Markov regenerative stochastic Petri nets

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Abstract

Stochastic Petri nets of various types (SPN, GSPN, ESPN, DSPN etc.) are recognized as useful modeling tools for analyzing the performance and reliability of systems. The analysis of such Petri nets proceeds by utilizing the underlying continuous-time stochastic processes – continuous-time Markov chains for SPN and GSPN, semi-Markov processes for a subset of ESPNs and Markov regenerative processes for DSPN. In this paper, we introduce a new class of stochastic Petri nets, called Markov Regenerative Stochastic Petri Nets (MRSPNs), that can be analyzed by means of Markov regenerative processes and constitutes a true generalization of all the above classes. The MRSPNs allow immediate transitions, exponentially distributed timed transitions and generally distributed timed transitions. With a restriction that at most one generally distributed timed transition be enabled in each marking, the transient and steady state analysis of MRSPNs can be carried out analytically-numerically rather than by simulation. Equations for the solution of MRSPNs are developed in this paper, and are applied to an example.

Key words: MRSPN; System modelling; Performance; Transient analysis; Steady state analysis; Markov regenerative process

1. Introduction

Several classes of stochastic Petri nets have been proposed for performance and reliability analysis of systems. They include the Stochastic Petri Net (SPN) [20], the Generalized Stochastic Petri Net (GSPN) [3], the Extended Stochastic Petri Net (ESPN) [12], and the Deterministic and Stochastic Petri Net (DSPN) [2]. In the SPN, a transition fires after an exponentially distributed amount of time (firing time) once it is enabled. The GSPN allows

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transitions with zero firing times or exponentially distributed firing times. The stochastic process underlying the SPN or the GSPN is a continuous-time Markov chain (CTMC). The exponential assumption has been regarded as one of the main restrictions in the application of the SPNs (GSPNs). Non-exponentially distributed processing durations often occur in many practical problems. For instance, deterministic times arise while modeling time-outs in the communication protocols, lognormal distribution [19] is often used for repair times, and uniform distribution is adopted in Estelle [4]. In an effort to alleviate this restriction, the ESPN was defined to allow generally distributed firing times. Under some restrictions, the underlying stochastic process of the ESPN is a semi-Markov process (SMP) and therefore analytically solvable. If these restrictions are not met, the ESPN models are solved by discrete-event simulation [12]. In [11,15], generally distributed firing times are allowed, however, the resulting model requires simulation for its solution.

The DSPN allows transitions with zero firing times, exponentially distributed or deterministic firing times. The underlying stochastic process of DSPN is neither a Markov nor a semi-Markov process. However, DSPN models can be solved analytically with a restriction that at most one transition with deterministic firing time is enabled together with zero or more exponentially distributed timed transitions. A steady state solution method for DSPNs appears in [2,8], and a transient solution method is given in [8]. Algorithms for parametric sensitivity analysis of the steady state solution are presented in [9]. In [8], the stochastic process underlying a DSPN is shown to be a Markov regenerative process (MRGP).

In this paper, we introduce a new class of stochastic Petri nets by generalizing the deterministic firing times of the DSPN to generally distributed firing times. This is a natural extension of [8] and we follow the same structure here. We show that the underlying stochastic process for this new class of Petri nets continues to be MRGP, and hence we call them Markov Regenerative Stochastic Petri Nets (MRSPNs).

The main contributions of this paper include: (1) definition of a new and powerful class of stochastic Petri nets that is a superset of SPNs, GSPNs, ESPNs and DSPNs, (2) showing that the underlying stochastic process for the new class of Petri nets is the Markov regenerative process. We show a sufficient condition for Petri nets to be MRSPNs. For a class of MRSPNs which satisfies this condition, we derive (3) the kernel distributions of the underlying MRGP, (4) the equations for the steady state behavior, and (5) the equations for the transient behavior.

The paper is organized as follows. After defining MRSPN in Section 2, we show in Section 3 that the underlying stochastic process of the MRSPN is a Markov regenerative process. Equations for the transient analysis and the steady state analysis of MRSPNs are derived in Sections 4 and 5, respectively. Computational aspects of the solution are given in Section 6. As an example, the M/G/1/2/2 queueing system is analyzed in Section 7.

2. Definition of MRSPN

A Petri net is a directed bipartite graph with two types of nodes called places (represented by circles in a graphical representation) and transitions (represented by rectangles or bars). Directed arcs (represented by arrows) connect places to transitions, and vice versa. Inhibitor arcs (represented by lines terminating with a small circle) connect places to transitions. If an
arc exists from a place (transition) to a transition (place), then the place is called an input (output) place of that transition. Multiple arcs can connect a place to a transition (input arc) or a transition to a place (output arc). The number of arcs connecting a place to a transition (a transition to a place) is called the \textit{multiplicity} of that input (output) arc. When the multiplicity of an arc is more than one, a small bar with a number (equal to the multiplicity) is placed next to the arc. Places may contain \textit{tokens} (represented by dots). The state of a Petri net is defined by the number of tokens in each place, and is represented by a vector \( m = (\#(p_1), \#p(2), \ldots, \#(p_{n_1})) \), called a \textit{marking}, where \( \#(p_i) \) is the number of tokens in place \( i \) and \( n_1 \) is the number of places in the net.

A transition is said to be \textit{enabled} if each of its ordinary input places contains at least as many tokens as the multiplicity of the input arc and each of its inhibitor input places contains fewer tokens than the multiplicity of the corresponding inhibitor arc. An enabled transition can fire. When it fires, as many tokens as the corresponding input arc's multiplicity are removed from each ordinary input place, and as many tokens as the corresponding output arc's multiplicity are deposited in each output place. The number of tokens in an inhibitor input place does not change. The firing of a transition may change the allocation of tokens to the places and, thus, may create a new marking. A marking \( m_j \) is said to be \textit{reachable} from a marking \( m_i \) if, starting from \( m_i \), there exists a sequence of transitions whose firings generate \( m_j \). A \textit{reachability graph} can be constructed by connecting a marking \( m_i \) to a marking \( m_j \) with a directed arc if the marking \( m_i \) can result from the firing of some transition enabled in \( m_i \). From the given initial marking \( m_0 \), a unique reachability graph is obtained for a Petri net.

In stochastic Petri nets, a random firing time elapses after a transition is enabled until it fires. Transitions which have nonzero firing times are called \textit{timed transitions} and transitions with zero firing times are called \textit{immediate transitions}. We will call a timed transition whose firing time is exponentially (generally) distributed an \( \text{EXP} \) (\( \text{GEN} \)) transition and let \( F_g(\cdot) \) be the distribution of the firing time of a \( \text{GEN} \) transition \( g \). Immediate transitions have priority to fire over timed transitions. When two or more immediate transitions are enabled at the same time, one of them is selected probabilistically to fire. Among different firing policies of timed transitions [1], we use the \textit{race with enabling memory} firing policy, i.e., the process of sampling the firing time of a timed transition and of elapsing it is repeated in case that this transition is re-enabled after being disabled before it fires (preemptive repeat different) [12]. In case that a marking changes and the transition is not disabled in the new marking, the original firing time of this transition is unaffected.

