Gaussian beam reflection and refraction by a spherical or parabolic surface: comparison of vectorial-law calculation with lens approximation

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A ray-tracing approach is used to demonstrate efficient application of the vectorial laws of reflection and refraction to computational optics problems. Both the full width at half-maximum (fwhm) and offset of Gaussian beams resulting from off-center reflection and refraction are calculated for spherical and paraboloidal surfaces of revolution. It is found that the magnification and displacement depend nonlinearly on the miscentering. For these geometries, the limits of accuracy of the lens approximation are examined quantitatively. In contrast to the ray-tracing solution, this paraxial approximation would predict a magnification of a beam’s fwhm that is independent of miscentering, and an offset linearly proportional to the miscentering. The focusing property of paraboloidal surfaces of revolution is also derived in setting up the calculation.

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1. INTRODUCTION

The interaction of light rays with surfaces of revolution of conic sections, especially spheres and paraboloids, is a common problem in optics. One practical example is the use of spherical surfaces in retroreflectors [1]. On the other hand, parabolic surfaces [2,3] are more common in industrial applications, due to the focusing of parallel rays and collimation of the emission from the focal point (e.g., car head lamps). Here, we take a ray-tracing approach to demonstrate the efficient application of the vectorial laws of reflection and refraction to computational optics problems of these surfaces.

The vectorial laws of reflection and reflection, based on the cross product and dot product [4], can be applied for a complete ray-tracing solution to well-posed geometrical optics problems. For a subset of these problems, the traditional lens equation is an acceptable approximation. For example, in the imaging of a point by a spherical surface of limited aperture, the lens equation yields equivalent results [5] to the vectorial laws. The lens equation can often be applied with considerable accuracy outside of geometrical optics, as well. One notable application is the modified lens equation to describe focusing of a Gaussian beam, which is derived from the known change in wavefront radius of curvature induced by a lens of focal length f. If d_0 is the position of the beam waist of the original spherical Gaussian beam with Rayleigh range z_R, the position of the focused beam waist d is given by the modified lens equation [6]:

\[ d = d_0 + \frac{w^2}{z_R} (d_0 - f)^{-1} = f^{-1}. \]

For example, if a laser is aligned to the center of a spherical surface, this modified lens equation predicts the full width at half-maximum (fwhm) for a refracted beam in a plane of interest. The modified lens equation does not apply, however, in the case of misalignment. When the central propagation axis of the Gaussian beam does not intersect the lens’ center of curvature, the change in radius of curvature is not uniform across a wavefront.

In the present theoretical investigation, we develop the vectorial laws of reflection and refraction to follow wave vectors striking a convex or concave spherical or parabolic surface off-center, with arbitrary aperture size. As numerical examples, the fwhm and offset of refracted and reflected Gaussian beams are explored for these off-center surfaces. It is found that the magnification of the fwhm and offset by the reflection and refraction depend nonlinearly on the miscentering. For comparison, the same problems are analyzed with a lens approximation of each surface. By contrast to the ray-tracing solution, this paraxial approximation produces a magnification of a beam’s fwhm that is independent of miscentering, and an offset linearly proportional to the miscentering.

2. CALCULATION SETUP FOR A SPHERICAL SURFACE OR PARABOLOID OF REVOLUTION

Here we consider the case of a spherical Gaussian beam of wavelength \( \lambda \) with central axis and propagation direction \( \mathbf{z} \), where \( z = 0 \) is the position with the smallest \( (1/e^2) \) beam radius \( w_0 \). The Rayleigh range \( z_R = \pi w_0^2 / \lambda \) permits calculation at any location of the \( (1/e^2) \) beam width \( w(z) \), wavefront radius of curvature \( R(z) \), and intensity \( I(r, z) \) at a distance \( r \) from the \( z \) axis [6]:
First, consider the case of a spherical surface with center at point $C$. We would like to calculate the intensity profile of the reflected or refracted beam in a specific area, for instance, on a planar photodetector defined by its normal $p$ at a point $P$. The beam profile of interest is determined by the intersection $Q$ of each individual reflected wave vector $q$ or refracted wave vector $q'$. The incident beam can be decomposed into rays with known intensities at a set of points $K$, where the wave vector $k$ of each ray is directed from the center of the radius of curvature. Full accuracy is achieved by taking $K$ to be on the reflecting surface. Alternatively, the computation is simplified by choosing the points $K$ instead at a fixed $z$ position before the surface. Although this simplification does not account for the change in $k$ caused by diffraction before the surface, this “cross-sectional vectorial calculation” generally yields highly accurate results, as will be seen by the examples in the present investigation.

