METHOD OF OPTIMAL DIRECTIONS FOR FRAME DESIGN

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ABSTRACT

A frame design technique for use with vector selection algorithms, for example Matching Pursuits (MP), is presented. The design algorithm is iterative and requires a training set of signal vectors. The algorithm, called Method of Optimal Directions (MOD), is an improvement of the algorithm presented in [1]. The MOD is applied to speech and electrocardiogram (ECG) signals, and the designed frames are tested on signals outside the training sets. Experiments demonstrate that the approximation capabilities, in terms of mean squared error (MSE), of the optimized frames are significantly better than those obtained using frames designed by the algorithm in [1]. Experiments show typical reduction in MSE by $20 - 50\%$.

1. INTRODUCTION

Traditional transform based compression schemes use orthogonal bases, and the goal is to represent as much signal information with as few transform coefficients as possible. The optimal transform for a signal depends on the statistics of the stochastic process that produced the signal. For a Gaussian process and high resolution quantization the Karhunen-Loève Transform (KLT) is the optimal transform. If the process is not Gaussian, or the high resolution assumption does not hold, the KLT need not be the optimal transform. It is then a nontrivial task to find the optimal transform even if the statistics are known [2]. In addition to these difficulties the signal is often non-stationary, and consequently no fixed transform will be optimal in all signal regions. One way to overcome these problems is to use a weighted sum of vectors from an overcomplete set of vectors. For a finite dimensional space, any finite overcomplete set of vectors which span the space form a *frame* [3].

The basic idea when using a frame instead of an orthogonal transform is that we have more vectors and thus a better chance of finding a small number of vectors whose linear combination match the signal vector well. Since a linearly dependent set of vectors is used, an expansion is no longer unique. In a compression scheme the goal is to use as few vectors as possible to obtain a good approximation of each signal vector. Finding the optimal vectors to use in an approximation is an NP-hard problem and requires extensive calculation [4].

The use of frames in compression schemes have been given some attention [3, 5, 6] whereas the problem of *frame design* in this context is largely unexplored. We presented an algorithm for frame design using a training set in [1]. In this paper we present a significantly improved version of the frame design algorithm, and we call it the Method of Optimal Directions (MOD).

2. BASES AND FRAMES

If an N-dimensional vector space V contains a linearly independent set $B = \{\mathbf{b}_i\}$ of N vectors, then B is called a *basis* for V, and it spans the space. Any vector, $\mathbf{v}$, in V can be expanded as a linear combination of the basis vectors: $\mathbf{v} = \sum_{j=1}^{N} \alpha_j \mathbf{b}_j$, and the expansion is unique. If the set of vectors is orthogonal, that is $b_i \perp b_j$ when $i \neq j$, then B is called an *orthogonal basis* for V.

A vector can also be written as a linear combination of an overcomplete set of vectors. If the N-dimensional vector space V contains a set $F = \{\mathbf{f}_j\}$ of K vectors where $K > N$, and F spans the space V, F is an overcomplete set. The vectors $\mathbf{f}_j$ are not independent, and F is not a basis but a *frame* [7]. Any vector, $\mathbf{v}$, in the set V can be expanded as a linear combination of the frame vectors: $\mathbf{v} = \sum_{j=1}^{K} \alpha_j \mathbf{f}_j$, but because of the linear dependence of the frame vectors, the expansion is not unique any more.

The term *frame* covers both a basis and an overcomplete set of vectors. We use the term *frame* for a general linearly dependent set of vectors, mostly overcomplete, which spans the space. Other terms, like dictionary or codebook have been used for similar sets, but these terms are often associated with vector quantization or classification, and we avoid them in this paper.

Let $\mathbf{F}$ denote an $N \times K$ matrix where $K \geq N$. The columns, $\{\mathbf{f}_j\}$, $j = 1, \ldots, K$, constitute a frame. Let $\mathbf{x}_l$ be a real signal vector, $\mathbf{x}_l \in R^N$, $\mathbf{x}_l$ can then be represented or approximated as

