Continuous Optimization

**Mixed symmetric duality in non-differentiable multiobjective mathematical programming**

S.K. Mishra a, S.Y. Wang b,*, K.K. Lai c, F.M. Yang d

---

**Abstract**

Two mixed symmetric dual models for a class of non-differentiable multiobjective nonlinear programming problems with multiple arguments are introduced in this paper. These two mixed symmetric dual models unify the four existing multiobjective symmetric dual models in the literature. Weak and strong duality theorems are established for these models under some mild assumptions of generalized convexity. Several special cases are also obtained.

© 2006 Elsevier B.V. All rights reserved.

**Keywords:** Symmetric duality; Non-differentiable nonlinear programming; Generalized convexity; Support function

---

**1. Introduction**

Dorn [7] introduced symmetric duality in nonlinear programming by defining a program and its dual to be symmetric if the dual of the dual is the original problem. The symmetric duality for scalar programming has been studied extensively in the literature; one can refer to Dantzig et al. [5], Devi [6], Mishra [11,12], Mond [15], Mond and Weir [17].

Mond and Schechter [16] studied non-differentiable symmetric duality for a class of optimization problems in which the objective functions consist of support functions. Following Mond and Schechter [16], Chen [4], Hou and Yang [8], and Yang et al. [21], studied symmetric duality for such problems.

Weir and Mond [20] presented two models for multiobjective symmetric duality. Several authors, such as the ones of [1,2,4,9,10,13,14,18], studied multiobjective second and higher order symmetric duality, motivated by Weir and Mond [20].

---

* This research is supported by the Grant-in-Aid (25/0132/04/EMR-II) from the Council of Scientific and Industrial Research, New Delhi, the National Natural Science Foundation of China and the Research Grants Council of Hong Kong.

* Corresponding author. Fax: +86 10 62621304.

E-mail address: sywang@amss.ac.cn (S.Y. Wang).

0377-2217/ - see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.ejor.2006.04.041
Very recently, Yang et al. [22] presented a mixed symmetric dual formulation for a non-differentiable non-linear programming problem. Bector et al. [3] introduced a mixed symmetric dual model for a class of non-linear multiobjective programming problems. However, the models given by Bector et al. [3] as well as by Yang et al. [22] do not allow the further weakening of generalized convexity assumptions on a part of the objective functions.

In this paper, we introduce two models of mixed symmetric duality for a class of non-differentiable multiobjective programming problems with multiple arguments. The first model is a multiobjective version of the model given by Yang et al. [22]. However, the second model is new. Mixed symmetric duality for this model has not been given so far by any other author. The advantage of the second model over the first one is that it allows further weakening of convexity on the functions involved. We establish weak and strong duality theorems for these two models and discuss several special cases of these models. The results of Yang et al. [22] as well as that of Bector et al. [3] are particular cases of the results obtained in the present paper.

2. Preliminaries

For \( x, y \in \mathbb{R}^n \), by \( x \preceq y \) we mean \( x_i \leq y_i \) for all \( i \), \( x \preceq y \) means \( x_i \leq y_i \) for all \( i \) and \( x_j < y_j \) for at least one \( j \), \( 1 < j \leq n \). By \( x < y \) we mean \( x_j < y_j \) for all \( i \) and by \( x \not< y \) we mean the negation of \( x \leq y \).

Let \( f(x,y) \) be real valued twice differentiable function defined on \( \mathbb{R}^n \times \mathbb{R}^m \). Let \( \nabla_x f(\bar{x},\bar{y}) \) and \( \nabla_y f(\bar{x},\bar{y}) \) denote the partial derivatives of \( f(x,y) \) with respect to \( x \) and \( y \) at \((\bar{x},\bar{y})\). The symbols \( \nabla_x f(\bar{x},\bar{y}) \), \( \nabla_y f(\bar{x},\bar{y}) \) and \( \nabla_{y,x} f(\bar{x},\bar{y}) \) are defined similarly. Consider the following multiobjective programming problem (VP):

\[
\min(f_1(x), f_2(x), \ldots, f_p(x))
\]

\[\text{s.t.}\quad h(x) \leq 0,\]

where \( f_i: \mathbb{R}^n \rightarrow \mathbb{R}, i = 1,2,\ldots,p \) and \( h: \mathbb{R}^n \rightarrow \mathbb{R}^m \).

Let us denote the feasible region of problem (VP) by \( X_0 = \{ x \in \mathbb{R}^n : h_j(x) = 0, j = 1,2,\ldots,m \} \). For problem (VP), an efficient solution and a properly efficient solution are defined as follows.