Let $J$ be the incident wave vector’s intersection with the sphere. Define the unit normal $n$ with direction outward from the reflecting surface (i.e., $k \cdot n \leq 0$). Define the curvature of the sphere $\rho$ to have magnitude equal to the radius of the sphere, with positive sign if the incident ray strikes a convex surface ($K$ opposite $C$ relative to the surface) [Fig. 1(a)], and negative sign for a concave surface ($K$ on same side as $C$ relative to the surface) [Fig. 1(b)].

The beam profile in the photodetector plane is determined in five steps, including only dot products, scalar operations, vector additions, and subtractions, and a quadratic solution. The vectorial law of reflection and refraction is central to the calculation, which explicitly determines the reflected $q$ or refracted wave vector $q'$. Details of the calculations and algebra are presented in Appendix A.
A similar calculation holds for a paraboloid of revolution, which by definition is the set of points equidistant from a focal point $F$ and a plane that is the directrix. Define $\rho$ with magnitude equal to the distance from the focus to the directrix, retaining the convention that $\rho > 0$ for the convex surface; $\rho < 0$ for the concave surface. Define $\hat{u}$ as the axis of the paraboloid of revolution, which is a unit vector perpendicular to the directrix, with $\hat{u}$ pointing from the directrix to the paraboloid for $\rho > 0$ and in the opposite direction for $\rho < 0$.

The focusing property of paraboloids of revolution, which is a primary reason for their widespread application in optics and antenna technology, follows directly from the equation for the normal presented in Appendix A. In particular, an incoming ray parallel to the axis is reflected through the focus $F$, and vice versa, wave vectors emitted from the focal point propagate parallel to the $z$ axis after reflection.

Note that spheres and paraboloids will have similar reflecting and refracting behaviors in the first-order approximation. Indeed, in Appendix A, we prove that a paraboloid of revolution is merely the second-order Taylor series approximation for a half-sphere.

3. ACCOUNTING FOR DIFFRACTION BY EXACT CALCULATION OF INTERSECTIONS WITH A SURFACE

Under conditions where diffraction effects are nonnegligible, the cross-sectional vectorial calculation (as defined in Section 2) introduces error to the extent that the wave vectors change direction in propagating to the striking the surface. In other words, if $K$ is far (in terms of diffraction) from the surface, $k$ is chosen incorrectly, and Eqs. (A8), (A5), (A3), (A2'),...
and (A2) yield inaccurate results. For example, under the tight focusing condition depicted in Fig. 2, the wave vectors are in reality diverging at the parabolic reflecting surface, an effect which would be entirely neglected by a cross-sectional calculation based on the beam profile before the waist. In such a situation, it is necessary to explicitly calculate wave vectors at the reflecting or refracting surface by Eq. (1). It may be possible to choose the points $K$ of known intensity and wave vectors of the incident laser beam to be on the surface, thereby fixing $t = 0$ in Eqs. (A3)–(A8). Alternatively, if the calculation is nevertheless set up with $K$ far from the reflecting surface (for instance, because of a real measurement there), it is possible to determine the exact intersection of each diffracting Gaussian ray with the surface (sphere or approximating parabola) in the precise mathematical approach outlined in Appendix A, which we shall refer to as the “exact vectorial calculation.”

4. LENS APPROXIMATION

It is often unnecessary to employ precise vectorial-law calculations. For example, in the case of small apertures, it is well known that reflection and refraction by spherical surfaces can, in the paraxial approximation, be described by the lens equation [7,8]. For any reflecting or refracting surface, let the “approximating lens” designate a hypothetical lens placed at the apex of the surface, which obeys the same lens equation as the paraxial approximation for reflection or refraction at the surface. This is a negative lens in the case of a convex reflecting surface. Mathematically rigorous details of this lens approximation are described in appendix Subsection A.7. In the present investigation, we explore the validity of this lens approximation for Gaussian beam reflection and refraction in the general case, not necessarily of small aperture. Large-aperture reflection and refraction off a spherical surface both

Figure 5. (Color online) (a) Ratio of the exact to the lens-approximated reflected beam magnification in the photodetector plane for different convex reflecting sphere center displacements $y_C$. The ratio of the fwhm measurements is shown in both the $y$ (the direction of the sphere center displacement from the axis) (no crosses) and the $x$ directions (crosses). The plots are for beam waist size fwhm$_0 = 0.5$ mm (solid curve) or fwhm$_0 = 2.0$ mm (dotted curve). The remaining calculation parameters are the same as in Fig. 4. Note that, by symmetry, the sign of $\rho$ (convex or concave surface) does not affect the calculation. (b) Reflection by the paraboloid most closely approximating the sphere. (c) Refraction by a sphere. (d) Reflection by concave sphere.
occurred, for example, in the recently-published cataract measuring device [9].