$$\tilde{\mathbf{x}}_l = \sum_j w_l(j)\mathbf{f}_j, \tag{1}$$

where $w_l(j)$ is the coefficient corresponding to vector $\mathbf{f}_j$. In a good compression scheme, many of the $w_l(j)$'s will be zero. The corresponding error energy is $\|\mathbf{r}_l\|^2 = \|\mathbf{x}_l - \tilde{\mathbf{x}}_l\|^2$, where $\|\cdot\|$ denotes the Euclidean norm in $\mathbf{R}^N$. For a set of M signal vectors, the mean squared error (MSE) can be calculated as

$$MSE = \frac{1}{NM} \sum_{l=0}^{M-1} \|\mathbf{r}_l\|^2. \tag{2}$$

We need to select the frame vectors to be used for approximating a given signal vector $\mathbf{x}$. Since finding the optimal solution is an NP-hard problem, a suboptimal technique is preferable in order to limit the computational complexity. There exist several different vector selection methods dealing with this problem. They can be divided into sequential (greedy) and parallel vector selection methods [8]. We use Orthogonal Matching Pursuit (OMP) [9] as the vector selection algorithm in this paper, but the MOD can also be used with other vector selection algorithms.

3. METHOD OF OPTIMAL DIRECTIONS (MOD)

The iterative algorithm used to design frames is inspired by the Generalized Lloyd Algorithm (GLA) used for designing VQ codebooks [10]. The main steps in the GLA are:

1. Find an initial codebook C_1 with K vectors in the codebook, each with dimension N. Set $i = 1$.
2. Given the codebook, C_i, perform the Lloyd Iteration to generate the improved codebook C_{i+1}.
3. If the average distortion for C_{i+1},

$$D = \sum_{l=1}^{K} \int_{R_l} d(\mathbf{x}, \mathbf{c}_l) f_x(\mathbf{x}) d\mathbf{x} \quad (3)$$

$$d(\mathbf{x}, \mathbf{y}) \triangleq \|\mathbf{x} - \mathbf{y}\|^2 \quad (4)$$

has changed by a small enough amount since the last iteration, stop. Otherwise: $i = i + 1$, and go to Step 2.

The Lloyd Iteration for empirical data is as follows:

a) Given the codebook, $C_i = \{\mathbf{c}_l\}$, partition the training set, $\mathcal{T}$, into sets R_l using the Nearest Neighbor Condition:

$$R_l = \{\mathbf{x} \in \mathcal{T} \;:\; d(\mathbf{x}, \mathbf{c}_l) \leq d(\mathbf{x}, \mathbf{c}_j);\ \text{all } j \neq l\} \quad (5)$$

b) The Centroid for a cell R_l is:

$$\text{cent}(R_l) = \frac{1}{\#R_l} \sum_{k=1}^{\#R_l} \mathbf{x}_{l_k}, \quad (6)$$

where $\mathbf{x}_{l_k}$ are elements in R_l, and $\#R_l$ is the number of elements in R_l. Compute the centroids for the sets, R_l, to obtain the new codebook, $C_{i+1} = \{\text{cent}(R_l)\}$.

The GLA is designed for optimizing a codebook for VQ. Part a) in the Lloyd iteration finds the optimal classification for the training set using a given codebook. In the context of VQ, classification corresponds to finding the best vector in the codebook representing the training vector. This vector is the approximation of the training vector. Part b) finds a better codebook for that classification. It follows that the new codebook is guaranteed to be better than the previous, and the GLA will eventually find at least a local optimum.

In the context of frame design, part a) in the Lloyd Iteration involves finding approximations for all the vectors in the training set. We will not call this *classification* since it includes both finding the frame vectors to be used when approximating a signal vector *and* their associated coefficients. Thus, part a) in the Lloyd Iteration for frame design is to find an *approximation* for each training vector.

Part b) in the Lloyd iteration is to use the current approximations to construct a new frame. We can not use the centroids of the training vectors to compute a new codebook, as in the original Lloyd iteration, because the approximation of each training vector includes several vectors and coefficients. We have to construct the new frame vectors in some other way.

The GLA requires that each new frame performs better, in terms of MSE, than the previous one using the existing approximation. In this context existing approximation means the approximation achieved using the *adjusted version* of the frame vectors selected, but with the old coefficients. GLA also requires the frame to perform better after the new approximation vectors and corresponding coefficients are found using the new frame. Then the new frame will always be better, or as good as the previous one, and the algorithm at least guarantees a local optimum. To be able to guarantee improvement in each iteration, part a) in the Lloyd iteration would have to be done by complete search for the optimal approximation. This is an NP-hard problem, and a suboptimal vector selection algorithm has to be used instead. With this approach there is no guarantee for the frame after an iteration to be better than the previous frame. Thus the algorithm presented here is not a GLA, but it is an iterative algorithm inspired by the GLA. Even though we can not guarantee improvement in each iteration, experiments show that the algorithm works remarkably well.