The markings of a stochastic Petri net can be classified into \textit{vanishing markings} and \textit{tangible markings}. In a vanishing marking at least one immediate transition is enabled and in a tangible marking no immediate transition is enabled.

Let \( M(t) \) be the tangible marking of a stochastic Petri net at time \( t \). The right-continuous, piecewise constant, continuous-time stochastic process underlying the stochastic Petri net is called its \textit{marking-process} \( \{ M(t), t \geq 0 \} \). Study of the marking-process \( \{ M(t), t \geq 0 \} \) is the thrust of the analysis of a stochastic Petri net.

\textbf{Definition 1} [17]. A sequence of bivariate random variables \( \{(Y_n, T_n), n \geq 0\} \) is called a \textit{Markov renewal sequence} if

1. \( T_0 = 0, \forall n > 0, T_{n+1} > T_n \) and \( Y_n \in \{-2, -1, 0, 1, 2, \ldots\} \),
(2) \( \forall i, j \in \{Y_n, n \geq 0\} \),

\[
P\{Y_{n+1} = j, T_{n+1} - T_n \leq t \mid Y_n = i, T_n, Y_{n-1}, T_{n-1}, \ldots, Y_0, T_0\} = P\{Y_{n+1} = j, T_{n+1} - T_n \leq t \mid Y_n = i\} \quad \text{(Markov Property)}
\]

\[
P\{Y_1 = j, T_1 \leq t \mid Y_0 = i\} \quad \text{(Time Homogeneity)}.
\]

**Definition 2** [5,17]. A stochastic process \( \{Z(t), t \geq 0\} \) is called a *Markov regenerative process* (also known as a semi-regenerative process) if there exists a Markov renewal sequence \( \{(Y_n, T_n), n \geq 0\} \) of random variables such that all the conditional finite dimensional distributions of \( \{Z(T_n + t), t \geq 0\} \) given \( \{Z(u), 0 \leq u \leq T_n, Y_n = i\} \) are the same as those of \( \{Z(t), t \geq 0\} \) given \( Y_0 = i \).

This definition implies that

\[
P\{Z(T_n + t) = j \mid Z(u), 0 \leq u \leq T_n, Y_n = i\} = P\{Z(t) = j \mid Y_0 = i\}.
\]

The process between \( T_n \) and \( T_{n+1} \) may be any continuous-time stochastic process, including a Markov regenerative process itself.

**Definition 3.** A stochastic Petri net is called a Markov Regenerative Stochastic Petri Net (MRSPN) if its marking-process \( \{M(t), t \geq 0\} \) is a Markov regenerative process.

It is not easy to check whether a given Petri net is an MRSPN. However, we can show that an MRSPN is a very large class which includes previously studied classes of Petri nets.

**Theorem 1.** A stochastic Petri net is an MRSPN if it is either an SPN or a GSPN or an ESPN\(^1\) or a DSPN.

**Proof.** We know that the marking-process of an SPN or a GSPN is a CTMC (see [20] and [3] respectively), that of an ESPN is an SMP (see [12]), and that of DSPN is an MRGP (see [8]). All these processes are special cases of the MRGP, thus the theorem follows. \( \square \)

In the next section, we study a sufficient condition under which a Petri net is an MRSPN.

**3. The class MRSPN*\)**

**Definition 4.** A stochastic Petri net is said to be in \( \mathcal{P}N^* \) if

- at most one GEN transition is enabled in a marking
- the firing time of the GEN transition is sampled at the time the transition is enabled; the firing time distribution may depend upon the marking at the time the transition is enabled.

Firing time cannot change until the transition either fires or is disabled.

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\(^1\) By ESPN, we mean a subset of ESPNs which can be solved analytically.
The main result of this section is that Petri nets in $\mathcal{P}N^*$ are MRSPNs. For this reason a stochastic Petri net in $\mathcal{P}N^*$ will be denoted by MRSPN*. Figs. 1(a) and (c) show two example MRSPNs* with their reachability graphs (b) and (d), respectively. In the figures, filled rectangles denote GEN transitions and empty rectangles denote EXP transitions. Each immediate transition is assigned a weight of firing, each EXP transition is assigned a firing rate, and each GEN transition is assigned a firing time distribution. In the reachability graphs, solid arcs denote GEN transitions by GEN transitions and dotted arcs denote state transitions by EXP transitions. Each marking of Fig. 1(b) is a 4-tuple: $(#(p1), #(p2), #(p3), #(p4))$. Similarly, each marking of Fig. 1(d) is a 3-tuple: $(#(p5), #(p6), #(p7))$. Transition $t_j$ represents the submission of a job by one of two customers. The firing time of this transition is exponentially distributed with the marking-dependent firing rate, i.e., $#(p1)$ times $\lambda$. The firing time of transition $ts$ has a phase type distribution, and that of transition $tv$ has a gamma distribution. Transition $td$ takes a fixed amount of time $\tau$ to fire. But it is disabled when the EXP transition $te$ fires before $\tau$. When transition $tr$ fires, a token is put back in place $p5$ making $td$ enabled again. This models the reset of a deterministic timer by a preemptable event that is represented by $te$ followed by a reset operation that is represented by $tr$.

The reduced reachability graph is obtained from the reachability graph by merging the vanishing markings into their successor tangible markings. Suppose a timed transition $t$ fires in a tangible marking $m_i$ and the successor marking is a vanishing marking $m_j$. Then some immediate transition enabled in $m_j$ will fire and the next successor marking may be either

\begin{tabular}{|c|c|c|}
\hline
Transition & Distribution & Parameters \\
\hline
$t_j$ & EXP & $\#(p1)\lambda$ \\
$ts$ & Phase Type & $(\alpha, M)$ \\
$tv$ & GAM & $(\lambda, \alpha)$ \\
\hline
\end{tabular}

\begin{tabular}{|c|c|c|}
\hline
Transition & Distribution & Parameters \\
\hline
$td$ & DET & $\tau$ \\
$te$ & EXP & $\lambda$ \\
$tr$ & EXP & $\mu$ \\
\hline
\end{tabular}

Fig. 1. Examples of MRSPNs* and their markings.
vanishing or tangible. The probability of visiting a tangible marking \( m_k \) after visiting an arbitrary number (including 0) of vanishing markings, given that the current marking is the vanishing marking \( m_j \), is \( W_{jk} \) that is a solution to the system of linear equations [21]:

\[
(I - B^{VV})W = B^{YT}.
\]

The \( B^{VV} \) (\( B^{YT} \)) is the matrix of one-step transition probabilities from vanishing to vanishing (tangible) markings in the reachability graph. The entire sequence of vanishing markings between tangible markings \( m_i \) and \( m_k \) initiated by the firing of a timed transition \( t \) in \( m_i \) is substituted by the branching probability \( W_{jk} \) in the following way:

1. If \( t \) is an EXP transition with firing rate \( \lambda_t(m_i) \), then each of the successor tangible markings \( m_k \) is connected from \( m_i \) directly with the transition rate

\[
\lambda(m_i, m_k) = \sum_{t \in \mathcal{E}(m_i)} \lambda_t(m_i)W_{jk}
\]

where \( \mathcal{E}(m_i) \) is the set of EXP transitions that are enabled in \( m_i \).