5. CROSS-SECTIONAL VECTORIAL CALCULATION OF THE INTENSITY PROFILE OF A BEAM REFLECTED BY A SPHERE

First, we perform the “cross-sectional vectorial calculations” described in Section 2, in which the points $K$ of known wave vectors and intensity for Eqs. (A5)–(A11) are taken at the waist of the Gaussian laser beam. Figure 3(a) shows the intensity profile after reflection by a spherical surface onto a planar photodetector [10]. The radius $\rho = 8 \text{ mm}$ and photodetector-to-spherical-vertex distance $d_{PD} = 4 \text{ cm}$ are chosen to approximate the human cornea’s specular reflection in a recently developed cataract measuring device [9]. Asymmetry in the reflected beam becomes more pronounced with greater displacements of the sphere’s center from the laser’s central axis $z$. As a result, the reflected beam width increases with increased displacements. This asymmetry and consequent dependence of the reflected beam size on displacement is not captured by the lens approximation for the same calculation [Fig. 3(b)]. Indeed, the magnification of the laser beam at the photodetector $M_{PD}$ [Eq. (A18)] is independent of any displacements perpendicular to the photodetector’s normal $p = z$.

6. EXACT VECTORIAL CALCULATION OF THE DISPLACEMENT AND MAGNIFICATION OF A REFLECTED OR REFRACED BEAM

While the lens approximation gives fairly accurate results in some geometries (such as the solid curve in Fig. 2), vectorial calculations are required for large displacements of the reflecting surface center from the laser beam’s axis (dashed-dotted curve in Fig. 2). Therefore, we investigated the deviation of the exact vectorial calculation results from the lens approximation for reflection and refraction. The beam center, as well as the fwhm in the photodetector plane, was calculated for different offsets of the sphere or approximating paraboloid from the laser beam’s central axis. In particular, the images of the center and half-maximum points are found by using $r = 0$ and $\pm 1/\sqrt{r}$ in Eqs. (A9)–(A12).

The displacement of the reflected laser beam center for a convex reflecting sphere, as calculated vectorially or with a lens approximation, is shown in Fig. 4(a). Figure 4(b) uses the nearest paraboloid to the sphere [Eqs. (A11) and (A5)–(A6)]. The displacement is always in the opposite (same) direction as the displacement $y$ of the convex reflecting (refracting) sphere center from the laser beam axis. By symmetry, calculations with the corresponding concave surfaces ($\rho < 0$) maintaining photodetector and beam waist to surface separations simply inverts the displacement plots [Fig. 4(c)]. Figure 4(d) illustrates the case of refraction by the sphere.

Although the lens law predicts a lateral magnification of the beam $M_{PD}$, which is independent of displacement [Eqs. (A18) and (A18')] and, therefore, the same in both the $x$ and $y$ directions, the magnification of the reflected or refracted beam is generally significantly larger in the direction of displacement of the sphere center than in the perpendicular direction (Fig. 5). For a small fwhm spot size at the waist (the solid curve corresponding to fwhm$_0 = 5 \text{ mm}$), there is virtually no error in using the lens approximation if there is no displacement of thereflecting sphere from the laser axis. This error grows dramatically for increasing displacements. A larger spot size deviates to a larger extent from the paraxial situation and introduces a slightly larger error, which is evident even for no displacement. The dotted curves corresponding to fwhm$_0 = 2 \text{ mm}$ in Fig. 5 are always above unity. Because the vectorially calculated magnifications have the same dependence on the sign of $\rho$ as the lens-law approximation predicts, the shape of the relative magnification curve is independent of the convexity or concavity of the surface [Figs. 5(a) and 5(d) are identical].

7. COMPARISON OF CROSS-SECTIONAL AND EXACT VECTORIAL CALCULATIONS

In the calculations of Figs. 4 and 5, the points of intersection with the surface are determined exactly, but the effects of diffraction are sufficiently small that the visual appearance of the curves is unchanged if the "cross-sectional vectorial

<table>
<thead>
<tr>
<th>Beam and Surface</th>
<th>Lens Law</th>
<th>No Displacement Magnification</th>
<th>x Magnification</th>
<th>y Magnification</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>exact</td>
<td>cross</td>
<td>exact</td>
</tr>
<tr>
<td>2 mm Convex sphere</td>
<td>11.000</td>
<td>11.258</td>
<td>11.258</td>
<td>11.900</td>
</tr>
<tr>
<td>Convex sphere (refraction)</td>
<td>0.875</td>
<td>0.876</td>
<td>0.876</td>
<td>0.879</td>
</tr>
<tr>
<td>0.5 mm Convex sphere</td>
<td>11.000</td>
<td>11.029</td>
<td>11.016</td>
<td>11.636</td>
</tr>
<tr>
<td>Convex sphere (refraction)</td>
<td>0.875</td>
<td>0.876</td>
<td>0.875</td>
<td>0.879</td>
</tr>
</tbody>
</table>

The fwhm of the beam at the waist is 0.5 mm for the top half or 2.0 mm for the bottom half of the table. The remaining calculation parameters are the same as in Fig. 4.