When a signal vector is approximated using a frame, the number of frame vectors to be used in the approximation has to be chosen. In the frame design algorithm presented here the number of frame vectors to be used, m, is constant for all training vectors and iterations. The main steps in of algorithm are as follows:

1. Begin with an initial frame $\mathbf{F}_0$ of size $N \times K$, and decide the number of frame vectors to be used in each approximation, m. Assign counter variable $i = 1$.
2. Approximate each training vector, $\mathbf{x}_l$, using a vector selection algorithm:

$$\tilde{\mathbf{x}}_l = \sum_{j=1}^{K} w_l(j) \mathbf{f}_j. \quad (7)$$

   where $w_l(j)$ is the coefficient corresponding to vector $\mathbf{f}_j$, and only m of the $w_l(j)$'s are different from zero.

   Find the residuals.

3. Given the approximations and residuals, adjust the frame vectors $\Rightarrow \mathbf{F}_i$.
4. Find the new approximations, and calculate the new residuals. If (stop-criterion = FALSE) $\Rightarrow i = i + 1$, go to step 3. Otherwise stop.

The difference in the MOD and the frame design algorithm presented in [1] is the way that the frame vectors are adjusted. In the following the attention is focused on step 3 in the algorithm above.

Consider a scheme where m frame vectors are selected for approximating each training vector, i.e. $\mathbf{x}_l$ is approximated as in Equation 7. The residual is:

$$\mathbf{r}_l = \mathbf{x}_l - \tilde{\mathbf{x}}_l, \quad (8)$$

The idea is now to adjust all frame vectors in such a manner that the total MSE, given by $\sum_l \|\mathbf{r}_l\|^2$, becomes as small as possible. Denote by $\boldsymbol{\delta}_j$ the adjustment of frame vector $\mathbf{f}_j$:

$$\tilde{\mathbf{f}}_j = \mathbf{f}_j + \boldsymbol{\delta}_j, \quad j = 1, 2 \ldots K. \quad (9)$$

In the following we show how to find the optimal vectors $\boldsymbol{\delta}_j$, $j = 1, 2, \ldots K$. Since we find the optimal directions in Equation 9, we call the design algorithm *the method of optimal directions*. The new residual for a training vector $\mathbf{x}_l$ is:

$$\mathbf{r}'_l = \mathbf{r}_l - \sum_{j=1}^{K} w_l(j)\boldsymbol{\delta}_j, \qquad (10)$$

where $w_l(j)$ is the coefficient corresponding to the non adjusted vector, $\mathbf{f}_j$, for the approximation of training vector $\mathbf{x}_l$. Only m of the $w_l(j)$'s are different to zero. A reduction of the total MSE over all training vectors is wanted:

$$\sum_l \|\mathbf{r}'_l\|^2 \le \sum_l \|\mathbf{r}_l\|^2. \qquad (11)$$

The resulting MSE after adjusting the frame vectors is investigated:

$$\sum_l \|\mathbf{r}'_l\|^2 = \sum_l \|\mathbf{r}_l - \sum_{j=1}^{K} w_l(j)\boldsymbol{\delta}_j\|^2 \qquad (12)$$

$$= \sum_l (\mathbf{r}_l - \sum_{j=1}^{K} w_l(j)\boldsymbol{\delta}_j)^T(\mathbf{r}_l - \sum_{j=1}^{K} w_l(j)\boldsymbol{\delta}_j) \qquad (13)$$

$$= \sum_l \|\mathbf{r}_l\|^2 - 2\sum_l \sum_{j=1}^{K} w_l(j)\boldsymbol{\delta}_j^T\mathbf{r}_l \qquad (14)$$

$$+ \sum_l \sum_{j=1}^{K}\sum_{k=1}^{K} w_l(j)w_l(k)\boldsymbol{\delta}_j^T\boldsymbol{\delta}_k. \qquad (15)$$