2. If \( t \) is a GEN transition with the random firing time \( X \), each of the successor tangible markings \( m_k \) is connected from \( m_i \) directly with the firing time \( X \). The branching probabilities in this case are kept in the branching-probability matrix \( \Delta = [\Delta(i, k)] \) where

\[
\Delta(i, k) = W_{jk}.
\]

For all other entries, \( \Delta(i, k) \) is set to 1 if \( i = k \) and 0 otherwise. In case there are no vanishing markings between \( m_i \) and \( m_k \), then \( \Delta(i, k) \) is set to 1.

A timed transition \( t \) is said to be exclusive \(^2\) in a marking \( m \) if \( t \) is the only transition enabled in \( m \). A timed transition \( t \) is said to be competitive with respect to another timed transition \( t' \) in the marking \( m \) if both \( t \) and \( t' \) are enabled in \( m \) and the firing of \( t \) disables \( t' \); otherwise it is said to be concurrent with \( t' \). Markings \( m_1, m_2 \) and \( m_5 \) of Fig. 1(b) are the ones in which a GEN transition and an EXP transition are enabled concurrently. Markings \( m_3 \) and \( m_6 \) (\( m_4 \)) are the ones in which a GEN (EXP) transition is enabled exclusively. Marking \( m_7 \) of Fig. 1(d) is the one in which a GEN transition and an EXP transition are enabled competitively.

We shall show that, under the condition of \( \mathcal{P}N^* \), the marking-process \( \{M(t), t \geq 0\} \) of MRSPN* is a Markov regenerative process (MRGP). We need to introduce the following notation in order to prove this result.

Let \( \Omega \) be the set of all tangible markings of the reduced reachability graph of the MRSPN*.

Then \( \Omega \) is also the state space of the marking-process of the MRSPN*. For example, \( \Omega = \{m_1, m_2, m_3, m_4, m_5, m_6\} \) is the state space of the marking-process of the MRSPN* shown in Fig. 1(a). Consider a sequence \( \{T_n, n \geq 0\} \), of epochs when the MRSPN* is observed. Let \( T_0 = 0 \). Define \( \{T_n, n \geq 0\} \) recursively as follows: Suppose \( M(T_n+) = m \).

1. If no GEN transition is enabled in state \( m \), define \( T_{n+1} \) to be the first time after \( T_n \) that a state change occurs. If no such time exists, we set \( T_{n+1} = \infty \).

---

\(^2\) This definition of exclusive, competitive, and concurrent transitions is slightly different from the original definition given in [12].
(2) If a GEN transition is enabled in state \( m \), define \( T_{n+1} \) to be the time when the GEN transition fires or is disabled. Note that there cannot be more than one GEN transition enabled in state \( m \), and hence the above cases cover all possibilities. With the above definition of \( \{T_n, n \geq 0\} \), let \( Y_n = M(T_n+) \). We now show the following result.

**Theorem 2.** The marking-process \( \{M(t), t \geq 0\} \) of an MRSPN* is a Markov regenerative process, i.e., MRSPN* is an MRSPN.

**Proof.** First we shall show that \( \{(Y_n, T_n), n \geq 0\} \) embedded in the marking process of the MRSPN* is a Markov renewal sequence, i.e.,

\[
\forall i, j \in \Omega, \quad P\{Y_{n+1} = j, T_{n+1} - T_n \leq t \mid Y_n = i, T_n, Y_{n-1}, T_{n-1}, \ldots, Y_0, T_0\} = P\{Y_{n+1} = j, T_{n+1} - T_n \leq t \mid Y_n = i\} = P\{Y_1 = j, T_1 \leq t \mid Y_0 = i\}.
\]

Suppose the past history \( Y_0, T_0, \ldots, Y_{n-1}, T_{n-1}, Y_n, T_n \) is given and \( Y_n = i \). Consider two cases.

1. No GEN transition is enabled, i.e., all the transitions that are enabled in state \( i \) at time \( T_n \) are EXP transitions. In this case, due to the memoryless property of the exponential random variable, the future of the marking-process depends only on the current state \( i \) and does not depend upon the past history or the time index \( n \).

2. Exactly one GEN transition is enabled in state \( i \). There may be other EXP transitions enabled in state \( i \). In this case, \( T_{n+1} \) is the next time when the GEN transition fires or is disabled. The joint distribution of \( Y_{n+1} \) and \( (T_{n+1} - T_n) \) will depend only on the \( Y_n \). That is, \( T_{n+1} \) depends on the destination state \( Y_{n+1} \) and the state \( Y_{n+1} \) will be decided from the current state \( Y_n \). It is thus independent of the past history and the time index \( n \).

This proves that \( \{(Y_n, T_n), n \geq 0\} \) is a Markov renewal sequence.

Now consider the marking-process from time \( T_n \) onwards, namely \( \{M(T_n + t), t \geq 0\} \). Given the history \( \{M(u), 0 \leq u \leq T_n, M(T_n) = i\} \), it is clear from the above argument that the stochastic behavior of \( \{M(T_n + t), t \geq 0\} \) depends only on \( M(T_n) = i \). Thus,

\[
\{M(T_n + t), t \geq 0 \mid M(u), 0 \leq u \leq T_n, M(T_n) = i\} \overset{d}{=} \{M(T_n + t), t \geq 0 \mid M(t) = i\} \overset{d}{=} \{M(t), t \geq 0 \mid M(0) = i\}, \quad \forall i \in \Omega,
\]

where \( \overset{d}{=} \) denotes equality in distribution. This proves that \( \{M(t), t \geq 0\} \) is an MRGP. \( \square \)

From Theorem 2, we see that the following conditional probabilities play an important role:

\[
K_{ij}(t) = P\{Y_1 = j, T_1 \leq t \mid Y_0 = i\}, \quad i, j \in \Omega. \tag{4}
\]

The matrix \( K(t) = [K_{ij}(t)] \) is called the kernel of the MRGP. The distribution function of \( T_1 \) starting from state \( i \) is defined as

\[
H_i(t) = P\{T_1 \leq t \mid Y_0 = i\} = \sum_{j \in \Omega} K_{ij}(t), \quad t \geq 0, \quad i \in \Omega. \tag{5}
\]
**Corollary 1.** \(\{Y_n, n \geq 0\}\) is a discrete-time Markov chain (DTMC) with one-step transition probability matrix \(K(\infty)\).