Table 1. Absolute Magnification of fwhm of Beam by Reflection (Refraction in Final Row) to Photodetector from Various Surfaces, in the Direction of Displacement (y Magnification) (italics) and the Perpendicular Direction (x Magnification), by Exact Vectorial Calculation (exact) or Approximation with the Lens Law or Cross-Sectional Vectorial Calculation (cross)
calculation" is used instead. The precise difference between vectorial calculation based on exact intersections and cross-sectional analysis is illustrated in Table 1, which shows selected numerical values of calculations illustrated in Fig. 5. Table 1 uses absolute magnification, in distinction to the normalized magnification in Fig. 5. The deviation of the sphere’s magnification from the lens approximation is greater than that of the paraboloid, especially for large displacements. This is not surprising, as the paraboloid possesses a true focal point, whereas the sphere does not (Fig. 2).

Finally, it is illustrative to examine the positions where the vectorial law deviates a specific amount from the lens approximation. Table 2 shows the value of \( y_c \) (the displacement of the surface center from the laser axis) where the vectorial calculation of fwhm magnification in the \( y \) direction (\( x \) direction) is 5\%, 10\%, or 25\% larger than the lens-law-approximated magnification. Again, as compared to the paraboloid of revolution, the results for the sphere differ to a greater extent from the lens approximation, reaching each level of error with a smaller displacement.

### 8. DISCUSSION AND CONCLUSION

Detailed electromagnetic calculations that generally describe the electric field in optics problems of spheres \([11,12]\) and paraboloids \([13]\) have been previously published. Nevertheless, paraxial approximations, such as the lens equation, remain a mainstay in optics due to their simplicity and sufficient accuracy. In situations where the lens approximation may have insufficient accuracy, the ray-tracing approach of the present study enables computationally efficient calculation. However, in distinction to the standard paraxial approximation of the lens equation, the vectorial-law-based calculation reveals that the magnification of the fwhm and the offset of Gaussian beams by reflection and refraction depend nonlinearly on the miscentering. The ray-tracing theory developed here applies, as well, to both diverging and nondiverging incoherent collimated beams of light. The calculation for paraboloids and spheres is mathematically similar, since paraboloids are in fact a second-order Taylor approximation of spheres. Even under a fairly small radius of curvature and fairly small laser spot size, it is possible to further simplify the ray-tracing calculation to follow wave vectors from a fixed beam cross section with results close to the exact calculation of the Gaussian beam’s intersection with the surfaces. Generalized laws of reflection and refraction are generating increasing interest in the optics community \([14]\). The current study demonstrates the capability of the vectorial laws of reflection and refraction to efficiently solve computational optics problems.

**APPENDIX A: CALCULATIONS AND ALGEBRA DETAILS**

1. **Definitions**

For vectors \( \mathbf{A}, \mathbf{E}, \) and \( \mathbf{v} \) we use here notations \( (A), \hat{a}, A_v, e_a, \) \( \mathbf{AE}, (AE)_v, \) and \( (AE) \) meaning

\[
A \equiv |A|, \quad \hat{a} \equiv \mathbf{A}/|A|, \quad A_v \equiv \mathbf{v} \cdot \mathbf{A}, \quad e_a \equiv \hat{b} \cdot \hat{a} \quad \mathbf{AE} \equiv \mathbf{E} - \mathbf{A} \quad (AE)_v \equiv \mathbf{AE} \cdot \mathbf{v} \\
(\mathbf{AE}) \equiv |\mathbf{AE}|.
\]

For points \( P, N \) in the coordinate system with origin \( O \) we use notations

\[
P \equiv OP \equiv \text{the vector from } O \text{ to the point } P \\
(PN) \equiv |PN| \equiv \text{distance from } P \text{ to } N.
\]
For example, the $y$ coordinate of point $P$ is given equivalently by any of
\[ y_P = P \cdot y = OP \cdot y = (P - O) \cdot y. \]

2. Vectorial-Law Calculation on a Spherical Surface

The beam profile in the photodetector plane is determined in five steps. First, note that the point $Q$ where the vector $q$ from point $J$ strikes the plane with normal $p$ at point $P$ is given by
\[ Q = J + ((JP)_p / q_p)q. \]  

(A1)

Proof: As $q$ is directed from $J$ to $Q$, there is a constant $m$ such that $mq = JQ = JP + PQ$.