If Equation 11 is satisfied, then:

$$\sum_{j=1}^{K}\sum_{k=1}^{K} a_{jk}\boldsymbol{\delta}_j^T\boldsymbol{\delta}_k - 2\sum_{j=1}^{K}\boldsymbol{\delta}_j^T\mathbf{b}_j \le 0 \qquad (16)$$

where

$$a_{jk} = \sum_l w_l(j)w_l(k) \qquad (17)$$

$$\mathbf{b}_j = \sum_l w_l(j)\mathbf{r}_l. \qquad (18)$$

We want to find the minimum of $\sum_l \|\mathbf{r}'_l\|^2$, and this is equivalent to finding the minimum of the left side of Equation 16:

$$\frac{\partial}{\partial\delta_q(p)}\left(\sum_{j=1}^{K}\sum_{k=1}^{K}\sum_{i=1}^{N} a_{jk}\delta_j(i)\delta_k(i) - 2\sum_{j=1}^{K}\sum_{i=1}^{N}\delta_j(i)b_j(i)\right) = 0, \qquad (19)$$

where $q = 1, 2 \ldots K$, and $p = 1, 2 \ldots N$. After some manipulations we get:

$$\sum_{j=1}^{K} a_{jq}\delta_j(p) - b_q(p) = 0. \qquad (20)$$

This can be written as a matrix equation:

$$\mathbf{A}\boldsymbol{\Delta} = \mathbf{B}, \qquad (21)$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots \\ \vdots & \ddots & \\ a_{K1} & & a_{KK} \end{bmatrix} \qquad (22)$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{b}_1^T \\ \vdots \\ \mathbf{b}_K^T \end{bmatrix} \qquad (23)$$

According to Equation 17, $\mathbf{A}$ is symmetric. The $\boldsymbol{\Delta}$ matrix contains the optimal adjustment vectors:

$$\boldsymbol{\Delta}^T = \begin{bmatrix} \boldsymbol{\delta}_1 & \ldots & \boldsymbol{\delta}_K \end{bmatrix} \qquad (24)$$

Assuming $\mathbf{A}$ to be full rank, we get:

$$\boldsymbol{\Delta} = \mathbf{A}^{-1}\mathbf{B}. \qquad (25)$$

$\sum_l \|\mathbf{r}'_l\|^2$ can not be less than 0, thus we know that the problem has a minimum solution. Since Equation 25 has only one solution when $\mathbf{A}$ is full rank, this is the minimum solution.

For each iteration, if the frame vectors are adjusted according to Equations 9 and 25 this gives the optimal improvement in MSE for the existing vector selection and corresponding coefficients.

We have now focused on point 3 in the algorithm. If an optimal vector selection algorithm had been used in point 4 in the algorithm, the new frame would always be better than the previous, with respect to MSE. Selection algorithms for frames are suboptimal, so there is no way to guarantee a better frame when using a practical selection algorithm, but test results show that this scheme works remarkably well and produces frames that are well suited for a given class of input data. In summary, the algorithm for frame design works as follows:

1. Begin with an initial frame $\mathbf{F}_0$, $i = 1$.
2. A vector selection algorithm is used to find an approximation for each training vector, and all the residuals are calculated.
3. All frame vectors, $\mathbf{f}_j$, are adjusted according to Equation 9 and Equation 25. The frame vectors are then normalized to unit length $\Rightarrow \mathbf{F}_i$.
4. A vector selection algorithm is used to find the new approximations and residuals.
   If (stop-criterion = FALSE) $\Rightarrow i = i + 1$, go to step 3, else terminate.

Suggested stop-criteria can be: Maximum number of iterations or almost constant MSE. Due to the lack of guarantee for the new frame to be better than the previous, the algorithm should allow the MSE to grow for several iterations without terminating the training. This can be seen by carefully inspection of the training results in the next section.

4. EXPERIMENTS AND RESULTS

The MOD is applied to ECG and speech signals. The ECG signals used are signals from the MIT arrhythmia database [11]. The records are represented with 12 bit per sample, and the sampling frequency is 360 Hz. The ECG signal used for training is MIT100, 0:00 to 5:00 minutes, and the test signal is MIT100 5:30 to 10:30 minutes. The speech signals used are recorded at 16 kHz in a room without echo, and downsampled to 8 kHz. 8.75 seconds of speech data is used for training, and the test signal is 8.75 seconds of speech recorded under the same conditions as the speech used for training.