**Definition 1.** A feasible solution \( x^0 \) is said to be an efficient solution for (VP) if there exists no other feasible solution \( x \) such that

\[ f(x) \preceq f(x^0). \]

Let \( C \) be a compact convex set in \( \mathbb{R}^n \). The support function of \( C \) is defined by

\[ s(x \mid C) = \max\{ x^T y : y \in C \}. \]

A support function, being convex and everywhere finite, has a subdifferential [19], that is, there exists \( z \in \mathbb{R}^n \) such that

\[ s(y \mid C) \geq s(x \mid C) + z^T(y-x), \quad \forall y \in C. \]

The subdifferential of \( s(x \mid C) \) is given by

\[ \partial s(x \mid C) = \{ z \in \mathbb{R}^n : z^T x = s(x \mid C) \}. \]

For any set \( D \subset \mathbb{R}^n \), the normal cone to \( D \) at a point \( x \in D \) is defined by

\[ N_D(x) = \{ y \in \mathbb{R}^n : y^T(z-x) \leq 0, \forall z \in D \}. \]

It is obvious that for a compact convex set \( C \), \( y \in N_C(x) \) if and only if \( s(y \mid C) = x^T y \), or equivalently, \( x \in \partial s(y \mid C) \).

The following definitions will be needed in the sequel.

**Definition 2.** Let \( X \subset \mathbb{R}^n \). A functional \( F: X \times X \times \mathbb{R}^n \rightarrow \mathbb{R} \) is said to be sublinear with respect to its third argument if for any \( x, y \in X \)

(A) \( F(x, y; a_1 + a_2) \leq F(x, y; a_1) + F(x, y; a_2) \) for any \( a_1, a_2 \in \mathbb{R}^n \);

(B) \( F(x, y; az) \leq \alpha F(x, y; a) \) for any \( z \in \mathbb{R}_+ \) and \( a \in \mathbb{R}^n \).
Definition 3. Let \( X \subseteq \mathbb{R}^n \), \( Y \subseteq \mathbb{R}^m \) and \( F: X \times Y \times \mathbb{R}^n \rightarrow \mathbb{R} \) be sublinear with respect to its third argument. For \( \forall y \in Y \), if
\[
f(x, y) - f(x, y_0) \geq F(x, x; \nabla_x f(x, y)), \quad \forall x \in X.
\]

Definition 4. Let \( X \subseteq \mathbb{R}^n \), \( Y \subseteq \mathbb{R}^m \) and \( F: X \times Y \times \mathbb{R}^n \rightarrow \mathbb{R} \) be sublinear with respect to its third argument. For \( \forall x \in X \), if
\[
f(x, y) - f(x, y_0) \geq F(y, y_0; -\nabla_y f(x, y)), \quad \forall y \in Y.
\]

Definition 5. Let \( X \subseteq \mathbb{R}^n \), \( Y \subseteq \mathbb{R}^m \) and \( F: X \times Y \times \mathbb{R}^n \rightarrow \mathbb{R} \) be sublinear with respect to its third argument. For \( \forall x \in X \), if
\[
F(x, x; \nabla_x f(x, y)) \geq 0 \quad \Rightarrow \quad f(x, y) \geq f(x, y_0), \quad \forall x \in X.
\]

Definition 6. Let \( X \subseteq \mathbb{R}^n \), \( Y \subseteq \mathbb{R}^m \) and \( F: X \times Y \times \mathbb{R}^n \rightarrow \mathbb{R} \) be sublinear with respect to its third argument. For \( \forall y \in Y \), if
\[
F(y, y_0; -\nabla_y f(x, y)) \geq 0 \quad \Rightarrow \quad f(x, y) \geq f(x, y_0), \quad \forall y \in Y.
\]

