The influence of Hall current and heat transfer on the flow of a fourth grade fluid

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Abstract: In this paper a mathematical model is analyzed to investigate the effects of Hall current and heat transfer on the flow of a fourth grade fluid between two heated and rotating disks. The corresponding problem of velocity and temperature are solved analytically by homotopy analysis method (HAM). Comparison is given between the results of velocity in fourth grade, third grade, second grade and viscous fluids. The variation of pertinent parameters are graphed and discussed.

Key words: Flow, heat transfer, fourth grade fluid.

1 Introduction

The study of non-Newtonian fluids in technological applications has become of increasing importance in the recent years. Unlike the Newtonian fluids there is not yet a single constitutive equation available which can describe features of all the non-Newtonian fluids. In view of the variety of non-Newtonian fluids many fluid models have been suggested to characterize the rheological characteristics. Navier Stokes equations are inadequate to describe the response of such fluids. The resulting equations of non-Newtonian fluids in general are of higher order and more non-linear than the Navier Stokes equations \cite{1,2} and present various challenges to the workers in the field. The non-Newtonian fluids are usually classified under the categories of differential type, rate type and integral type. A simplest subclass of differential type fluids is known as second grade. Although the second grade fluids exhibit normal stress effects but these cannot possess the shear thinning and thickening effects. For this reason the fluid models of third grade or fourth grade are used. These fluids are able to predict the shear thinning and shear thickening effects. The literature regarding fourth grade fluid is scarce. Some useful investigations in this direction are mentioned through the studies \cite{3-9}. Moreover, the heat transfer analysis has key importance in engineering applications especially during the handling and processing of non-Newtonian fluids.

The main purpose of the present investigation is to put forward the analysis of fourth grade fluid. For that we consider the steady flow between two eccentric rotating disks of different temperatures. The flow is induced because of a sudden pull of eccentric disks. The fluid is electrically conducting in the presence of a uniform magnetic field applied transversely to the disks. The induced magnetic field is negligible and the Hall effects are taken into account. Analytic solution of the velocity and temperature are constructed by a homotopy analysis method \cite{10-15}. The obtained results reveal that HAM is very effective and simple. Some most recent studies involving HAM may be mentioned through the investigations \cite{16-30}. Interesting observations of the present analysis are summarized in the concluding remarks.

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2 Definition of the problem

Let us consider the steady flow of an electrically conducting fourth grade fluid between two infinite disks rotating non-coaxially with constant angular velocity $\Omega$. A constant magnetic field of strength $B_0$ is applied parallel to the $z-axis$ (chosen normal to the disks). The electric and induced magnetic fields are assumed negligible. However the Hall effect is taken into account. Furthermore, the disks at $z = h$ and $z = -h$ are pulled with velocities $U$ and $-U$ respectively. Consider the electrically conducting fourth grade fluid between two infinite disks rotating with angular velocity $\Omega$ about two axes non-coaxially. The distance between the axis is $2l$. A constant magnetic field is applied in the direction of $z-axis$. The disks at $z = h$ and $z = -h$ are pulled with constant velocities $U$ and $-U$ respectively. The corresponding boundary conditions are

$$
egin{align*}
  u &= -\Omega(y-l) + U_1, & v &= \Omega x + U_2, & w &= 0 & \text{at } z = h, \\
  u &= -\Omega(y+l) - U_1, & v &= \Omega x - U_2, & w &= 0 & \text{at } z = -h,
\end{align*}
$$

in which $V = (u, v, w)$ is the velocity field. The above boundary conditions suggest the following form of the velocity components

$$
egin{align*}
u = -\Omega y + f(z), & \quad v = \Omega x + g(z), & \quad w = 0.
\end{align*}
$$

and the continuity equation is identically satisfied.

The equations which govern the steady flow in absence of electric displacement are

$$
\rho(V, \nabla) V = \text{div} T + J \times B, \tag{3}
$$

$$
\nabla \cdot B = 0, \quad \nabla \times B = \mu_m J, \quad \nabla \times E = 0, \tag{4}
$$

$$
J + \frac{\omega_e \tau_e}{B_0} (J \times B_0) = \sigma \left[ E + V \times B + \frac{1}{en_e} \nabla p_e \right], \tag{5}
$$

where $\rho$ is the fluid density, $B$ is the total magnetic field, $B_0$ is an applied magnetic field, $E(= 0)$ is the total electric field, $J$ is the current density, $\mu_m$ is the magnetic permeability, $\sigma$ is the electrical conductivity, $e$ is the electron charge, $\omega_e$ is the cyclotron frequency of electrons, $\tau_e$ is the electron collision time, $n_e$ is the number density of the electron and $p_e$ is the electron pressure. Here ion-slip and thermoelectric effects are not taken into account. Also $\omega_e \tau_e \approx O(1)$ and $\omega_i \tau_i << 1$ ($\omega_i$ and $\tau_i$ are the cyclotron frequency and collision time for ions respectively).

