Optimality and duality for multiple-objective optimization under generalized type I univexity

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Abstract

In this paper, we extend the classes of generalized type I vector-valued functions introduced by Aghezzaf and Hachimi in [J. Global Optim. 18 (2000) 91–101] to generalized univex type I vector-valued functions and consider a multiple-objective optimization problem involving generalized type I univex functions. A number of Kuhn–Tucker type sufficient optimality conditions are obtained for a feasible solution to be an efficient solution. The Mond–Weir and general Mond–Weir type duality results are also presented.

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1. Introduction

The field of multiple-objective optimization, also known as multiobjective programming, has grown remarkably in different directions in the setting of optimality conditions and duality theory since the 1980s. It has been enriched by the applications of various types of generalizations of convexity theory, with and without differentiability assumptions, and in the framework of continuous time programming, fractional programming, inverse vector optimization, saddle point theory, symmetric duality, variational problems and control problems. A new reader may like to consult Mishra [25] and Pini and Singh [37] for relatively more exhaustive references on the subject. More specifically, some of the recent work in the area can be found in Aghezzaf and Hachimi [1], Antczak [2–4], Brandao et al. [6,7], Chen [8], Hanson et al. [13], Kim and Kim [16], Kim and Lee [17,18], Kim et al. [19], Kuk et al. [20], Mishra [26–31], Mishra and Giorgi [32], Mishra and Mukherjee [33,34], Mishra and Rueda [35,36], Rueda et al. [39], and Zhian and Qingkai [40].

Parallel to the above development in multiple-objective optimization, there has been a very popular growth and application of invexity theory which was originated by Hanson [11] but so named by Craven [9]. Later Hanson and Mond [12] introduced type I and type II invexities which have been further generalized to pseudo type I, and quasi type I functions by Rueda and Hanson [38] and pseudoquasi type I, quasi pseudo type I and strictly pseudoquasi type I functions by Kaul et al. [15]. Rueda et al. [39] obtained optimality and duality results for several mathematical programs by combining the concepts of type I functions and univex functions [5]. Mishra [28] obtained optimality, duality and saddle point results for a multiple-objective program by combining the concepts of quasi pseudo type I, quasi pseudo type I, strictly pseudoquasi type I and univex functions.

Recently, Hanson et al. [13] extended the concept of type I functions to vector type I functions by combining the concepts of type I functions and V-invex functions introduced in Jeyakumar and Mond [14]. V-invex functions have been studied by Mishra [25] to the context of nonsmooth programs [33] and to variational problems [26]. Moreover, Aghezzaf and Hachimi [1] introduced a new class of generalized type I vector-valued functions and established the Mond–Weir and general Mond–Weir type duality results under the class of functions.

In this paper, we introduce new classes of generalized type I univex functions on the lines of Rueda et al. [39] and Aghezzaf and Hachimi [1] by extending weak strictly pseudoquasi type I, strong pseudoquasi type I, weak quasistrictly-pseudo type I and weak strictly pseudo type I functions of Aghezzaf and Hachimi [1]. In Section 2, we introduce some preliminaries. Some sufficient optimality results are established in Section 3. A number of duality theorems in the Mond–Weir setting [10] are shown in Section 4. In Section 5, we give two results on general Mond–Weir type duality.

2. Preliminaries

To compare vectors along the lines of Mangasarian [23], we will distinguish between $\leq$ and $\preceq$ or between $\geq$ and $\succeq$. Specifically,

$$x \in \mathbb{R}^n, \quad y \in \mathbb{R}^n, \quad x \leq y \iff x_i \leq y_i, \quad \forall i = 1, \ldots, n, \quad x \neq y.$$
x ∈ ℜ^n, y ∈ ℜ^n, x ≤ y ⇔ xi ≤ yi Vi = 1, ..., n.

Similar notations are applied to distinguish between ≥ and ≥.