3. Mixed type multiobjective symmetric duality

For \( N = \{1, 2, \ldots, n\} \) and \( M = \{1, 2, \ldots, m\} \) let \( J_1 \subset N \), \( K_1 \subset M \) and \( J_2 = N \setminus J_1 \) and \( K_2 = M \setminus K_1 \). Let \( |J_1| \) denote the number of elements in the set \( J_1 \). The numbers \( |J_2|, |K_1| \) and \( |K_2| \) are defined similarly. Notice that if \( J_1 = \emptyset \), then \( J_2 = N \), that is, \( |J_1| = 0 \) and \( |J_2| = n \). Hence, \( R^{|J_1|} \) is zero dimensional Euclidean space and \( R^{|J_2|} \) is \( n \)-dimensional Euclidean space. It is clear that any \( x \in \mathbb{R}^n \) can be written as \( x = (x^1, x^2) \), \( x^1 \in R^{|J_1|} \), \( x^2 \in R^{|J_2|} \). Similarly, any \( y \in \mathbb{R}^m \) can be written as \( y = (y^1, y^2) \), \( y^1 \in R^{|K_1|} \), \( y^2 \in R^{|K_2|} \). Let \( f : R^{|J_1|} \times R^{|K_1|} \rightarrow R^l \) and \( g : R^{|J_2|} \times R^{|K_2|} \rightarrow R^l \) be twice differentiable functions and \( e = (1, 1, \ldots, 1)^T \in R^l \).

Now we can introduce the following two pairs of non-differentiable multiobjective programs and discuss their duality theorems under some mild assumptions of generalized convexity.

3.1. First model

3.1.1. Primal problem (MP1)

\[
\min \quad H(x^1, x^2, y^1, y^2, z_1^1, z_2^1, \lambda) = (H_1(x^1, x^2, y^1, y^2, z_1^1, z_2^2, \lambda), \ldots, H_l(x^1, x^2, y^1, y^2, z_1^1, z_2^1, \lambda)) \quad \text{subject to} \quad (x^1, x^2, y^1, y^2, z_1^1, z_2^1, \lambda) \in R^{|J_1|} \times R^{|J_2|} \times R^{|K_1|} \times R^{|K_2|} \times R^{|K_2|} \times R^{|J_2|} \times R_+^l
\]

\[
\sum_{i=1}^l \lambda_i [\nabla_{y^1} f_i(x^i, y^1) - z_1^1] \leq 0, \quad (1)
\]

\[
\sum_{i=1}^l \lambda_i [\nabla_{y^2} g_i(x^i, y^2) - z_2^1] \leq 0, \quad (2)
\]

\[
(y^2)^T \sum_{i=1}^l \lambda_i [\nabla_{y^2} g_i(x^i, y^2) - z_2^1] \geq 0, \quad (3)
\]

\[
(x^1, x^2) \geq 0, \quad (4)
\]

\[
z_1^1 \in D_1^1, \quad z_2^1 \in D_2^1, \quad i = 1, 2, \ldots, l, \quad (5)
\]

\[
\lambda > 0, \quad \sum_{i=1}^l \lambda_i = 1. \quad (6)
\]
3.2.1. Dual problem (MD1)

\[
\begin{align*}
\max & \ G(u^1, u^2, v^1, v^2, w^1, w^2, \lambda) = (G_1(u^1, u^2, v^1, v^2, w^1, \lambda), \ldots, G_l(u^1, u^2, v^1, v^2, w^1, w^2, \lambda)) \\
& \text{subject to} (u^1, u^2, v^1, v^2, w^1, w^2, \lambda) \in R^{2l} \times R^{2l} \times R^{l} \times R^{l} \times R^{l} \times R^{l} \times R^l
\end{align*}
\]

where

\[
\begin{align*}
\sum_{i=1}^{l} \lambda_i [\nabla_i f_i(u^1, v^1) + w^2_i] & \geq 0, \\
\sum_{i=1}^{l} \lambda_i [\nabla_i g_i(u^2, v^2) + w^2_i] & \geq 0, \\
(u^2)^T \sum_{i=1}^{l} \lambda_i [\nabla_i g_i(u^2, v^2) + w^2_i] & \leq 0, \\
(v^1, v^2) & \geq 0, \\
w^1_i & \in C_i, \quad \text{and} \quad w^2_i \in C_i^2, \quad i = 1, 2, \ldots, l, \\
\lambda & > 0, \quad \sum_{i=1}^{l} \lambda_i = 1,
\end{align*}
\]

and \( C_i \) is a compact and convex subset of \( R^{d_i} \) for \( i = 1, 2, \ldots, l \) and \( C_i^2 \) is a compact and convex subset of \( R^{d_i^2} \) for \( i = 1, 2, \ldots, l \). Similarly, \( D_i^1 \) is a compact and convex subset of \( R^{d_i^1} \) for \( i = 1, 2, \ldots, l \) and \( D_i^2 \) is a compact and convex subset of \( R^{d_i^2} \) for \( i = 1, 2, \ldots, l \).