In Figure 1 the training curves are compared to the training curves for the same training signals using the frame design method presented in [1]. For all the training experiments in this paper, the initial frames consist of normalized versions of the first signal vectors in the training set.

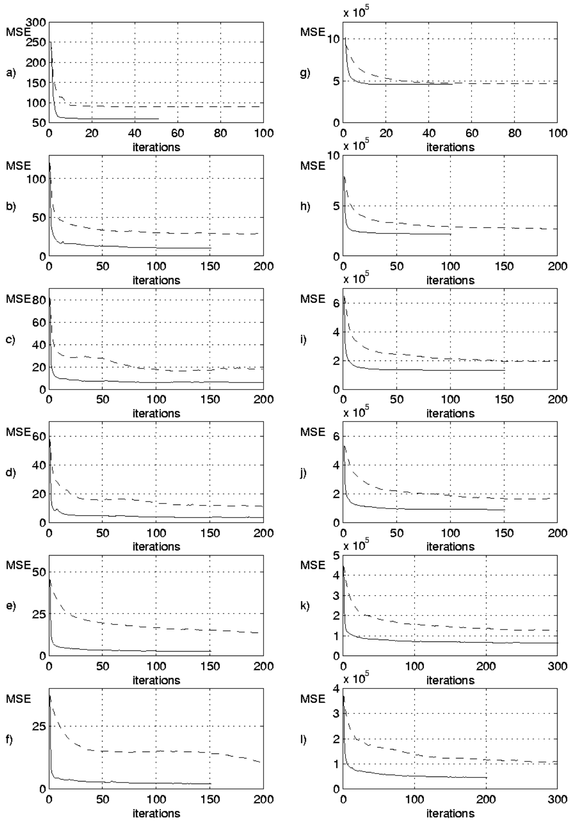

Figure 1: MSE is plotted as a function of training iterations. The dashed curves are taken from [1], and the solid curves are obtained using MOD. In a), b), c), d), e), and f) the training signal is an ECG signal, and 1,2,3,4,5, and 6 frame vectors are used in each approximation, respectively. In g), h), i), j), k), and l) the training signal is a speech signal, and 1,2,3,4,5, and 6 frame vectors are used in each approximation, respectively.

The optimized frames are tested on the test signals. Figure 2 compares the results of the frames optimized using the algorithm in [1] with the results of the frames optimized using MOD.

5. CONCLUSION

From the experiments it is seen that the MOD performs significantly better than the frame design method presented in [1]. In the old method we made assumptions on the direction to adjust the frame vectors. With MOD no assumptions are made. For a given frame and approximation, MOD gives the *optimal* adjustment of the frame vectors in each iteration. The suboptimal vector selection algorithm still makes it impossible to guarantee improvement in each iteration.

The training experiments show that the MOD provides better convergence properties than the old method. Using the MOD, the MSE decays faster and converges on a lower level than the resulting MSE using the old method.

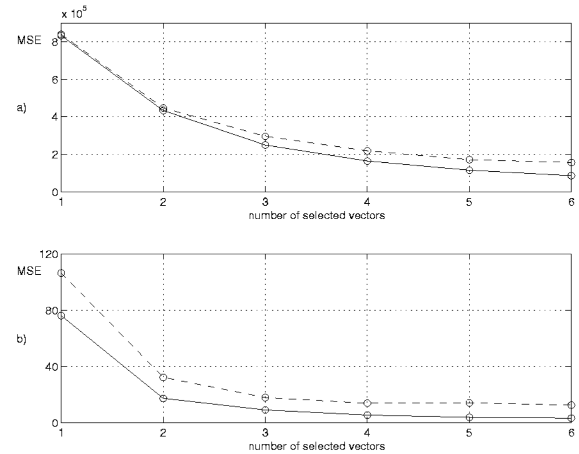

Figure 2: MSE is plotted as a function of different numbers of vectors in an approximation. Test signal is used. Dotted: frames optimized by the old frame design algorithm, solid: frames optimized using MOD a) speech signal, b) ECG signal, MIT100.

In future work complete compression results using the MOD frames will be presented. Compression using frames designed by the old method performed well at low bitrates [12, 13], and we have reason to belive that using MOD frames the rate-distortion results will be even better.

6. REFERENCES

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