We consider the following multiple-objective optimization problem:

(VP) minimize f(x) = (f_1(x), ..., f_p(x))
subject to g(x) ≤ 0, x ∈ X ≤ ℜ^n,

where f : X → ℜ^p and g : X → ℜ^m are differentiable functions and X ⊆ ℜ^n is an open set.

Let X_0 be the set of all feasible solutions of (VP). We quote some definitions and also give some new ones.

Definition 2.1. A point a ∈ X_0 is said to be an efficient solution of problem (VP) if there exists no x ∈ X_0 such that f(x) ≤ f(a).

Following Rueda et al. [39] and Aghezzaf and Hachimi [1], we define a generalized type I univex problem. In the following definitions b_0, b_1 : X × X × [0, 1] → ℜ^+, b(x, a) = lim_{λ→0} b(x, a, λ) ≥ 0, and b does not depend on λ, if the corresponding functions are differentiable, φ_0, φ_1 : R → R and η : X × X → ℜ^n is an n-dimensional vector-valued function.

In following definitions, we assume that φ_0, φ_1 : R → R satisfy u ≤ 0 ⇒ φ_0(u) ≤ 0 and u ≤ 0 ⇒ φ_1(u) ≤ 0; and b_0(x, a) > 0 and b_1(x, a) ≥ 0.

Definition 2.2. The problem (VP) is said to be weak strictly pseudo type I univex at a ∈ X_0 if there exist real-valued functions b_0, b_1, φ_0 and η such that

b_0(x, a)φ_0[f(x) − f(a)] ≤ 0 ⇒ (∇ f(a))η(x, a) < 0,

−b_1(x, a)φ_1[g(a)] ≤ 0 ⇒ (∇ g(a))η(x, a) ≤ 0,

for all x ∈ X_0 and for all i = 1, ..., p, and j = 1, ..., m. If (VP) is weak strictly pseudo type I univex at each a ∈ X, (VP) is said to be weak strictly pseudo type I univex on X.

This definition is an extension of that of the weak strictly pseudoquasi type I functions in Aghezzaf and Hachimi [1].

There exist functions which are weak strictly pseudoquasi type I univex but not strictly pseudoquasi type I univex.

Example 2.1. f(x) = (x_1 \sin x_2, x_2(x_2 - 1) \cos x_1) and g(x) = 2x_1 + x_2 - 2 are weak strictly pseudoquasi type I univex with respect to b_0 = 1 = b_1, φ_0 and φ_1 are the identity function on R and η(x, a) = (x_1 + x_2 - 1, x_2 - x_1) at a = (0, 0).

Definition 2.3. The problem (VP) is said to be strong pseudoquasi type I univex at a ∈ X_0 if there exist real-valued functions b_0, b_1, φ_0, φ_1 and η such that

b_0(x, a)φ_0[f(x) − f(a)] ≤ 0 ⇒ (∇ f(a))η(x, a) ≤ 0,

−b_1(x, a)φ_1[g(a)] ≤ 0 ⇒ (∇ g(a))η(x, a) ≤ 0,
for all \( x \in X_0 \) and for all \( i = 1, \ldots, p \), and \( j = 1, \ldots, m \). If (VP) is strong pseudoquasi type I univex at each \( a \in X \), (VP) is said to be strong pseudoquasi type I univex on \( X \).

**Example 2.2.** \( f(x) = (x_1(x_1 - 1)^2, x_2(x_2 - 1)^2(x_3^2 + 2)) \) and \( g(x) = x_1^2 + x_2^2 - 9 \) are strong pseudoquasi type I univex function on \( \mathbb{R}^2 \) and \( \eta(x, a) = (x_1 - 1, x_2 - 1) \) at \( a = (0, 0) \), but \((f, g)\) is not weak strictly pseudoquasi type I univex at each \( b_0, b_1, \phi_0, \phi_1 \) and \( \eta \) as can be seen by taking \( x = (1, -1) \).