As $PQ$ is in the plane perpendicular to $p$, then $mq \cdot p = JP \cdot p + 0$. Thus, $m = (JP)_p / q_p$. The reflected $q$ or refracted wave vector $q'$ is given, in turn, by the vectorial law of reflection and refraction, where $\mu$ is the ratio of the refractive index of the entered medium relative to the originating medium [4]:
\[ q = k - 2n\mu n. \]  

(A2)

Q = k - (sqrt(k^n + (\mu^2 - 1)|k|^2) + n)n. \]  

(A2')

The unit normal $n$ and point of intersection $J$ of the light ray with the sphere are found by setting $KJ = tk$:
\[ J = K + tk. \]  

(A3)

\[ n = CJ / \rho = (CK + tk) / \rho. \]  

(A4)

Further, $\rho^2 = (CK + tk) \cdot (CK + tk)$, and thus $t$ is the root of
\[ (k^2 + 2(CK)t + (CK)^2 - \rho^2 = 0. \]  

(A5)

Note that this quadratic is solvable for $t$ if and only if $(CK)^2 / (k^2) \leq (CK)^2 - \rho^2$, with a unique solution in the case of equality, and two distinct solutions for inequality. Only non-negative solutions are physically meaningful, in that light must propagate forward to reach the surface. In the event of two solutions, the larger value of $t$ corresponds to intersection with the concave surface of the sphere.

Dot products, quadratic solutions, scalar operations, and vector additions and subtractions are computationally efficient operations. Thus, after determination of the wave vectors $k$ at points $K$, the profile of the reflected or refracted beam is quickly determined by solving Eqs. (A1)–(A5) in reverse order.

3. Vectorial-Law Calculation on a Paraboloid of Revolution

Define (Fig. 6):

$f_I = -\rho / 2$, the focal length (for reflection);

$F' = F - \rho \hat{u}$, the point of the directrix plane on the axis of the paraboloid of revolution; and

$J' \equiv J - (\hat{u} \cdot J) \hat{u} + (\hat{u} \cdot F') \hat{u}$, the projection of the paraboloid’s point $J$ in the directrix plane.

First, we determine the outward unit normal $n$ to the paraboloid at $J$. Define a right-handed $x$-$y$-$z$ coordinate system with origin at the vertex of the paraboloid (the point closest to the directrix), $y = \text{sign}(\rho) \hat{u}$, and $z$ such that $J_z = 0$, i.e., $J = (x, y, 0)$. By definition, $y + |f_I| = (J'J)$, the distance from $J$ to the directrix, must equal sqrt($x^2 + (y - |f_I|)^2$) = ($F'J$), the distance to the focus. Therefore, the parabola is a level set of $x^2 + 4f_Iy = 0$. Thus, the paraboloid’s normal is collinear with the gradient 2$x$ + $4f_Iy = 2F'J - 2|\rho|\text{sign}(\rho) \hat{u} = 2F'J$.

Note that, for the case of reflection or refraction by a concave surface ($\rho < 0$), the surface’s outward unit normal $n$ for the optical problem points inward to the parabola. Therefore, the outward unit normal can be expressed compactly as
\[ n = \text{sign}(\rho)F'J/|F'J|. \]  

(A6)

The projection $J'$ is calculated from $J$ by the definition above:
\[ J' = J - (\hat{u} \cdot FJ + \rho) \hat{u}. \]  

(A7)

To find the constant $t$ of Eq. (A3), we note that $J_\mu = F_\mu$, $(J'F')$ in the directrix plane implies $(J' - F') \cdot \hat{u} = 0$. We use the paraboloid definition to obtain two expressions for $(J'J)^2 = (FJ)^2$:
\[ (J'J)^2 = JJ' \cdot \hat{u} = (JK + JK) \cdot \hat{u} = (J\hat{K})_\mu + tk_u = (F\hat{K})_\mu + tk_u, \]
\[ \rho t^2 = (k_u^2 + 2(F\hat{K})_\mu t + (F\hat{K})^2. \]

Taking the difference of the two expressions yields the quadratic for $t$ of the paraboloid, which is analogous to Eq. (A5) for the sphere:
\[ [(k_u)^2 - (k^2)]t^2 + 2[(F\hat{K})_\mu t + (F\hat{K})^2 = 0. \]  

(A8)

4. Derivation of the Focusing Property of Paraboloids of Revolution

The focusing property of paraboloids of revolution, which is a primary reason for their widespread application in optics and antenna technology, follows directly from the equation for the normal [Eq. (A6)]. In particular, an incoming ray parallel to the axis is reflected through the focus $F'$, and, vice versa, wave
vectors emitted from the focal point propagate parallel to the 
z axis after reflection.

