, $w(k)$ is dependent on the past noise vectors $\eta_0(k-1), \eta_0(k-2), \ldots$, and the past input matrices $U(k-1), U(k-2), \ldots$. However, the current noise vector $\eta_0(k)$ is uncorrelated with the pass noises as indicated by (35). Furthermore, $\eta_0(k)$ is also assumed to be statistically independent of the input matrices, $U(k)$ for all $k$. Therefore, $\eta_0(k)$ is uncorrelated with the current and all the previous weight estimates, $w(k), w(k-1), w(k-2), \ldots$.

REFERENCES