**Definition 2.4.** The problem (VP) is said to be weak quasistrictly pseudo type I univex if there exist real-valued functions \( b_0, b_1, \phi_0, \phi_1 \) and \( \eta \) at each \( a \in X \) if there exist real-valued functions \( b_0, b_1, \phi_0, \phi_1 \) and \( \eta \) such that

\[
\begin{align*}
\quad b_0(x, a)\phi_0[|f(x)| - |f(a)|] & \leq 0 \quad \Rightarrow \quad (\nabla f(a))\eta(x, a) \leq 0, \\
\quad -b_1(x, a)\phi_1[|g(a)|] & \leq 0 \quad \Rightarrow \quad (\nabla g(a))\eta(x, a) \leq 0,
\end{align*}
\]

for all \( x \in X_0 \) and for all \( i = 1, \ldots, p \), and \( j = 1, \ldots, m \). If (VP) is weak quasistrictly pseudo type I univex at each \( a \in X \), (VP) is said to be weak quasistrictly pseudo type I univex on \( X \).

**Example 2.3.** \( f(x) = (x_1^2(x_1^2 + 1), x_2^3(x_2 - 1)^3) \) and \( g(x) = ((2x_1 - 4)x_1^2 - x_2^2, x_1 + x_2 - 2)(x_1^2 + 2x_1 + 4) \) are weak quasistrictly pseudo type I univex function with respect to \( b_0 = 1 = b_1, \phi_0 \) and \( \phi_1 \) are the identity function on \( \mathbb{R}^2 \) and \( \eta(x, a) = (x_1, x_2(1 - x_2)) \) at \( a = (0, 0) \), but \((f, g)\) is not type I univex with respect to same \( b_0, b_1, \phi_0, \phi_1 \) and \( \eta \) as can be seen by taking \( x = (1, 0) \). Type I univex functions are defined in Rueda et al. [39].

**Definition 2.5.** The problem (VP) is said to be weak strictly pseudo type I univex with respect to \( b_0, b_1, \phi_0, \phi_1 \) and \( \eta \) at each \( a \in X \) if there exist real-valued functions \( b_0, b_1, \phi_0, \phi_1 \) and \( \eta \) such that

\[
\begin{align*}
\quad b_0(x, a)\phi_0[|f(x)| - |f(a)|] & \leq 0 \quad \Rightarrow \quad (\nabla f(a))\eta(x, a) < 0, \\
\quad -b_1(x, a)\phi_1[|g(a)|] & \leq 0 \quad \Rightarrow \quad (\nabla g(a))\eta(x, a) < 0,
\end{align*}
\]

for all \( x \in X_0 \) and for all \( i = 1, \ldots, p \), and \( j = 1, \ldots, m \). If (VP) is weak strictly pseudo type I univex at each \( a \in X \), (VP) is said to be weak strictly pseudo type I univex on \( X \).

### 3. Optimality conditions

In this section, we establish some sufficient optimality conditions for an \( a \in X_0 \) to be an efficient solution of problem (VP) under various generalized type I univex functions defined in the previous section.

**Theorem 3.1** (Sufficiency). Suppose that

(i) \( a \in X_0 \);

(ii) there exist \( \tau^0 \in \mathbb{R}^p, \tau^0 > 0, \lambda \in \mathbb{R}^m \) and \( \lambda^0 \geq 0 \) such that
(a) $\tau^0 \nabla f(a) + \lambda^0 \nabla g(a) = 0$,
(b) $\lambda^0 g(a) = 0$,
(c) $\tau^0 e = 1$, where $e = (1, \ldots, 1)^T \in R^p$;
(iii) problem (VP) is strong pseudoquasi type I univex at $a \in X_0$ with respect to some $b_0$, $b_1$, $\phi_0$, $\phi_1$ and $\eta$ for all feasible $x$.

Then $a$ is an efficient solution to (VP).

Proof. Suppose contrary to the result that $a$ is not an efficient solution to (VP). Then there exists a feasible solution $x$ to (VP) such that $f(x) \leq f(a)$.