3.2. Second model

3.2.1. Primal problem (MP2)

\[
\begin{align*}
\min & \ H^*(x^1, x^2, y^1, y^2, z^1, z^2, \lambda) = (H_1^*(x^1, x^2, y^1, y^2, z^1, z^2, \lambda), \ldots, H_l^*(x^1, x^2, y^1, y^2, z^1, z^2, \lambda)) \\
& \text{subject to} (x^1, x^2, y^1, y^2, z^1, z^2, \lambda) \in R^{2l} \times R^{2l} \times R^{l} \times R^{l} \times R^{l} \times R^l \times R^l
\end{align*}
\]

where

\[
\begin{align*}
\sum_{i=1}^{l} \lambda_i [\nabla_i f_i(x^1, y^1) - z^1_i] & \leq 0, \\
\sum_{i=1}^{l} \lambda_i [\nabla_i g_i(x^2, y^2) - z^2_i] & \leq 0, \\
(y^1)^T \sum_{i=1}^{l} \lambda_i [\nabla_i f_i(x^1, y^1) - z^1_i] & \geq 0, \\
(y^2)^T \sum_{i=1}^{l} \lambda_i [\nabla_i g_i(x^2, y^2) - z^2_i] & \geq 0, \\
(x^1, x^2) & \geq 0, \\
z^1_i & \in D_i^1, \quad \text{and} \quad z^2_i \in D_i^2, \quad i = 1, 2, \ldots, l, \\
\lambda & > 0, \quad \sum_{i=1}^{l} \lambda_i = 1.
\end{align*}
\]

3.2.2. Dual problem (MD2)

\[
\begin{align*}
\max & \ G^*(u^1, u^2, v^1, v^2, w^1, w^2, \lambda) = (G_1(u^1, u^2, v^1, v^2, w^1, \lambda), \ldots, G_l(u^1, u^2, v^1, v^2, w^1, w^2, \lambda)) \\
& \text{subject to} (u^1, u^2, v^1, v^2, w^1, w^2, \lambda) \in R^{2l} \times R^{2l} \times R^{l} \times R^{l} \times R^{l} \times R^l \times R^l
\end{align*}
\]
\[
\sum_{i=1}^{l} \lambda_i [\nabla_x f_i(u^i, v^1) + w_1^i] \geq 0, \tag{20}
\]
\[
\sum_{i=1}^{l} \lambda_i [\nabla_x g_i(u^2, v^2) + w_2^i] \geq 0, \tag{21}
\]
\[
(u^1)^T \sum_{i=1}^{l} \lambda_i [\nabla_x f_i(u^i, v^1) + w_1^i] \leq 0, \tag{22}
\]
\[
(u^2)^T \sum_{i=1}^{l} \lambda_i [\nabla_x g_i(u^2, v^2) + w_2^i] \leq 0, \tag{23}
\]
\[
(v^1, v^2) \geq 0, \tag{24}
\]
\[
w_1^i \in C_1^i, \quad \text{and} \quad w_2^i \in C_2^i, \quad i = 1, 2, \ldots, l, \tag{25}
\]
\[
\lambda > 0, \quad \sum_{i=1}^{l} \lambda_i = 1, \tag{26}
\]

where
\[
H^I_1(x^1, x^2, y^1, y^2, z, \lambda) = f_i(x^1, y^1) + g_i(x^2, y^2) + s(x^1 | C_1^i) + s(x^2 | C_2^i) - (v^1)^T z_1^i - (v^2)^T z_2^i,
\]
\[
G^I_1(u^1, u^2, v^1, v^2, w, \lambda) = f_i(u^1, v^1) + g_i(u^2, v^2) - s(v^1 | D_1^i) - s(v^2 | D_2^i) - (u^1)^T w_1^i + (u^2)^T w_2^i
\]
and \(C_1^i\) is a compact and convex subset of \(R^{r_1}\) for \(i = 1, 2, \ldots, l\) and \(C_2^i\) is a compact and convex subsets of \(R^{r_2}\) for \(i = 1, 2, \ldots, l\), similarly, \(D_1^i\) is a compact and convex subset of \(R^{r_1}\) for \(i = 1, 2, \ldots, l\) and \(D_2^i\) is a compact and convex subset of \(R^{r_2}\) for \(i = 1, 2, \ldots, l\).

For the first model, we give the following weak duality theorem.