Consider the incident ray (with wave vector \( \mathbf{k} \)), line segments \( FJ' \) and \( FJ \), and the line through \( F \) perpendicular to \( FJ' \) [Fig. 6(b)]. Let \( H \) be the intersection of this line with the incident ray. Draw \( \mu \), the line through \( J \) with direction of the normal \( \mathbf{n} \). According to Eq. (A6), this line \( \mu \) is parallel to \( FJ' \). The focusing property is equivalent to stating that \( \mu \) bisects \( \angle HJF \). This is clear when the bisection statement is alternatively written \( \mathbf{n} \times \mathbf{k} = \mathbf{n} \times \mathbf{JF} \), which, by the vectorial law of reflection [4], would imply parallelism of \( JF \) with the reflected wave vector. To prove the bisection, define

\[
\alpha \equiv \angle HJF, \quad \beta \equiv \angle FHJ, \quad \gamma \equiv \angle JFH.
\]

Since \( \Delta JFH \) was constructed with a right angle at \( F \), \( \beta = 90 - \alpha \). By the definition of the paraboloid of revolution, \( (FJ) = (JJ') \), and so \( \Delta JFJ' \) is isosceles, implying that \( \angle JFJ' = \alpha \). Thus \( \gamma = \angle JFH - \angle JFJ = 90 - \alpha \). Therefore \( \gamma = \beta \) and \( \Delta HJF \) is isosceles. In turn, \( \lambda \), the perpendicular to the base \( HF \), must bisect the vertex angle \( \angle HJF \). The focusing property is illustrated in Fig. 7.

5. Paraboloid and Taylor Expansion of a Sphere

Note that, near its vertex, the paraboloid of revolution (with equation \( y = (x^2 + z^2)/2\rho \) in the \( x-y-z \) system set up in Subsection A.3) is simply the second-order Taylor series approximation for a half-sphere of radius \( \rho \) centered at \( C = F + (\rho / 2) \hat{a} \). This is evident by the half-space’s equation \( y = \rho - \sqrt{\rho^2 - z^2} \). Additionally, near the vertex, the directions of the respective normals \( \mathbf{CJ} \) and \( \mathbf{FJ} \) will be similar. Thus, the vectorial-law [Eq. (A2)] implies similar behavior for reflection by either surface near the vertex. Indeed, in the paraxial approximation, both surfaces have the same focal point \( F \) and the same focal length \( f_i = \rho / 2 \). Figure 7 traces reflected wave vectors of the half-maxima and center of a Gaussian laser beam near its waist for both surfaces, in a large-aperture situation [10]. In this situation, where the paraxial approximation is compromised, the sphere does not have a true focal point [Fig. 7(a)], although the paraboloid [Fig. 7(b)] does.

6. Accounting for Diffraction by Exact Calculation of Intersections with a Surface

First, setup the \( x-y-z \) coordinate system (by appropriate rotation and translation) so that the laser beam’s axis is \( z \), and the direction of displacement of the ray of interest is \( +y \).

In this system, \( J \), the point of junction of the laser with the surface, has coordinates \( (x_j = 0, y_j, z_j) \), and \( y_J \) must be positive. Define \( \tau \) such that the product \( \tau \cdot w_0 \) is the coordinate of the ray of interest at the beam waist \( z = 0 \). For example, the half-maximum point has \( \tau = 0.8493218(1 / \tau = \text{sqr}(2 \sin 2 \lambda)) \). At all locations, \( \tau \) specifies how many \( 1/\rho^2 \) beam widths the ray of interest is from the central axis, and, in particular, \( y_J = \tau \cdot w(z_j) \). By Eq. (1), then,

\[
y_j^2 = (\tau w_0)^2 + (\tau w_0 / z_R)^2 \cdot z_j^2.
\]

Once \( z_j \) is determined, \( y_j \) can be found directly as the positive root of Eq. (A9). In this coordinate system, the intersection of the beam with the sphere centered at \( C \) is

\[
(0 - x_c)^2 + (y_j - y_C)^2 + (z_j - z_C)^2 = \rho^2.
\]

Subtracting this from Eq. (A9) yields, for \( y_j \), a quadatic of \( z_j \):

\[
2y_C \cdot y_j = 2y_C \cdot a_2 \cdot z_j^2 + 2y_C \cdot a_1 \cdot z_j + 2y_C \cdot a_0.
\]

where constants \( a_{i=0,1,2} \) are defined such that

\[
\begin{align*}
2 \cdot y_C \cdot a_2 &= 1 + (\tau \cdot w_0 / z_R)^2, \\
2 \cdot y_C \cdot a_1 &= -2 \cdot z_C, \\
2 \cdot y_C \cdot a_0 &= (\tau \cdot w_0)^2 + \left( z_C^2 + y_C^2 + z_C^2 - \rho^2 \right).
\end{align*}
\]

At times, solving the homogeneous approximation to Eq. (A10) for \( z_j \) will have negligible error. In particular, if \( y_C \) is sufficiently small that one can be certain that the perturbation \( 2 \cdot y_C \cdot y_j \ll |z_C| \), then \( z_j \) will be a root of the quadratic with coefficients \( y_C \cdot a_2, y_C \cdot a_1 \), and \( y_C \cdot a_0 \). In this case, there can be two real solutions for \( z_j \), with \( z_j > z_C \) corresponding to the intersection with the concave surface of the sphere (designated formally with \( \rho < 0 \)), and \( z_j < z_C \) corresponding to the intersection with the convex surface (\( \rho > 0 \)).