By conditions (iv), (v) and the above inequality, we have

$$b_0(x, a)\phi_0[f(x) - f(a)] \leq 0. \quad (1)$$

By the feasibility of $a$, we have

$$-\lambda^0 g(a) \leq 0.$$  

By conditions (iv), (v) and the above inequality, we have

$$-b_1(x, a)\phi_1[\lambda^0 g(a)] \leq 0. \quad (2)$$

By inequalities (1), (2) and condition (iii), we have

$$(\nabla f(a))\eta(x, a) \leq 0 \quad \text{and} \quad \lambda^0 \nabla g(a)\eta(x, a) \leq 0.$$  

Since $\tau^0 > 0$, the above inequalities give

$$[\tau^0 \nabla f(a) + \lambda^0 \nabla g(a)]\eta(x, a) < 0, \quad (3)$$

which contradicts condition (iii). This completes the proof. \qed

Theorem 3.2 (Sufficiency). Suppose that

(i) $a \in X_0$;
(ii) there exist $\tau^0 \in R^p$, $\tau^0 \geq 0$, $\lambda \in R^m$ and $\lambda^0 \geq 0$ such that
(a) $\tau^0 \nabla f(a) + \lambda^0 \nabla g(a) = 0$,  
(b) $\lambda^0 g(a) = 0$,  
(c) $\tau^0 e = 1$, where $e = (1, \ldots, 1)^T \in R^p$;
(iii) problem (VP) is weak strictly pseudoquasi type I univex at $a \in X_0$ with respect to some $b_0$, $b_1$, $\phi_0$, $\phi_1$ and $\eta$ for all feasible $x$.

Then $a$ is an efficient solution to (VP).

Proof. Suppose contrary to the result that $a$ is not an efficient solution to (VP). Then there exists a feasible solution $x$ to (VP) such that $f(x) \leq f(a)$.
By conditions (iv), (v) and the above inequality, we get (1). By the feasibility of $a$, conditions (iv) and (v), we get (2). By inequalities (1), (2) and condition (iii), we have

$$(\nabla f(a))^T \eta(x,a) < 0 \quad \text{and} \quad \lambda^0 \nabla g(a)^T \eta(x,a) \leq 0.$$ 

Since $\tau^0 \geq 0$, the above inequalities give

$$[\tau^0 \nabla f(a) + \lambda^0 \nabla g(a)] \eta(x,a) < 0,$$

which contradicts condition (iii). This completes the proof.

\section*{4. Mond–Weir type duality}

In this section, we present some weak and strong duality theorems for (VP) and the following Mond–Weir dual problem suggested by Egudo [10]:

\begin{equation}
\begin{aligned}
\text{(MWD)} \quad & \text{maximize} \quad f(y) \\
\text{subject to} \quad & \tau \nabla f(y) + \lambda \nabla g(y) = 0, \\
& \lambda^T g(y) \geq 0, \\
& \lambda \geq 0, \quad \tau \geq 0 \quad \text{and} \quad \tau^T e = 1,
\end{aligned}
\end{equation}

where $e = (1, \ldots, 1)^T \in \mathbb{R}^p$. 

Then $a$ is an efficient solution to (VP).

\textbf{Proof.} Suppose contrary to the result that $a$ is not an efficient solution to (VP). Then there exists a feasible solution $x$ to (VP) such that

$$f(x) \leq f(a).$$

By conditions (iv), (v) and the above inequality, we get (1). By the feasibility of $a$, conditions (iv) and (v), we get (2). By inequalities (1), (2) and condition (iii), we have

$$(\nabla f(a))^T \eta(x,a) < 0 \quad \text{and} \quad \lambda^0 \nabla g(a)^T \eta(x,a) < 0.$$ 

Since $\tau^0 \geq 0$, the above inequalities give

$$[\tau^0 \nabla f(a) + \lambda^0 \nabla g(a)] \eta(x,a) < 0,$$

which contradicts condition (iii). This completes the proof. \hfill \Box


Denote by $Y^0$ the set of all the feasible solutions of problem (MWD); i.e.,

$$Y^0 = \{ (y, \tau, \lambda) : \tau \nabla f(y) + \lambda \nabla g(y) = 0, \lambda g(y) \geq 0, \tau \in R^p, \lambda \in R^m, \lambda \geq 0 \}.$$

**Theorem 4.1** (Weak duality). Suppose that

(i) $x \in X_0$;
(ii) $(y, \tau, \lambda) \in Y^0$ and $\tau > 0$;
(iii) problem (VP) is strong pseudoquasi type I univex at $y$ with respect to some $b_0, b_1, \phi_0, \phi_1$ and $\eta$;

then $f(x) \leq f(y)$.