**Proof.** Follows from Eq. (4) by letting \(t \to \infty\). □

**Remark.** \(\{Y_n, n \geq 0\}\) is called the *embedded Markov chain* (EMC) for the MRSPN*.

Let \(N(t) = \sup\{n \geq 0: T_n \leq t\}\). The process \(\{X(t), t \geq 0\}\) such that

\[
X(t) = Y_{N(t)}, \quad t \geq 0
\]

is a *semi-Markov process* (SMP) of the MRSPN* with kernel \(K(\cdot)\).

Using the theory of Markov regenerative process, we carry out the transient analysis and the steady state analysis in the following sections.

**4. Transient analysis of MRSPN*\)**

In this section, we develop equations for the transient probabilities of the marking-process \(\{M(t), t \geq 0\}\). Define the transition probability

\[
V_{ij}(t) = P\{M(t) = j \mid M(0) = Y_0 = i\}, \quad i, j \in \Omega,
\]

and let \(V(t) = [V_{ij}(t)]\). Let

\[
K_{iu} * V_{uj}(t) = \int_0^t dK_{iu}(x)V_{uj}(t-x),
\]

and \(K * V(t)\) be a matrix whose \((i, j)\) element is \(\sum_u K_{iu} * V_{uj}(t)\).

The transient analysis of the marking-process of the MRSPN* is based on the following basic theorem on MRGPS with kernel \(K(\cdot)\). We include the proof for completeness.

**Theorem 3.** The transition probability matrix \(V(t)\) satisfies the following generalized Markov renewal equation:

\[
V(t) = E(t) + K * V(t)
\]

where \(E(t) = [E_{ij}(t)]\) such that \(E_{ij}(t) = P\{M(t) = j, T_1 > t \mid Y_0 = i\}\).

**Proof.** Conditioning on \(Y_1\) and \(T_1\) and using the Markov regenerative properties of \(\{M(t), t \geq 0\}\), we obtain

\[
P\{M(t) = j \mid Y_0 = i, Y_1 = k, T_1 = x\} = \begin{cases} P\{M(t) = j \mid Y_0 = i, T_1 = x\}, & t < x, \\ V_{kj}(t-x), & t \geq x. \end{cases}
\]
Unconditioning, we get:

\[ V_{ij}(t) = \sum_{k \in \Omega} \int_0^\infty P[M(t) = j \mid Y_0 = i, Y_1 = k, T_1 = x] \, dK_{ik}(x) \]

\[ = \sum_{k \in \Omega} \int_0^t P[M(t) = j \mid Y_0 = i, T_1 > t \mid Y_0 = i] \, dK_{ik}(x) + \sum_{k \in \Omega} \int_0^t V_{kj}(t - x) \, dK_{ik}(x) \]

\[ = P[M(t) = j, T_1 > t \mid Y_0 = i] + \sum_{k \in \Omega} K_{ik} * V_{kj}(t) \]

\[ = E_{ij}(t) + [K * V]_{ij}(t) \]

which in matrix form is Eq. (9). \[ \square \]

Note that \( E_{ij}(t) \) describes the behavior of the marking-process between two transition epochs of the EMC, i.e., over the time interval \([0, T_1)\). We will call the matrix \( E(t) \) the local kernel, as opposed to the global kernel \( K(t) \).

In order to use the above theorem, we need to specify \( E(t) = [E_{ij}(t)] \) and \( K(t) = [K_{ij}(t)] \) matrices for the MRSPN*. We give the following definitions for this purpose.

Consider a state \( m \) in \( \Omega \). Let \( \mathcal{G}(m) \) be the set of GEN transitions and \( \mathcal{G}(m) \) be the set of EXP transitions enabled in state \( m \). For example, \( \mathcal{G}(m_1) = \{t_1\} \), \( \mathcal{G}(m_2) = \{t_2\} \) for state \( m_1 \) of Fig. 1(b). Consider the following cases.

Case 1 (\( \mathcal{G}(m) = \emptyset \), i.e., no GEN transition is enabled in \( m \)). Given \( Y_0 = m, T_1 \) is exponentially distributed with rate \( \Lambda_0(m) \) such that

\[ \Lambda_0(m) = \sum_{t \in \mathcal{G}(m)} \lambda_t(m) \]  \hspace{1cm} \( \text{(10)} \)

and the state at time \( T_1 + \) is \( n \) with probability \( \lambda(m, n)/\Lambda_0(m) \), where \( \lambda(m, n) \) is defined in Eq. (2). Furthermore \( M(t) = Y_0 \) for \( 0 < t < T_1 \).

Case 2 (\( \mathcal{G}(m) = \{g\} \), i.e., exactly one GEN transition \( g \) is enabled). Suppose \( Y_0 = m \). In this case \( T_1 \) is the time when \( g \) fires or is disabled due to the firing of a competitive EXP transition. Define \( \Omega(m) \) to be the set of all states reachable from \( m \) in which the marking-process can spend a non-zero time before the next EMC transition occurs, i.e., during \([0, T_1)\). For example, \( \Omega(m_1) = \{m_1, m_2, m_3\} \) for \( m_1 \) of Fig. 1(b), and \( \Omega(m_7) = \{m_7\} \) for \( m_7 \) of Fig. 1(d). The marking-process during \([0, T_1)\) is a CTMC on state space \( \Omega \), which is called the subordinated CTMC [18]. The infinitesimal generator matrix of the subordinated CTMC with initial state \( m \) will be denoted by \( Q(m) \), and is formed as follows: for any \( n \in \Omega(m) \), the entries from \( n \) to \( n' \in \Omega \) are given by \( \lambda(n, n') \) if \( n \neq n' \), and \(-\sum_{n' \in \Omega} \lambda(n, n') \) if \( n = n' \), and when \( n \notin \Omega(m) \), the entries from \( n \) are zeros \(^3\). The subordinated CTMC and its generator matrix for state \( m_1 \) of Fig. 1(b) are shown in Fig. 2 for example. This CTMC is subordinated to the GEN transition \( t_1 \) and describes possible state changes by EXP transition(s) while \( t_1 \) is enabled.

\(^3\) Even though \( Q(m) \) is defined for each \( m \), it does not have to be distinct for each \( m \). For instance, \( Q(m_1) = Q(m_2) = Q(m_3) \) for states \( m_1, m_2 \) and \( m_3 \) in Fig. 1(b).
Next, define $\Omega_x(m)$ to be the set of states which are reachable starting from $m$ (not necessarily directly) by firing of a competitive EXP transition. Similarly, define $\Omega_y(m)$ to be the set of states reachable by firing of the GEN transition $g$. For example, $\Omega_x(m_7) = \{m_8\}$, $\Omega_y(m_7) = \{m_9\}$ for state $m_7$ of Fig. 1(d) and $\Omega_x(m_1) = \{m_4, m_5, m_6\}$, $\Omega_y(m_1) = \emptyset$, for state $m_1$ of Fig. 1(b). Note that $Q(m)$ consists of the transition rates from a state $n$ in $\Omega(m)$ to a state $n'$ in $\Omega$. This means that $n'$ may not necessarily belong to $\Omega(m)$, it may be a state in $\Omega_y(m)$ instead. For simplicity, we assume $4$ that $\Omega(m) \cap \Omega_y(m) = \emptyset$.