**Theorem 1** (Weak duality). Let \((x^1, x^2, y^1, y^2, z^1, z^2, \lambda)\) be feasible for (MP1) and \((u^1, u^2, v^1, v^2, w^1, w^2, \lambda)\) be feasible for (MD1). Suppose that for \(i = 1, 2, \ldots, l, f_i(\cdot, y^1)\) is \(F_1\)-convex for fixed \(y^1, f_i(x^1, \cdot)\) is \(F_2\)-concave for fixed \(x^1, g_i(\cdot, y^2) + T w^2_i\) is \(G_1\)-convex for fixed \(y^2\) and \(g_i(x^2, \cdot) - T z_2^i\) is \(G_2\)-concave for fixed \(x^2\), and the following conditions are satisfied:

(i) \(F_1(x^1, u^i; \nabla_x f_i(u^i, v^1)) + (u^1)^T \nabla_x f_i(u^i, v^1) + (x^1)^T w_1^i \geq 0;\)

(ii) \(G_1(x^2, u^i; \nabla_x g_i(u^2, v^2) + w_2^i) + (u^2)^T (\nabla_x g_i(u^2, v^2) + w_2^i) \geq 0;\)

(iii) \(F_2(y^1, v^i; \nabla_y f_i(x^1, y^1)) + (y^1)^T \nabla_y f_i(x^1, y^1) - (v^1)^T z_1^i \leq 0;\)

(iv) \(G_2(y^2, v^i; \nabla_y g_i(x^2, y^2) - z_2^i) + (y^2)^T (\nabla_y g_i(x^2, y^2) - z_2^i) \leq 0.\)

Then \(H(x^1, x^2, y^1, y^2, z, \lambda) \notin G_i(u^1, u^2, v^1, v^2, w, \lambda).\)

**Proof.** Suppose \((x^1, x^2, y^1, y^2, z^1, z^2, \lambda)\) is feasible for (MP1) and \((u^1, u^2, v^1, v^2, w^1, w^2, \lambda)\) is feasible for (MD1). By the \(F_1\)-convexity of \(f_i(\cdot, y^1)\) and \(F_2\)-concavity of \(f_i(x^1, \cdot)\), for \(i = 1, 2, \ldots, l\), we have
\[
f_i(x^1, v^1) - f_i(u^i, v^1) \geq F_1(x^1, u^i; \nabla_x f_i(u^i, v^1)) \quad \text{for } i = 1, 2, \ldots, l,
\]
and
\[
f_i(x^1, v^1) - f_i(x^1, y^1) \leq F_2(v^1, y^1; \nabla_y f_i(x^1, y^1)) \quad \text{for } i = 1, 2, \ldots, l.
\]
Rearranging the above two inequalities, and by using the conditions (i) and (iii), we obtain
\[
f_i(x^1, y^1) - f_i(u^i, v^1) \geq -(u^1)^T \nabla_x f_i(u^i, v^1) - (x^1)^T w_1^i + (y^1)^T \nabla_y f_i(x^1, y^1) - (v^1)^T z_1^i \quad \text{for } i = 1, 2, \ldots, l.
\]
Using \((v^1)^T z_1^i \leq s(v^1 | D_1^i)\) and \((x^1)^T w_1^i \leq s(x^1 | C_1^i)\), for \(i = 1, 2, \ldots, l\), we have
\[
f_i(x^1, y^1) + s(x^1 | C_1^i) - (y^1)^T \nabla_y f_i(x^1, y^1) \geq f_i(u^i, v^1) - s(v^1 | D_1^i) - (u^1)^T \nabla_x f_i(u^i, v^1) \quad \text{for } i = 1, 2, \ldots, l.
\]
Because of (6) and (12), the above inequalities yield
\[
\sum_{i=1}^{I} \lambda_i [f_i(x^i, y^i) + s(x^i | C_i^1) - (y^i)^T \nabla_{\nu_i} f_i(x^i, y^i)] \geq \sum_{i=1}^{I} \lambda_i [f_i(u^i, v^i) - s(v^i | D_i^1) - (u^i)^T \nabla_{\nu_i} f_i(u^i, v^i)].
\] (27)
By $G_1$-convexity of $g_i(\cdot, v^2) + T w_i^2$, and condition (ii), we get
\[
[g_i(x^i, v^2) + (x^i)^T w_i^2] - [g_i(u^i, v^2) + (u^i)^T w_i^2] \geq G_1(x^i, u^i; \nabla_{\nu_i} g_i(u^i, v^2) + w_i^2) \geq -(u^i)^T [\nabla_{\nu_i} g_i(u^i, v^2) + w_i^2].
\] (28)
Using (6), (12) and (28), we get
\[
\sum_{i=1}^{I} \lambda_i [g_i(x^i, v^2) + (x^i)^T w_i^2] - \sum_{i=1}^{I} \lambda_i [g_i(u^i, v^2) + (u^i)^T w_i^2] \geq -(u^i)^T \sum_{i=1}^{I} \lambda_i [\nabla_{\nu_i} g_i(u^i, v^2) + w_i^2].
\] (29)
From (9) and (29), we get
\[
\sum_{i=1}^{I} \lambda_i [g_i(x^i, v^2) + (x^i)^T w_i^2] - \sum_{i=1}^{I} \lambda_i [g_i(u^i, v^2) + (u^i)^T w_i^2] \geq 0.
\] (30)
Similarly, by $G_2$-concavity of $g_i(x^2, \cdot) - T z_i^2$ and condition (iv), we get
\[
[g_i(x^2, v^2) - (v^2)^T z_i^2] - [g_i(x^2, y^2) - (y^2)^T z_i^2] \leq G_2(y^2, v^2; \nabla_{\nu_i} g_i(x^2, y^2) - z_i^2) \leq -(y^2)^T [\nabla_{\nu_i} g_i(x^2, y^2) - z_i^2].
\]
Using (6), (12) and (3) the above inequality yields
\[
\sum_{i=1}^{I} \lambda_i [g_i(x^i, v^2) - (v^2)^T z_i^2] - \sum_{i=1}^{I} \lambda_i [g_i(x^i, y^2) - (y^2)^T z_i^2] \leq 0.
\] (31)
Rearranging (30) and (31), we get
\[
\sum_{i=1}^{I} \lambda_i [g_i(x^i, y^2) - g_i(u^i, v^2) + (x^i)^T w_i^2 - (u^i)^T w_i^2 - (y^2)^T z_i^2 + (v^2)^T z_i^2] \geq 0.
\]
Using $(x^i)^T w_i^2 \leq s(x^i | C_i^2)$ and $(v^2)^T z_i^2 \leq s(v^2 | D_i^1)$ for $i = 1, 2, \ldots, I$, we have
\[
\sum_{i=1}^{I} \lambda_i [(x^i)^T w_i^2 - g_i(u^i, v^2) + s(x^i | C_i^2) - (y^2)^T z_i^2 - g_i(u^i, v^2) + s(v^2 | D_i^1) - (u^i)^T w_i^2] \geq 0.
\] (32)
Finally, from (6), (12), (27) and (32), we have
\[
H(x^1, x^2, y^1, y^2, z_1, z_2, \lambda) \nsubseteq G(u^1, u^2, v^1, v^2, w^1, w^2, \lambda).
\]
\[\square\]