**Proof.** Suppose contrary to the result, i.e.,

$$f(x) \leq f(y).$$

By conditions (iv), (v) and the above inequality, we have

$$b_0(x, y) \phi_0 \left[ f(x) - f(y) \right] \leq 0.$$  \(4\)

By the feasibility of $(y, \tau, \lambda)$, we have

$$-\lambda g(y) \leq 0.$$  \(5\)

By conditions (iv), (v) and the above inequality, we get

$$-b_1(x, y) \phi_1 \left[ \lambda g(y) \right] \leq 0.$$  \(6\)

By inequalities (4), (5) and condition (iii), we have

$$\left( \nabla f(y) \right) \eta(x, y) \leq 0 \quad \text{and} \quad \lambda \nabla g(y) \eta(x, y) \leq 0.$$

Since $\tau > 0$, the above inequalities give

$$\left[ \tau \nabla f(y) + \lambda \nabla g(y) \right] \eta(x, y) < 0,$$

which contradicts condition (iii). This completes the proof. \(\square\)

**Theorem 4.2** (Weak duality). Suppose that

(i) $x \in X_0$;
(ii) $(y, \tau, \lambda) \in Y^0$ and $\tau > 0$;
(iii) problem (VP) is weak strictly pseudoquasi type I univex at $y$ with respect to some $b_0, b_1, \phi_0, \phi_1$ and $\eta$;

then $f(x) \leq f(y)$.

**Proof.** Suppose contrary to the result, i.e.,

$$f(x) \leq f(y).$$
By conditions (iv), (v) and the above inequality, we get (4). By the feasibility of \((y,\tau,\lambda)\), conditions (iv) and (v), we get (5). By inequalities (4), (5) and condition (iii), we have
\[
(\nabla f(y))\eta(x, y) < 0 \quad \text{and} \quad \lambda \nabla g(y)\eta(x, y) \leq 0.
\]
Since \(\tau^0 \geq 0\), the above inequalities give
\[
[\tau \nabla f(y) + \lambda \nabla g(y)]\eta(x, y) < 0,
\]
which contradicts condition (iii). This completes the proof.

Theorem 4.3 (Weak duality). Suppose that
\begin{enumerate}
  \item \(x \in X_0\);
  \item \((y,\tau,\lambda) \in Y_0^0\);
  \item problem \((\text{VP})\) is weak strictly pseudo type I univex at \(y\) with respect to some \(b_0, b_1, \phi_0, \phi_1\) and \(\eta\);
\end{enumerate}
then \(f(x) \leq f(y)\).

Proof. Suppose contrary to the result, i.e.,
\[
f(x) \leq f(y).
\]
By conditions (iv), (v) and the above inequality, we get (4). By the feasibility of \((y,\tau,\lambda)\), conditions (iv) and (v), we get (5). By inequalities (4), (5) and condition (iii), we have
\[
(\nabla f(y))\eta(x, y) < 0 \quad \text{and} \quad \lambda \nabla g(y)\eta(x, y) < 0.
\]
Since \(\tau \geq 0\), the above inequalities give
\[
[\tau \nabla f(y) + \lambda \nabla g(y)]\eta(x, y) < 0,
\]
which contradicts condition (iii). This completes the proof.