Otherwise, in the event of finite \( y_C \), Eq. (A10) can be divided by \( y_C \) and substituted into Eq. (A9) to yield a fourth degree polynomial to solve for \( z_j \):

\[
0 = b_1 \cdot z_j^4 + b_3 \cdot z_j^3 + b_2 \cdot z_j^2 + b_1 \cdot z_j + b_0, \quad \text{with}
\]

\[
\begin{align*}
b_1 &= a_2^2, \\
b_3 &= 3 \cdot a_2 \cdot a_1, \\
b_2 &= 2 \cdot a_2 \cdot a_0 + a_2^2 - (\tau \cdot w_0 / z_R)^2, \\
b_1 &= 2 \cdot a_1 \cdot a_0, \\
b_0 &= a_0^2 - (\tau \cdot w_0)^2.
\end{align*}
\]

In practice for small offsets, division by a small number \( y_C \) introduces errors in determination of coefficients \( a_2, a_1, \) and \( a_0 \), and it is useful instead to multiply Eq. (A12) by \( y_C^2 \) before finding roots \( z_j \). In general, Eq. (A12) can yield up to four distinct real solutions for \( z_j \). At most two of these will meet the \( y_J > 0 \) requirement (thicker red branch in Fig. 2) of the problem statement, with \( z_j < z_C \) for the convex surface and \( z_j > z_C \) for the concave surface.

If we wish to find the intersection \( J \) with the parabolic surface approximating the sphere, we have instead (by some

![Fig. 7](Color online) Reflection of three parallel rays from spherical and paraboloidal surfaces. Surfaces and their normals are shown in black; wave vectors in red. This calculation accurately represents the axis and half-maximum edges of a Gaussian laser beam (in red) with \( j \ll \rho \). (a) A concave spherical reflecting surface does not reflect all three rays to the same point. (b) The sphere’s nearest approximating paraboloid reflects all three rays to its focal point. The generating MATLAB script is available [10].
algebra manipulations extending from the earlier discussion on the paraboloid’s normal)

\[(0 - x_C)^2 + (y_J - y_C)^2 - 2\rho(z_J - z_C) = 2\rho^2.\]

Subtracting this from Eq. (A9) reproduces Eq. (A10) with new constants \(a_i:\)

\[
\begin{align*}
2 \cdot y_C \cdot a_2 &= (\tau \cdot w_0 / \rho)^2, \\
2 \cdot y_C \cdot a_1 &= -2 \cdot \rho, \\
2 \cdot y_C \cdot a_0 &= (\tau \cdot w_0)^2 + (x_C^2 + y_C^2 + 2 \cdot \rho \cdot z_C - 2\rho^2). \quad \text{(A11')}
\end{align*}
\]

The remainder of the solution is identical. Note that, for simplicity, here we have only considered the exact paraboloid intersection for the case of \(\hat{u}\) parallel to the laser axis \(z\).

The exact intersection of the hyperboloid [Eq. (A9)] with a non-parallel paraboloid can be determined by extending the above approach to use the general solution for intersection of conics [15].

Note that the exact intersection calculation does not remove errors resulting from diffraction effects after the surface, since Eq. (A1) does not account for changing wave vectors \(\mathbf{q}\). However, even with a small spot size at the beam waist, the exact intersection calculation’s diffraction error will be minimal if either (1) the surface is far from the beam waist or (2) the photodetector is near the surface.

7. Lens Approximation

Consider reflection of a Gaussian beam by a spherical or paraboloidal surface centered at \(C\) to a planar photodetector with normal \(p\) parallel to the laser axis \(z\). We retain the notations from Section 2, which solved the problem generally. The “approximating lens” defined above has focal length for reflection \(f_i = -\rho / 2\) [8]. This paraxial approximation is equivalent to assuming a small acceptance aperture for the surface such that \(z_J = z_C - \rho\). The \(x\)-\(y\)-\(z\) coordinate system is chosen with \(z\) parallel to the laser axis and the \(y\)-\(z\) plane containing the incident vector \(k\). The origin \(O\) is chosen such that \(OK\) is parallel to \(k\). Note that this origin will generally be displaced along \(z\) from the origin for vectorial-law calculations (which is taken at the laser beam waist). Let \(I\) be the image of the origin by this approximating lens. If the photodetector is a distance \(d_{PD}\) before the interface’s surface, then the detection plane’s reference point is \(P = (0, 0, P_z)\) with \(P_z = C_z - \rho - d_{PD}\). The quantity of interest is \(y_Q = (QP)\).