Corollary 1. Let $(x^1, x^2, y^1, y^2, z_1, z_2, \lambda)$ be feasible for (MP1) and let $(u^1, u^2, v^1, v^2, w^1, w^2, \lambda)$ be feasible for (MD1) with the corresponding objective function values being equal. If the convexity and concavity assumptions and conditions (i)-(iv) of Theorem 1 are satisfied, then $(x^1, x^2, y^1, y^2, z_1, z_2, \lambda)$ and $(u^1, u^2, v^1, v^2, w^1, w^2, \lambda)$ are an efficient solution for (MP1) and (MD1), respectively.

Theorem 2 (Weak duality). Let $(x^1, x^2, y^1, y^2, z_1, z_2, \lambda)$ be feasible for (MP2) and $(u^1, u^2, v^1, v^2, w^1, w^2, \lambda)$ be feasible for (MD2). Suppose that for $i = 1, 2, \ldots, I$, $f_i(\cdot, y^1) + T w_i^1$ is $F_1$-convex for fixed $y^1$, $f_i(x^i, \cdot) - T z_i^1$ is $F_2$-concave for fixed $x^i$, $g_i(\cdot, y^2) + T w_i^2$ is $G_1$-convex for fixed $y^2$ and $g_i(y^2, \cdot) - T z_i^2$ is $G_2$-concave for fixed $x^2$, and the following conditions are satisfied:

(i) $F_i(x^1, u^1; a) + (u^1)^T a \geq 0$ if $a \geq 0$;
(ii) $G_i(x^2, u^2; b) + (u^2)^T b \geq 0$ if $b \geq 0$;
(iii) $F_2(y^1, v^1; c) + (y^1)^T c \leq 0$ if $c \leq 0$; and
(iv) $G_2(y^2, v^2; d) + (y^2)^T d \leq 0$ if $d \leq 0$.