Theorem 4.4 (Strong duality). Let \(\bar{x}\) be an efficient solution for \((\text{VP})\) and \(\bar{x}\) satisfies a constraint qualification for \((\text{VP})\) in Marusciac [24]. Then there exist \(\bar{\tau} \in \mathbb{R}^p\) and \(\bar{\lambda} \in \mathbb{R}^m\) such that \((\bar{x}, \bar{\tau}, \bar{\lambda})\) is feasible for \((\text{MWD})\). If any of the weak duality in Theorems 4.1–4.3 also holds, then \((\bar{x}, \bar{\tau}, \bar{\lambda})\) is efficient solution for \((\text{MWD})\).

Proof. Since \(\bar{x}\) is efficient for \((\text{VP})\) and satisfies the constraint qualification for \((\text{VP})\), then from the Kuhn–Tucker necessary optimality condition, we obtain \(\bar{\tau} > 0\) and \(\bar{\lambda} \geq 0\) such that
\[
\bar{\tau} \nabla f(\bar{x}) + \bar{\lambda} \nabla g(\bar{x}) = 0, \quad \bar{\lambda} g(\bar{x}) = 0.
\]
The vector \(\bar{\tau}\) may be normalized according to \(\bar{\tau} e = 1, \bar{\tau} > 0\), which gives that the triplet \((\bar{x}, \bar{\tau}, \bar{\lambda})\) is feasible for \((\text{MWD})\). The efficiency of \((\bar{x}, \bar{\tau}, \bar{\lambda})\) for \((\text{MWD})\) follows from weak duality theorem. This completes the proof.
5. General Mond–Weir type duality

In this section, we consider a general Mond–Weir type of dual problem to (VP) and establish weak and strong duality theorems under some mild assumption. We consider the following general Mond–Weir type dual problem:

\[ \begin{align*}
\text{(GMWD)} \quad & \max f(y) + \lambda \sum_{j=0}^{J} g_j(y) e_j \\
\text{subject to} \quad & \tau \nabla f(y) + \lambda \nabla g(y) = 0, \\
& \lambda_j g_j \geq 0, \quad 1 \leq j \leq r, \\
& \lambda \geq 0, \quad \tau \geq 0 \quad \text{and} \quad \tau e = 1,
\end{align*} \]

where \( e = (1, \ldots, 1)^T \in \mathbb{R}^p \) and \( J_t, 0 \leq t \leq r \), are partitions of the set \( M \).

**Theorem 5.1** (Weak duality). Suppose that for all feasible \( x \) for (VP) and all feasible \( (y, \tau, \lambda) \) for (GMWD):

(a) \( \tau > 0 \), and \( (f + \lambda \sum_{j=0}^{J} g_j(\cdot)e_j, \lambda_j g_j(\cdot)) \) is strong pseudoquasi type I univex at \( y \) for each \( t, 1 \leq t \leq r \), with respect to \( b_0, b_1, \phi_0, \phi_1 \) and \( \eta \);

(b) \( (f + \lambda \sum_{j=0}^{J} g_j(\cdot)e_j, \lambda_j g_j(\cdot)) \) is weak strictly pseudoquasi type I univex at \( y \) for each \( t, 1 \leq t \leq r \), with respect to \( b_0, b_1, \phi_0, \phi_1 \) and \( \eta \);

(c) \( (f + \lambda \sum_{j=0}^{J} g_j(\cdot)e_j, \lambda_j g_j(\cdot)) \) is weak strictly pseudo type I univex at \( y \) for each \( t, 1 \leq t \leq r \), with respect to \( b_0, b_1, \phi_0, \phi_1 \) and \( \eta \);

then \( f(x) \leq f(y) + \lambda \sum_{j=0}^{J} g_j(y) e_j \).