To emphasize geometrical optics, define also object and image height \(h_o, h_i\), and distances \(d_o, d_i\) [Fig. 8]:

\[
\begin{align*}
\ h_o \equiv (CO)_{y}, \quad & h_i \equiv (CI)_{y}, \quad \text{or} \quad y_J = (h_i - y_C), \\
\ d_o \equiv (OJ)_{z}, \quad & d_i \equiv (IJ)_{z}. \quad \text{(A13)}
\end{align*}
\]

The reflected ray appears from the photodetector to originate from \(I\), the virtual image of \(O\). By similarity of triangles (darker shaded triangle in Fig. 8), we have \((IQ)_y / (IQ)_{-z} = (IJ)_y / -d_i\), which can be rewritten

\[
(y_Q - h_i + h_o) / (-d_{PD}) = (y_J - (h_i - h_o)) / (-d_i).
\]

By similarity of the lighter shaded triangle in Fig. 8:

\[
y_J / d_o = y_K / z_K.
\]

Combining these two equations yields

\[
(y_Q - h_i + h_o) / (-d_{PD}) = (d_o y_K - (h_i - h_o)z_K) / (-d_i z_K).
\]

This can be rewritten

\[
y_Q = d_o y_K (-d_i + d_{PD}) / (-d_i z_K) + [M_o - 1] (d_{PD} h_o / d_i), \quad \text{(A14)}
\]

where we have defined origin magnification for the lens \(M_o \equiv -h_i / h_o\).

Examining Eq. (A14), it is convenient to define a photodetector magnification, which is independent of \(y_K\):

\[
M_{PD} \equiv d_o (-d_i + d_{PD}) / (-d_i z_K). \quad \text{(A15)}
\]

The origin magnification can be calculated from geometry via the lens equation, \(d_i^{-1} + d_{0}^{-1} = f_i^{-1}\):

\[
M_o = -d_i / d_o = f_i / (f_i - d_o). \quad \text{(A16)}
\]

By substituting Eq. (A16) into Eq. (A14), we obtain

\[
y_Q = M_{PD} y_K - (d_{PD} / f_i) h_o, \quad \text{(A17)}
\]

The lens equation can also be used to calculate the photodetector magnification [Eq. (A15)]:

\[
M_{PD} = (f_i d_o + f_i d_{PD} - d_o d_{PD}) / (f_i z_K), \quad \text{with} \quad f_i = -\rho / 2. \quad \text{(A18)}
\]

The same calculations results [Eqs. (A14), (A15), and (A17)] hold for refracted rays, where the photodetector is a distance \(d_{PD}\) after the refracting surface, except the refracted image \(I\) (Fig. 9) is now determined by the lens equation for a refractive interface \(n_i d_i^{-1} + n_o d_o^{-1} = f^{-1}\), where the focal length for refraction is \(f = \rho / (n_i - n_o)\) [8], and the sign for the image distance is reversed:

\[
d_o \equiv (OJ)_z, \quad -d_i \equiv (IJ)_z. \quad \text{(A13')}
\]

![Fig. 8. Geometry of light emanating from a point \(K\), striking a reflecting (a) convex or (b) concave spherical or paraboloidal surface at \(J\), and reflecting to a point \(Q\) on a photodetector a distance \(d_{PD}\) from the apex of the surface.](image)
The magnification can then be expressed in terms of the back (image) focal distance $f_i = n_i f = n_i \rho / (n_i - n_o)$:

$$M_o = -d_i / (d_o n_i / n_o) = f_i / (f_i - d_o n_i / n_o). \quad (A16')$$

Substituting Eq. (A16') into Eq. (A14), we again obtain Eq. (A17), where the effective magnification defined in Eq. (A15) for refraction is

$$M_{PD} = (n_f d_o + n_o f f_{PD} - d_o f_{PD}) / (n_f d_o) \quad \text{with} \quad f = \rho / (n_i - n_o). \quad (A18')$$

Note that the magnification for reflection [Eq. (A18)] is the special case of Eq. (A18') with $f = -\rho / 2$ and $n_i = n_o = 1$.

It is convenient to base calculations on the profile of the beam in the photodetector plane, in which case $z_K = d_o - f_{PD}$. In the particular case that $C$ is far from $O$, Eqs. (A18) and (A18') for the limit of large $d_o$ are considerably simplified:

$$M_{PD} = 1 - (d_{PD} / f_i). \quad (A19')$$

Alternatively, solution by ray transfer matrix analysis [8] yields the same results for reflection,

$$\begin{bmatrix} y_Q + h_o \\ \frac{\partial y}{\partial z} \bigg|_{Q} \end{bmatrix} = \begin{bmatrix} 1 & d_{PD} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \frac{\partial y}{\partial z} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & d_o - z_K \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} y_K + h_o \\ \frac{\partial y}{\partial z} \bigg|_{K} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

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REFERENCES AND NOTES

8. E. Hecht, Optics (Addison-Wesley, 1987).
10. The generating MATLAB scripts are available upon request.