This completes the description of the process in case 2.

Now we derive expression for the local kernel $E(t)$ in the following theorem.

**Theorem 4.** The local kernel $E(t) = [E_{m,n}(t)] (m, n \in \Omega)$ of the MRSPN* is given by:

1. when $\mathcal{E}(m) = \emptyset$:

   $$E_{m,n}(t) = \delta_{m,n} e^{-\lambda m t}$$

   where $\delta_{m,n}$ is the Kronecker $\delta$ defined by $\delta_{m,n} = 1$ if $m = n$ and $0$ otherwise,

2. when $\mathcal{E}(m) = \{g\}$:

   for $n \in \Omega(m)$,

   $$E_{m,n}(t) = [e^{Q(m) t}]_{m,n} (1 - F_g(t))$$

   for $n \notin \Omega(m)$, $E_{m,n}(t) = 0$.

**Proof.** As defined in Theorem 3:

$$E_{m,n}(t) = \Pr\{M(t) = n, T_1 > t | Y_0 = m\}$$

is the state transition probability of the marking-process between two transition epochs of the EMC. Starting with $M(0) = m$ and knowing that $T_1 > t$, the marking-process $\{M(u), 0 \leq u \leq t\}$ is a CTMC as explained below.

1. Suppose $\mathcal{E}(m) = \emptyset$. Then the firing of any EXP transition in $\mathcal{E}(m)$ triggers the state change of the EMC. The probability of marking-process being in state $n$ at time $t$ (before the EMC state change occurs) given that it entered state $m$ at time $0$ is the probability that

---

4 Even if this assumption does not hold, i.e., the next state that is reached from a state in $\Omega(m)$ by firing of a competitive EXP transition is also in $\Omega(m)$, we can still handle the case by modifying the generator matrix.
the marking-process stays at the initial state \( m \) until time \( t \), i.e., \( M(u) = M(0) \) for all \( 0 \leq u < T_1 \). The process \( \{M(u), 0 \leq u < T_1 \mid M(0) = m\} \) in this case is a degenerative CTMC that stays in state \( m \). Therefore:

\[
P\{M(t) = n, T_1 > t \mid Y_0 = m\} = \delta_{m,n}(1 - H_m(t)) = \delta_{m,n} e^{-A_m t},
\]

where \( H_m(t) \) is defined in Eq. (5).

(2) Suppose \( \mathcal{E}(m) = \{g\} \). Then the EMC state change is triggered either at the time of firing of \( g \) or when \( g \) is disabled. Recall that the marking-process stays in a state in \( \Omega(m) \), starting from \( m \), before the next EMC state change occurs. The marking-process is captured by the subordinated CTMC with the generator matrix \( Q(m) \). If we let the firing time \( X \) of the GEN transition be \( x \), \( E_{m,n}(t) \) is the transition probability of this CTMC to state \( n \) \((n \in \Omega(m))\) by time \( t \) \((t < x)\). That is, for state \( n \in \Omega(m) \) \( E_{m,n}(t) \) is evaluated to \([e^{Q(m)t}]_{m,n}\). If \( g \) is disabled before \( t \) by firing of a competitive EXP transition, the next state should be outside of \( \Omega(m) \). That is, for state \( n \notin \Omega(m) \) \( E_{m,n}(t) = 0 \).

Unconditioning on \( X \), we get

\[
E_{m,n}(t) = \int_0^\infty \left[ e^{Q(m)s} \right]_{m,n} dF_g(x) = \left[ e^{Q(m)s} \right]_{m,n}(1 - F_g(t)), \quad n \in \Omega(m).
\]

The above two cases cover all the possibilities. \( \Box \)

We now define the kernel of an MRSPN* in the following theorem.

**Theorem 5.** The kernel \( K(t) = [K_{m,n}(t)] \) \((m, n \in \Omega)\) of the marking-process of the MRSPN* is given by:

1. for state \( m \) such that \( \mathcal{E}(m) = \emptyset \),

\[
K_{m,n}(t) = \begin{cases} 0, & \text{if } n \in \Omega_\emptyset(m) \text{ and } n \notin \Omega_\emptyset(m) \\ \frac{\lambda(m,n)}{A_m}(1 - e^{-A_m t}), & \text{if } n \in \Omega_\emptyset(m) \text{ and } n \in \Omega_\emptyset(m) \\
\end{cases}, \quad A_m = 0,
\]

(13)

2. for state \( m \) such that \( \mathcal{E}(m) = \{g\} \),

- if \( n \in \Omega_g(m) \) but \( n \notin \Omega_g(m) \):

\[
K_{m,n}(t) = \left[ e^{Q(m)x} \right]_{m,n}(1 - F_g(t)) + \int_0^t \left[ e^{Q(m)x} \right]_{m,n} dF_g(x),
\]

(14)

- if \( n \notin \Omega_g(m) \) but \( n \in \Omega_g(m) \):

\[
K_{m,n}(t) = \sum_{m' \in \Omega(m)} \int_0^t \left[ e^{Q(m)x} \right]_{m,m'} dF_g(x) \Delta(m', n),
\]

(15)

- if \( n \in \Omega_g(m) \) and also \( n \in \Omega_\emptyset(m) \):

\[
K_{m,n}(t) = \left[ e^{Q(m)x} \right]_{m,n}(1 - F_g(t)) + \int_0^t \left[ e^{Q(m)x} \right]_{m,n} dF_g(x) + \sum_{m' \in \Omega(m)} \int_0^t \left[ e^{Q(m)x} \right]_{m,m'} dF_g(x) \Delta(m', n),
\]

(16)
If \( n \notin \Omega_\varnothing(m) \) and also \( n \notin \Omega_\varnothing(m) \):

\[
K_{m,n}(t) = 0, \quad t \geq 0.
\]

**Proof.** As given in Eq. (4), \( K_{m,n}(t) \) is defined by

\[
K_{m,n}(t) = P\{Y_1 = n, T_1 \leq t \mid Y_0 = m\}, \quad m, n \in \Omega.
\]

(1) When no GEN transition is enabled in state \( m \), the firing of any EXP transition in \( \mathcal{G}(m) \) triggers the state change of the EMC. Hence it is clear that

\[
P\{Y_1 = n, T_1 \leq t \mid Y_0 = m\} = P\{\text{state transition occurs before } t \mid Y_0 = m\} \times P\{n \text{ is reached by the transition } Y_0 = m\} = (1 - e^{-\lambda_m t}) \times \frac{\lambda(m, n)}{\Lambda_m}.
\]

If \( \mathcal{G}(m) = \emptyset \), then \( \lambda(m, n) = \Lambda_m = 0 \). In this case, further state transitions are not possible from the state \( m \), i.e., \( m \) is an absorbing state and \( P\{Y_1 = n, T_1 \leq t \mid Y_0 = m\} \) is computed to be 0 for all \( n \in \Omega \).