The constitutive equation for the Cauchy stress tensor ($T$) in a fourth grade fluid is [3]

$$
T = -pI + S_1 + S_2 + S_3 + S_4, \tag{6}
$$

$$
S_1 = \mu \mathbf{A}_1, \quad S_2 = \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2, \\
S_3 = \beta_1 \mathbf{A}_3 + \beta_2 \left( \mathbf{A}_2 \mathbf{A}_1 + \mathbf{A}_1 \mathbf{A}_2 \right) + \beta_3 \left( tr \mathbf{A}_1^2 \right) \mathbf{A}_1, \\
S_4 = \gamma_1 \mathbf{A}_4 + \gamma_2 \left( \mathbf{A}_3 \mathbf{A}_1 + \mathbf{A}_1 \mathbf{A}_3 \right) + \gamma_3 \mathbf{A}_2^2 + \gamma_4 \left( \mathbf{A}_2 \mathbf{A}_2^2 + \mathbf{A}_1^2 \mathbf{A}_2 \right) + \gamma_5 \left( tr \mathbf{A}_2 \right) \mathbf{A}_2 + \gamma_6 \left( tr \mathbf{A}_2 \right) \mathbf{A}_1^2 + \left( \gamma_7 \left( tr \mathbf{A}_3 \right) + \gamma_8 \left( tr \mathbf{A}_2 \right) \right) \mathbf{A}_1, \\
\mathbf{A}_1 = L + L^T, \quad L = \text{grad} V, \\
\mathbf{A}_n = \frac{d\mathbf{A}_{n-1}}{dt} + \mathbf{A}_{n-1} L + L^T \mathbf{A}_{n-1}, \quad n = 2, 3, \ldots
$$
From Eqs. (2) and (6) we can write

\[
T_{xx} = -p + 2\beta_2 \Omega \left( \frac{\partial f}{\partial z} \right) + \gamma_2 \Omega^2 \left( \frac{\partial g}{\partial z} \right)
+ \alpha_2 - 2\Omega^2 \gamma_2 + 2\gamma_6 \left( \left( \frac{\partial f}{\partial z} \right) + \left( \frac{\partial g}{\partial z} \right) \right) \right) \left( \frac{\partial f}{\partial z} \right)^2,
\]

\[
T_{yy} = -p + \alpha_2 \left( \frac{\partial g}{\partial z} \right)^2 - 2\beta_2 \Omega \left( \frac{\partial f}{\partial z} \right) + \gamma_3 \Omega^2 - 2\Omega^2 \gamma_2 + 2\gamma_6 \left( \left( \frac{\partial f}{\partial z} \right) + \left( \frac{\partial g}{\partial z} \right) \right) \right) \left( \frac{\partial f}{\partial z} \right)^2,
\]

\[
T_{zz} = -p + (\alpha_2 + 2\alpha_1) \left( \left( \frac{\partial g}{\partial z} \right) + \left( \frac{\partial f}{\partial z} \right) \right)
+ \left[ \gamma_3 \Omega^2 - 2\Omega^2 \gamma_2 + 2\gamma_6 \left( \left( \frac{\partial f}{\partial z} \right) + \left( \frac{\partial g}{\partial z} \right) \right) \right] \left( \frac{\partial f}{\partial z} \right)^2 + \left( \frac{\partial g}{\partial z} \right)^2,
\]

\[
T_{xy} = T_{yx} = -\beta_2 \Omega \left( \left( \frac{\partial f}{\partial z} \right) - \left( \frac{\partial g}{\partial z} \right) \right)
+ \left[ \alpha_2 - \gamma_3 \Omega^2 - 2\Omega^2 \gamma_2 + 2\gamma_6 \left( \left( \frac{\partial f}{\partial z} \right) + \left( \frac{\partial g}{\partial z} \right) \right) \right] \left( \frac{\partial f}{\partial z} \right) + \left( \frac{\partial g}{\partial z} \right),
\]

\[
T_{yz} = T_{zy} = \mu + 2(\beta_2 + \beta_3) \left( \left( \frac{\partial f}{\partial z} \right) + \left( \frac{\partial g}{\partial z} \right) \right) - \beta_1 \Omega^2 \left( \frac{\partial f}{\partial z} \right),
\]

\[
T_{zx} = T_{xz} = \mu + 2(\beta_2 + \beta_3) \left( \left( \frac{\partial f}{\partial z} \right) + \left( \frac{\partial g}{\partial z} \right) \right) - \beta_1 \Omega^2 \left( \frac{\partial f}{\partial z} \right),
\]