Then $H_i'(x^1, x^2, y^1, y^2, z, \lambda) \nsubseteq G_i'(u^1, u^2, v^1, v^2, w, \lambda)$. 

Proof. Suppose \((x^i, x^2, y^i, z^i, \lambda)\) be feasible for (MP2) and \((u^i, u^2, v^i, v^2, w^i, \lambda)\) be feasible for (MD2). Then using the \(F_1\)-convexity of \(f_i(\cdot, y^i) + \lambda^T w^i\) and \(F_2\)-concavity of \(g_i(x^i, \cdot) - \lambda^T z^i\), for \(i = 1, 2, \ldots, l\), we have

\[
f_i(x^i, v^i) + (x^i)^T w^i - f_i(u^i, v^i) - (u^i)^T w^i \geqslant F_1(x^i, u^i; \nabla_{x^i} f_i(u^i, v^i) + w^i),
\]

and

\[
f_i(x^i, v^i) - (v^i)^T z^i - f_i(x^i, y^i) + (y^i)^T z^i \leqslant F_2(v^i, y^i; \nabla_{y^i} f_i(x^i, y^i) - z^i).
\]

From (19), (26), the sublinearity of \(F_1\) and \(F_2\) and the above inequalities, we get

\[
\sum_{i=1}^l \lambda_i [f_i(x^i, v^i) + (x^i)^T w^i - f_i(u^i, v^i) - (u^i)^T w^i] \geqslant F_1\left(x^i, u^i; \sum_{i=1}^l \lambda_i [\nabla_{x^i} f_i(u^i, v^i) + w^i]\right),
\]

and

\[
\sum_{i=1}^l \lambda_i [f_i(x^i, v^i) - (v^i)^T z^i - f_i(x^i, y^i) + (y^i)^T z^i] \leqslant F_2\left(v^i, y^i; \sum_{i=1}^l \lambda_i [\nabla_{y^i} f_i(x^i, y^i) - z^i]\right).
\]

From constraints (13), (20), conditions (i), (iii) and the above inequalities, we get

\[
\sum_{i=1}^l \lambda_i [f_i(x^i, v^i) + (x^i)^T w^i - f_i(u^i, v^i) - (u^i)^T w^i] \geqslant 0,
\]

\[
\sum_{i=1}^l \lambda_i [f_i(x^i, v^i) - (v^i)^T z^i - f_i(x^i, y^i) + (y^i)^T z^i] \leqslant 0.
\]

Rearranging the above two inequalities, we obtain

\[
\sum_{i=1}^l \lambda_i [f_i(x^i, v^i) - f_i(u^i, v^i) + (x^i)^T w^i] - (u^i)^T w^i + (v^i)^T z^i] \geqslant 0.
\]

Using \((v^i)^T z^i \leqslant s(v^i | D^i)\) and \((x^i)^T w^i \leqslant s(x^i | C^i)\) for \(i = 1, 2, \ldots, l\), we have

\[
\sum_{i=1}^l \lambda_i [f_i(x^i, v^i) - f_i(u^i, v^i) + s(x^i | C^i) - (u^i)^T w^i] + s(v^i | D^i) - (v^i)^T z^i] \geqslant 0.
\]

Following as in the proof of Theorem 1, we get

\[
\sum_{i=1}^l \lambda_i [g_i(x^2, z^i) + s(x^i | C^i) - (y^2)^T z^i - g_i(u^2, v^2) + s(u^2 | D^i) - (u^2)^T w^i] \geqslant 0.
\]

Finally, from (19), (26), (33) and (34), we have

\[
H^*(x^i, x^2, y^i, z^2, \lambda) \notin G^*(u^i, u^2, v^i, v^2, w^i, \lambda).
\]

Corollary 2. Let \((x^1, x^2, y^1, z^1, \lambda)\) be feasible for (MP2) and \((u^1, u^2, v^1, v^2, w^1, \lambda)\) be feasible for (MD2) with the corresponding objective function values being equal. Let the convexity and concavity assumptions and conditions (i)–(iv) in Theorem 2 be satisfied, then \((x^1, x^2, y^1, z^2, \lambda)\) and \((u^1, u^2, v^1, v^2, w^1, \lambda)\) are an efficient solution for (MP2) and (MD2), respectively.

Remark 1. Theorems 1 and 2 can be established for more general classes of functions such as \(F_1\)-pseudo-concavity and \(F_2\)-pseudo-concavity, and \(G_1\)-pseudo-convexity and \(G_2\)-pseudo-convexity on the functions involved in the corresponding theorems. The proofs will follow the same lines as that of Theorems 1 and 2.