(2) When a GEN transition \( g \) is enabled, the EMC state change is triggered either at the time of firing of \( g \) or when \( g \) is disabled by a competitive EXP transition. Depending on the type of state \( n \), we have the following cases:

(a) If \( n \) is in \( \Omega_\varnothing(m) \) but not in \( \Omega_\varnothing(m) \), then \( n \) is only reachable by firing of a competitive EXP transition. Recall that the set \( \Omega_\varnothing(m) \) defines the states which are reachable from \( m \) by firing of the competitive EXP transitions. Let the firing time \( X \) of the GEN transition be \( x \).

If \( 0 \leq t < x \), the competitive EXP transitions may have fired during \([0, t]\):}

\[
P\{Y_1 = n, T_1 \leq t \mid Y_0 = m\} = P\{\text{the state of the subordinated CTMC is } n \text{ at time } t \mid Y_0 = m\} = \left[ e^{Q(m)t} \right]_{n, m, n \in \Omega_\varnothing(m)}.
\]

If \( t \geq x \), the competitive EXP transitions can fire only up to time \( x \) \( ([0, x]) \). Thus:

\[
K_{m,n}(t) = \begin{cases} 
\left[ e^{Q(m)t} \right]_{m, n}, & 0 \leq t < x, \\
\left[ e^{Q(m)x} \right]_{m, n}, & t \geq x.
\end{cases}
\]

Unconditioning,

\[
K_{m,n}(t) = \int_t^\infty \left[ e^{Q(m)t} \right]_{m, n} dF_g(x) + \int_0^t \left[ e^{Q(m)x} \right]_{m, n} dF_g(x)
\]

\[
= \left[ e^{Q(m)x} \right]_{m, n} (1 - F_g(t)) + \int_0^t \left[ e^{Q(m)x} \right]_{m, n} dF_g(x).
\]
(b) If \( n \) is in \( \Omega_g(m) \) but not in \( \Omega_g(m) \), then \( n \) is reachable by firing of \( g \). Suppose the GEN transition \( g \) fires at time \( X = x \). Then \( n \in \Omega_g(m) \) will be reached at time \( x \). The probability of this event is \( \sum_{m' \in \Omega(m)} [e^{Q(m)X}]_{m,m'} \Delta(m', n) \) which means that the marking-process is in the states \( m' \in \Omega(m) \) until \([0, x)\) and then jump to \( n \in \Omega_g(m) \) at time \( x \). Unconditioning on \( X \), we get Eq. (15).

(c) If \( n \) is both in \( \Omega_g(m) \) and in \( \Omega_g(m) \), \( n \) can be reached either by firing of a competitive EXP transition or by firing of \( g \). Eq. (16) is obtained by combining the previous two cases.

(d) If \( n \) is neither in \( \Omega_g(m) \) nor in \( \Omega_g(m) \), \( n \) is not reachable from \( m \) at the next EMC transition. Therefore \( K_{m,n}(t) = 0 \).

The above cases cover all the possibilities. □

**Corollary 2.** The one-step transition probability matrix \( P = [P_{m,n}] \) of the EMC is given by:

1. for state \( m \) such that \( \mathcal{G}(m) = \emptyset \),
   \[
P_{m,n} = \begin{cases} 
0, & \lambda_m = 0, \\
\lambda(m, n), & \lambda_m > 0,
\end{cases}
\]

(18)

2. for state \( m \) such that \( \mathcal{G}(m) = \{g\} \),
   - if \( n \in \Omega_g(m) \) but \( n \notin \Omega_g(m) \):
     \[
P_{m,n} = E\left[ e^{Q(m)X} \right]_{m, m'} \Delta(m', n),
\]
   - if \( n \notin \Omega_g(m) \) but \( n \in \Omega_g(m) \):
     \[
P_{m,n} = \sum_{m' \in \Omega(m)} E\left[ e^{Q(m)X} \right]_{m,m'} \Delta(m', n),
\]
   - if \( n \in \Omega_g(m) \) and also \( n \in \Omega_g(m) \):
     \[
P_{m,n} = E\left[ e^{Q(m)X} \right]_{m,n} + \sum_{m' \in \Omega(m)} E\left[ e^{Q(m)X} \right]_{m,m'} \Delta(m', n),
\]
   - if \( n \notin \Omega_g(m) \) and also \( n \notin \Omega_g(m) \):
     \[
P_{m,n} = 0.
\]

(19) (20) (21) (22)

**Proof.** These results follow from Theorem 5 using \( P = K(\infty) \) and \( E[X] = \int_0^\infty x \, dF_g(x) \). □

**Corollary 3.** Given the transition probability matrix \( V(t) \) and the initial probability distribution \( p(0) = (p_i(0)) \), the state probability at time \( t \) of the MRSPN* is computed by

\[
p_i(t) = \sum_{i \in \Omega} p_i(0)V_{ij}(t), \quad i, j \in \Omega.
\]

(23)
Proof. Follows by conditioning on the initial state $Y_0$. □

The $K_{m,n}(t)$, $P_{m,n}$ and $E_{m,n}(t)$ expressions for a state $m$ where a transition $g$ with uniformly distributed firing time over the interval $(a, b)$ is enabled are shown in Corollary 4, 5 and 6 of [7]. The expressions for a state where a GEN transition with deterministic firing time is enabled are given by Theorems 3, 4 and Corollary 2 of [8].

5. Steady state analysis of the MRSPN*

In this section, we consider the steady state analysis of an MRSPN* whose underlying SMP is finite and ergodic (irreducible, aperiodic and positive recurrent) so that the limiting probability distributions exist. These can be computed using the general theory of MRGP. We briefly review the theory below. Define

$$t_{zm} = E[T_{11}|Y_0 = m],$$

$$O_{lmn} = E[\text{time spent by the marking-process in state } n \text{ during } [0, T_1)|Y_0 = m],$$

and the steady state probability vector $v = (v_i)$ of the EMC:

$$v = vP, \quad \sum_{i \in \Omega} v_i = 1,$$

where $P = K(\infty)$ is the one-step transition probability matrix of the EMC defined in Corollary 2. When $\mathcal{F}(m) = \emptyset$, $T_1$ is the time until any of EXP transitions fires. Hence, $\mu_m = E[T_1 | Y_0 = m] = 1/\Lambda_m$ where $\Lambda_m$ is as defined in Eq. (10). When $\mathcal{F}(m) = \{g\}$, $\mu_m$ is computed by

$$\mu_m = E[T_1 | Y_0 = m] = \int_0^{\infty} \sum_{n \in \Omega(m)} \int_0^x e^{Q(m)u} \mu_{m,n} \, du \, dF_g(x). \quad (24)$$

The $\alpha_{mn}$ is computed by

$$\alpha_{mn} = \int_0^{\infty} P\{M(t) = n, T_1 > t | Y_0 = m\} \, dt = \int_0^{\infty} E_{m,n}(t) \, dt. \quad (25)$$

Note that $\mu_m = \sum_{n \in \Omega(m)} \alpha_{mn}$. By changing the order of integrations in Eq. (24), we get Eq. (25).