Invoking Eqs. (1)-(5) one arrives at

\[
\frac{\partial p}{\partial x} = \rho \Omega [\Omega x + g(z)] + \frac{\partial T_{xx}}{\partial z} + \frac{\sigma B_0^2(1 + i\phi)}{1 + \phi^2} \left( \frac{Q}{2h} - f(z) \right),
\]

\[
\frac{\partial p}{\partial y} = -\rho \Omega [-\Omega y + g(z)] + \frac{\partial T_{yz}}{\partial z} + \frac{\sigma B_0^2(1 + i\phi)}{1 + \phi^2} \left( \frac{P}{2h} - g(z) \right),
\]

where \( z \)-components of Eq.(2) indicates that \( p \neq p(z) \), \( \phi = \omega_e \tau_e \) is the Hall parameter and

\[
Q = \int_{-h}^{h} f(z)dz, \quad P = \int_{-h}^{h} g(z)dz,
\]

\[
f(z) = \Omega l + U_1, \quad g(z) = U_2, \quad \text{at} \quad z = h, \quad (11)
\]

\[
f(z) = -\Omega l - U_1, \quad g(z) = -U_2, \quad \text{at} \quad z = -h. \quad (12)
\]
Equations (9) and (10) yield
\[\Omega \rho g(z) + \frac{\partial T_x}{\partial z} - H f(z) = C_1, \quad (13)\]
\[-\Omega \rho f(z) + \frac{\partial T_y}{\partial z} - H g(z) = C_2, \quad (15)\]

where
\[H = \frac{\sigma B_0^2 (1 + i\phi)}{1 + \phi^2}.\]

and performing integration of Eqs. (9) and (10) we get
\[p = p_0 + \frac{1}{2}(x^2 + y^2) + \left[C_1 + \frac{H Q}{2h}\right] x + \left[C_2 + \frac{HP}{2h}\right] y, \quad (16)\]
in which \(p_0\) is the reference pressure, \(C_i (i = 1, 2)\) are the arbitrary constants and in absence of the pressure gradient
\[C_1 = -\frac{HQ}{2h}, \quad C_2 = -\frac{HP}{2h}, \quad (17)\]
\[F = f + ig, \quad \bar{F} = f - ig. \quad (18)\]

Substitution of Eqs. (7), (8) and (17) into Eqs. (13) and (14) and then combining the resulting equations and using the boundary conditions (11) and (12) we have
\[\left[\mu - \beta_1 \Omega^2 + i \left(\Omega^2 \gamma_1 - \alpha_1 \Omega\right)\right] \frac{d^2F}{dz^2} \]
\[+ \left[4 (\beta_2 + \beta_3) - 2i \Omega (2 \gamma_3 + 2 \gamma_5 + \gamma_4)\right] \frac{d^2F}{dz^2} \frac{dF}{dz} \frac{d\bar{F}}{dz} \]
\[+ \left[2 (\beta_2 + \beta_3) - i \Omega (2 \gamma_3 + 2 \gamma_5 + \gamma_4)\right] \frac{d^2\bar{F}}{dz^2} \left(\frac{dF}{dz}\right)^2 \]
\[+ \left(\frac{(\sigma B_0^2 (1 + i\phi)}{1 + \phi^2} - i \Omega \rho\right) F = \left(\frac{\sigma B_0^2 (1 + i\phi)}{1 + \phi^2} \right) \left(Q + iP\right), \quad (19)\]
\[F(h) = \Omega l + U_1 + iU_2, \quad F(-h) = -\left(\Omega l + U_1 + iU_2\right). \quad (20)\]