Strong duality theorems for the given models can be established on the lines of the proof of Theorem 2 of Yang et al. [22] in light of the discussions given above in this paper.
Theorem 3 (Strong duality). Let \((\bar{x}^1, \bar{x}^2, y^1, y^2, z_1, z_2, \lambda)\) be an efficient solution for (MP1). Let \(\bar{\lambda} = \lambda\) be fixed in (MD1) and the Hessian matrices \(\nabla^2_{y^1} \lambda^T f(x^1, y^1)\) and \(\nabla^2_{y^2} \lambda^T g(x^2, y^2)\) be either positive definite or negative definite and \(\nabla^2_{y^1} \lambda^T g(x^1, y^1) \neq \lambda^T \nabla_{y^2} z^2\). Also let the set \(\{\nabla_y g_1 - z_1, \ldots, \nabla_y g_l - z_l\}\) is linearly independent. If the generalized convexity hypotheses and conditions (i)–(iv) of Theorem 1 are satisfied, then \((\bar{x}^1, \bar{x}^2, y^1, y^2, z_1, z_2, \bar{\lambda})\) is an efficient solution for (MD1).

Theorem 4 (Strong duality). Let \((\bar{x}^1, \bar{x}^2, y^1, y^2, z_1, z_2, \lambda)\) be an efficient solution for (MP2). Let \(\bar{\lambda} = \lambda\) be fixed in (MD2). Suppose that the Hessian matrix \(\nabla^2_{y^1} \lambda^T f(x^1, y^1)\) is positive definite for \(i = 1, 2, \ldots, l\) and \(\sum_{i=1}^l \lambda_i [\nabla_y g_i - z^i] \geq 0\); and \(\nabla^2_{y^2} \lambda^T g(x^2, y^2)\) is positive definite for \(i = 1, 2, \ldots, l\) and \(\sum_{i=1}^l \lambda_i [\nabla_y g_i - z^i] \geq 0\); or \(\nabla^2_{y^1} \lambda^T f(x^1, y^1)\) is negative definite for \(i = 1, 2, \ldots, l\) and \(\sum_{i=1}^l \lambda_i [\nabla_y g_i - z^i] \leq 0\); and \(\nabla^2_{y^2} \lambda^T g(x^2, y^2)\) is negative definite for \(i = 1, 2, \ldots, l\) and \(\sum_{i=1}^l \lambda_i [\nabla_y g_i - z^i] \leq 0\). Also suppose that the sets \(\{\nabla_y g_1 - z_1, \ldots, \nabla_y g_l - z_l\}\) and \(\{\nabla_y g_1 - z_1, \ldots, \nabla_y g_l - z_l\}\) are linearly independent. If the generalized convexity hypotheses and conditions (i)–(iv) of Theorem 2 are satisfied, then \((\bar{x}^1, \bar{x}^2, y^1, y^2, z_1, z_2, \bar{\lambda})\) is an efficient solution for (MD2).

4. Special cases

In this section, we consider some special cases of problems (MP1), (MD1), (MP2) and (MD2) by choosing particular forms of compact convex sets, and the number of objective and constraint functions:

(i) If \(C^1 = C^2 = D^1 = D^2 = \{0\}, i = 1, 2, \ldots, l\), then (MP1) and (MD1) reduce to the pair of problems studied in Bector et al. [3].

(ii) If \(l = 1\), then (MP1) and (MD1) reduce to the pair of problems studied in Yang et al. [22], and (MP2) and (MD2) become an extension of problems studied in Yang et al. [22].

(iii) If \(|J_2| = 0, |K_2| = 0\) and \(l = 1\), then (MP1) and (MD1) reduce to the pair of problems (P) and (D) of Mond and Schechter [16], and (MP2) and (MD2) reduce to the pair of problems (P1) and (D1) of Mond and Schechter [16].

(iv) If \(|J_2| = 0, |K_2| = 0\), then (MP1) and (MD1) become multiobjective extension of the pair of problems (P) and (D) of Mond and Schechter [16], and (MP2) and (MD2) becomes the multiobjective extension of the pair of problems (P1) and (D1) of Mond and Schechter [16].

(v) If \(l = 1\), then (MP2) and (MD2) are an extension of the pair of problems studied in Yang et al. [22].

(vi) These results can be extended to second case thus extend the results of [23,24].

(vii) These results can be extended to higher order case as well as to other generalized convexity assumptions.

References