When the GEN transition $g$ is enabled concurrently with EXP transitions, then $\mu_m$ is the same as $E[X]$, where $X$ is the time to fire of the GEN transition.

The following theorem describes the steady state probability distribution of the MRSPN*.

**Theorem 6.** Let $(M(t), t \geq 0)$ be an MRGP with an embedded Markov renewal sequence $(Y_n, T_n)$, $n \geq 0$. Suppose $v = (v_i)$ is a positive solution to $v = vP$ and the SMP of the MRGP is finite and ergodic. The limiting distribution $p = (p_j)$ of the state probabilities of the MRGP is given by

$$p_j = \lim_{t \to \infty} P\{M(t) = j | Y_0 = m\} = \frac{\sum_{k \in \Omega} v_k \alpha_{kj}}{\sum_{k \in \Omega} v_k \mu_k} = \frac{\sum_{k \in \Omega} \beta_k \alpha_{kj}}{\beta_k \mu_k} \quad (26)$$
where
\[ \beta_k = \frac{u_k \mu_k}{\sum_{r \in \Omega} v_r \mu_r}. \]

See [17] for the proof of the theorem. Intuitively \( p_j \) is the fraction of time the marking-process spends in state \( j \):

\[ p_j = \sum_{k \in \Omega} \left( \text{fraction of time the SMP of the MRSPN* spends in state } k \right) \times \left( \text{the time the marking-process spends in state } j \text{ per unit time of the SMP spent in state } k \right) \]

\[ = \sum_{k \in \Omega} \beta_k \frac{\alpha_{kj}}{\mu_k}. \]

6. Computational methods

6.1. Computational methods for transient analysis

We discuss methods of computing Eq. (9) in this section. A direct approach may be to solve the system of integral equations:

\[ V_i(t) = E_i(t) + \sum_{k \in \Omega} \int_0^t V_j(t-x) dK_{ik}(x) \]

that is discussed in detail in [13]. Numerical solution of a system of partial differential equations corresponding to the above system of integral equations is another alternative.

Laplace–Stieltjes transformation can be also employed. Define the Laplace transform (LT) of a function \( f(t) \) to be \( F^*(s) = \int_0^\infty e^{-st} f(t) \, dt \) and the Laplace–Stieltjes transform (LST) of a function \( f(t) \) to be \( F^-(s) = \int_0^\infty e^{-st} dF_g(t) \). Then \( F^*(s) = \int_0^\infty e^{-st} dF_g(t) \) is the LT of the firing time \( X \) of \( g \) and is known for most distributions. For a square matrix \( A \), define \( F^-(A) \) to be \( \int_0^\infty e^{-At} \, dF_g(t) \). The LT (LST) of a matrix of functions is defined to be the matrix of the LTs (LSTs) of elements.

Taking LST’s on both sides of Eq. (9), we get:

\[ V^-(s) = E^-(s) + K^-(s)V^-(s), \]

\[ V^-(s) = [I - K^-(s)]^{-1} E^-(s). \] (27)

Therefore, we can get \( V^-(s) \) once \( E^-(s) \) and \( K^-(s) \) are known. \( V(t) \) can be obtained either analytically or numerically inverting \( V^-(s) \). We now show that \( V(t) \) can be computed this way by deriving the expressions of the \( E^-(s), K^-(s) \) of the MRSPN*.
Theorem 7. The LST of $E(t) = [E_{m,n}(t)]$ $(m, n \in \Omega)$ of the MRSPN* is given by:

1. for state $m$ such that $\mathcal{G}(m) = \emptyset$:

$$E_{m,n}(s) = \delta_{m,n} \frac{s}{s + \Lambda_m},$$  \hspace{1cm} (28)

2. for state $m$ such that $\mathcal{G}(m) = \{g\}$, for $n \in \Omega(m)$:

$$E_{m,n}(s) = \left[ s(\lambda - Q(m))^{-1} (I - F\sim (\lambda - Q(m))) \right]_{m,n},$$  \hspace{1cm} (29)

for $n \not\in \Omega(m)$:

$$E_{m,n}(s) = 0.$$

Proof. Eq. (28) is directly obtained by taking LST directly on Eq. (11). Eq. (29) is obtained from Eq. (12) by using the fact [10]:

$$\int_{t_1}^{t_2} e^{-sx} e^{Q(m)x} \, dx = (\lambda - Q(m))^{-1} \left( e^{-(\lambda - Q(m)t_1)} - e^{-(\lambda - Q(m)t_2)} \right).$$

Theorem 8. The LST of $K(t) = [K_{m,n}(t)]$ $(m, n \in \Omega)$ of the MRSPN* is given by:

1. for state $m$ such that $\mathcal{G}(m) = \emptyset$:

$$K_{m,n}(s) = \frac{\lambda(m, n)}{s + \Lambda_m},$$  \hspace{1cm} (30)

2. for state $m$ such that $\mathcal{G}(m) = \{g\}$,

- if $n \in \Omega(g)(m)$ but $n \not\in \Omega_{\mathcal{G}}(m)$:

$$K_{m,n}(s) = \left[ s(\lambda - Q(m))^{-1} - Q(m)(\lambda - Q(m))^{-1} F\sim (\lambda - Q(m)) \right]_{m,n},$$  \hspace{1cm} (31)

- if $n \not\in \Omega_{\mathcal{G}}(m)$ but $n \in \Omega_{\mathcal{G}}(m)$:

$$K_{m,n}(s) = \sum_{m' \in \Omega(m)} F_{m,m'}(\lambda - Q(m)) \Delta(m', n),$$  \hspace{1cm} (32)

- if $n \in \Omega_{\mathcal{G}}(m)$ and also $n \in \Omega_{\mathcal{G}}(m)$:

$$K_{m,n}(s) = \left[ s(\lambda - Q(m))^{-1} - Q(m)(\lambda - Q(m))^{-1} F\sim (\lambda - Q(m)) \right]_{m,n}$$

$$+ \sum_{m' \in \Omega(m)} F_{m,m'}(\lambda - Q(m)) \Delta(m', n),$$  \hspace{1cm} (33)

- if $n \not\in \Omega_{\mathcal{G}}(m)$ and also $n \not\in \Omega_{\mathcal{G}}(m)$:

$$K_{m,n}(s) = 0.$$  \hspace{1cm} (34)

Proof. Eq. (30), (32) and (34) are obtained by taking LST’s directly on Eq. (13), (15) and (17), respectively. Eq. (31) and (33) are obtained by the same technique used for Eq. (29).
Efficient numerical methods are crucial for the computation of solutions for large models. The transformation method described here needs a matrix inversion that is costly for a large size matrix. As an alternative, an iterative linear system solution using the successive overrelaxation (SOR) is recommended. Subsequently, a numerical LST conversion is needed.