**Proof.** Suppose contrary to the result. Thus, we have

\[ f(x) \leq f(y) + \lambda \sum_{j=0}^{J} g_j(y) e_j. \]

Since \( x \) is feasible for (VP) and \( \lambda \geq 0 \), the above inequality implies that

\[ f(x) + \lambda \sum_{j=0}^{J} g_j(x) e \leq f(y) + \lambda \sum_{j=0}^{J} g_j(y) e. \] \hspace{1cm} (8)

By the feasibility of \( (y, \tau, \lambda) \) inequality (7) gives

\[ -\lambda \sum_{j=0}^{J} g_j(y) \leq 0, \quad 1 \leq t \leq r. \] \hspace{1cm} (9)

Since \( \phi_0 \) and \( \phi_1 \) are increasing, from (8) and (9), we have

\[ b_0(x, y) \phi_0 \left( (f(x) + \lambda \sum_{j=0}^{J} g_j(x) e) - (f(y) + \lambda \sum_{j=0}^{J} g_j(y) e) \right) \leq 0, \] \hspace{1cm} (10)

\[ -b_1(x, y) \phi_1 \left( \lambda \sum_{j=0}^{J} g_j(y) \right) \leq 0, \quad 1 \leq t \leq r. \] \hspace{1cm} (11)

By condition (a), from (10) and (11), we have

\[ (\nabla f(y) + \lambda \sum_{j=0}^{J} g_j(y) e) \eta(x, y) \leq 0, \]

\[ (\lambda \sum_{j=0}^{J} g_j(y)) \eta(x, y) \leq 0, \quad 1 \leq t \leq r. \]
Since $\tau > 0$, the above inequalities give

$$\left[ \tau \nabla f(y) + \sum_{t=0}^{r} \lambda_J \nabla g_J(y) \right] \eta(x, y) < 0. \tag{12}$$

Since $J_t$, $0 \leq t \leq r$, are partitions of the set $M$, (12) is equivalent to

$$\left[ \tau \nabla f(y) + \lambda \nabla g(y) \right] \eta(x, y) < 0,$$

which contradicts (6).

By condition (b), from (10) and (11), we have

$$\left( \nabla f(y) + \lambda_J \nabla g_J(y) \right) \eta(x, y) < 0,$$

$$\left( \lambda_J \nabla g_J(y) \right) \eta(x, y) \leq 0, \quad 1 \leq t \leq r.$$

Since $\tau \geq 0$, the above inequalities give (12), which again contradicts (6).

By condition (c), from (10) and (11), we have

$$\left( \nabla f(y) + \lambda_J \nabla g_J(y) \right) \eta(x, y) < 0,$$

$$\left( \lambda_J \nabla g_J(y) \right) \eta(x, y) < 0, \quad 1 \leq t \leq r.$$

Since $\tau \geq 0$, the above inequalities give (12), which again contradicts (6). This completes the proof. \qed

Theorem 5.2 (Strong duality). Let $\bar{x}$ be an efficient solution for (VP) and $\bar{x}$ satisfies a constraint qualification for (VP). Then there exist $\bar{\tau} \in \mathbb{R}^p$ and $\bar{\lambda} \in \mathbb{R}^m$ such that $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is feasible for (GMWD). If any of the weak duality in Theorem 5.1 holds, then $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is an efficient solution for (GMWD).

Proof. Since $\bar{x}$ is efficient for (VP) and satisfies a generalized constraint qualification, by the Kuhn–Tucker necessary condition (see Maeda [22]), there exist $\bar{\tau} > 0$ and $\bar{\lambda} \geq 0$ such that

$$\bar{\tau} \nabla f(\bar{x}) + \bar{\lambda} \nabla g(\bar{x}) = 0,$$

$$\bar{\lambda}_i g_i(\bar{x}) = 0, \quad 1 \leq i \leq p.$$

The vector $\bar{\tau}$ may be normalized according to $\bar{\tau} e = 1$, $\bar{\tau} > 0$, which gives that the triplet $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is feasible for (GMWD). The efficiency follows from the weak duality in Theorem 5.1. This completes the proof. \qed

6. Conclusion

In this paper, we have extended the corresponding results of Mishra [25] and Aghezzaf and Hachimi [1] to a wider class of functions. These results can also be extended to the case of nonsmooth functions with same proofs only one has to replace the derivatives with the subdifferentials.
Uncited references

[21]

References