6.2. Computational methods for steady state analysis

The steady state solution of Eq. (26) requires \( \nu, \mu, \) and \( \alpha_{mn} (m, n \in \Omega) \). The \( \mu_m \) is readily computable after \( \nu \) and \( \alpha_{mn} \) are known since \( \mu_m = \sum_{n \in \Omega(m)} \alpha_{mn} \). The steady state probability vector \( \nu = (\nu_m) \) of the EMC is a solution to \( \nu = \nu P \), and it can be obtained by solving this linear system.

The computation of \( \alpha_{mn} \) depends on the distribution function of the GEN transition \( g \) that is enabled in marking \( m \) and \( n \):

\[
\alpha_{mn} = \int_0^\infty E_{m,n}(t) \, dt = \int_0^\infty [e^{Q(m)}]_{m,n}(1 - F_g(t)) \, dt, \quad m, n \in \Omega(m),
\]

for some \( m \) such that \( \mathcal{E}(m) = \{g\} \). For example, if \( g \) has deterministic firing time \( \tau \), Eq. (35) reduces to

\[
\alpha_{mn} = \int_0^\tau [e^{Q(m)}]_{m,n} \, dt.
\]

This can be computed as follows. Define the integral of the state transition probability of a subordinate CTMC,

\[
L_{m,n}(t) = \int_0^t [e^{Q(m)}u]_{m,n} \, du = \int_0^t G_{m,n}(u) \, du
\]

and the corresponding matrix \( L(t) \). The integral \( L(t) \) of the transition probabilities is obtained by solving the following differential equation:

\[
\frac{dL(t)}{dt} = L(t)Q(m) + G(0), \quad L(0) = 0,
\]

where \( G(0) \) is the transition probability matrix at time 0 of the subordinate CTMC, hence \( G(0) = I \), the identity matrix of size \( |\Omega| \). The \( \alpha_{mn} \) is obtained by solving the above differential equation for \( L(t) \) at time \( \tau \). For methods of solving the above, see [14].

7. Numerical example

We illustrate the analysis methods through an example of the \( M/G/1/2/2 \) queueing system (the finite population \( M/G/1 \) queue with 2 buffers and 2 customers). Fig. 3 shows the MRSPN*, its reachability graph, and its reduced reachability graph.

Each customer is assumed to submit a job with an interval that is exponentially distributed with rate \( \lambda = 0.5 \) job/hour, and the service time of a job is assumed to be uniformly distributed in the interval (0.5, 1.0) hours. The \# sign above the transition \( ta \) indicates the marking-dependent firing rate as explained in Section 2. The transition \( ti \) is an immediate transition. Dotted (thick solid) arcs represent the state transitions by \( \text{EXP (GEN)} \) transitions and thin solid arcs represent the state transitions by immediate transitions. We compute the state probability...
vector \( p(t) = (p_j(t)) \), \((j = 1, 2, 3)\) at time \( t \) with given initial marking \( m_1 = (2001) \) so that \( p_j(t) = V_j(t) \).

The kernel \( K(t) = [K_{ij}(t)] \), \((i, j = 1, 2, 3)\) of this model is given as:

\[
K_{21}(t) = \begin{cases} 0, & 0 \leq t < 0.5, \\ 4(e^{-0.25} - e^{-0.5t}), & 0.5 \leq t \leq 1.0, \\ 4(e^{-0.25} - e^{-0.5}), & t > 1.0, \\ \end{cases}
\]

\[
K_{32}(t) = \begin{cases} 0, & 0 \leq t < 0.5, \\ 2(t - 0.5) - 4(e^{-0.25} - e^{-0.5t}), & 0.5 \leq t \leq 1.0, \\ 1 - e^{-t}, & t > 1.0, \\ \end{cases}
\]

\[
K_{22}(t) = \begin{cases} 0, & 0 \leq t < 0.5, \\ 2(t - 0.5) - 4(e^{-0.25} - e^{-0.5t}), & 0.5 \leq t \leq 1.0, \\ 1 - 4(e^{-0.25} - e^{-0.5}), & t > 1.0, \\ \end{cases}
\]

The local kernel \( E(t) = [E_{ij}(t)] \), \((i, j = 1, 2, 3)\) is given as:

\[
E_{23}(t) = \begin{cases} 1 - e^{-0.5t}, & 0 \leq t < 0.5, \\ 2(t - 0.5)(1 - e^{-0.5t}), & 0.5 \leq t \leq 1.0, \\ 0, & t > 1.0, \\ \end{cases}
\]

\[
E_{22}(t) = \begin{cases} e^{-0.5t}, & 0 \leq t < 0.5, \\ 2(t - 0.5)e^{-0.5t}, & 0.5 \leq t \leq 1.0, \\ 0, & t > 1.0, \\ \end{cases}
\]

\[
E_{11}(t) = e^{-t}, \quad E_{ij}(t) = 0 \quad \text{for all other} \ i, j.
\]

We computed LSTs of \( E(t) \) and \( K(t) \) according to Theorem 7 and Theorem 8 respectively and obtained the LST of \( V(t) \) using Eq. (27). Then we computed the LT of \( V(t) \) by \( V^*(s) = V^-(s)/s \). We obtained \( V(t) \) for a fixed \( t \) by inverting the LT of \( V(t) \) numerically using Jagerman’s method [16] as adapted by Chimento and Trivedi [6]. The plot in Fig. 4 shows the transient (time-dependent) state probabilities \( p_j(t) \) over a time interval \([0, 5]\) along with the
Fig. 4. Transient and steady state probabilities of the M/G/1/2/2 system.

steady state probabilities $p_j$ which are computed analytically by Theorem 6. As expected, the transient state probabilities approach the steady state probabilities as time approaches infinity.

8. Conclusions

We have defined a new class of stochastic Petri nets called MRSPNs and showed that the underlying stochastic process of an MRSPN is a Markov regenerative process. We have shown a sufficient condition for stochastic Petri nets to be MRSPNs. We also discussed analytical solution techniques for MRSPNs* which is a class of MRSPNs and satisfies this condition. We have derived the kernel distributions, the equations for the steady state behavior, and the equations for the transient behavior for this class of MRSPN. We illustrated our method by means a simple example.

We observe that the condition given in Definition 4 is a sufficient condition for a Petri net to be an MRSPN. More relaxed conditions are possible. For example, we can allow two or more GEN transitions to be enabled in a marking. The restriction is that all the GEN transitions enabled in that marking must have started their enabling epochs at the same time point.

MRSPNs have been shown to be useful in solving models with generally distributed event times which occur in many practical systems. There is a broad range of applications and much room for further exploration.

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References