Introducing
\[z^* = \frac{z}{h}, \quad F^* = \frac{F}{\Omega l}, \quad \alpha_1^* = \frac{\alpha_1 \Omega}{\mu}, \quad \beta_1^* = \frac{\beta_1 \Omega^2}{\mu}, \quad \gamma_1^* = \frac{\gamma_1 \Omega^3}{\mu}, \quad \gamma^* = \frac{(2 \gamma_3 + 2 \gamma_5 + \gamma_4) \Omega^3}{\mu}, \quad (21)\]
\[R = \frac{\Omega \rho h^2}{a}, \quad V_1 = \frac{U_1}{\Omega l}, \quad V_2 = \frac{U_2}{\Omega l}, \quad M^2 = \frac{\sigma B_0^2 h^2}{a}, \quad \]

the governing problem reduces to
\[\left(1 - \alpha_1 - \beta_1 + i \gamma_1\right) \frac{d^2F}{dz^2} \]
\[+ 2(\beta - i\gamma) \left[\frac{d^2F}{dz^2} \frac{dF}{dz} \frac{d\bar{F}}{dz} + \frac{dF}{dz} \frac{d^2\bar{F}}{dz^2}\right] \]
\[+ \left(\frac{M^2 (1 + i\phi)}{1 + \phi^2} + iR\right) F = \left(\frac{M^2 (1 + i\phi)}{1 + \phi^2}\right) \left(Q + iP\right), \quad (22)\]
\[F(1) = 1 + V_1 + iV_2, \quad F(-1) = -(1 + V_1 + iV_2), \quad (23)\]

where asterisks have been suppressed for brevity.
3 Analytic solution by homotopy analysis method

To solve Eq. (22) subject to boundary conditions (23) analytically, we use homotopy analysis method. For analytic solution we choose the initial guess $(F_0)$ and the auxiliary linear operator $(L_1)$ in the following form

$$F_0(z) = (1 + V_1 + iV_2)z,$$

$$L_1(F) = \frac{d^2F}{dz^2},$$

where

$$L_1 [A_1 + z A_2] = 0$$

and $A_1$ and $A_2$ are the arbitrary constants. If $p \in [0, 1]$ and $\hat{\epsilon}$ are the embedding and auxiliary parameters respectively then the zeroth order problem is

$$(1 - p) L_1 \hat{F}(z; p) - F_0(z) = p \hat{\epsilon} N_1 \hat{F}(z; p),$$

$$\hat{F}(1; p) = 1 + V_1 + iV_2, \quad \hat{F}(-1; p) = - (1 + V_1 + iV_2).$$

By Taylor’s theorem and Eq. (30) one can easily write

$$\hat{F}(z; p) = F_0(z) + \sum_{m=1}^{\infty} F_m(z) p^m,$$

$$F_m(z) = \frac{1}{m!} \left. \frac{\partial^m \hat{F}(z, p)}{\partial p^m} \right|_{p=0}.$$
\[ F_m(1) = F_m(-1) = 0, \]  

\[ \mathcal{R}_m(z) = (1 - \alpha_1 - \beta_1 + i\gamma_1) F''_{m-1}(z) + \left( \frac{M^2(1+i\phi)}{1+\phi^2} + iR \right) F_{m-1}(z) - (1 - \chi_m) \left( \frac{M^2(1+i\phi)}{1+\phi^2} \right) (Q + iP) \]

\[ + 2(\beta - i\gamma) \sum_{k=0}^{m-1} \sum_{l=0}^{k} [F''_{m-1-k}F''_{l} + F''_{m-1-k}F''_{l}], \]  

\[ \chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \]

Now the boundary value problem consisting of Eqs. (33) with boundary conditions (34) has been solved by "MATHEMATICA" up to first few order of approximations. Finally we get

\[ F_m(z) = \sum_{t=0}^{2m+1} C_{m,t} z^t, \quad \text{where} \quad C_{m,t} = a_{m,t} + ib_{m,t} \quad m \geq 0, \]  

and \( m \geq 1, \) and \( 0 \leq t \leq 2m + 1, \) we have

\[ C_{m,0} = \chi_m \chi_{2m+1} c_{m-1,0} \left( \frac{A}{2} - \sum_{t=0}^{m} \frac{\Gamma_{m,2t}}{(2t+1)(2t+2)} \right) \]

\[ C_{m,1} = \chi_m \chi_{2m} c_{m-1,1} - \sum_{t=0}^{m} \frac{\Gamma_{m,2t+1}}{(2+2t)(3+2t)}, \]

\[ C_{m,2} = \chi_m \chi_{2m-1} c_{m-1,2} + \frac{1}{2} (-A + \Gamma_{m,0}), \]

\[ C_{m,t} = \chi_m \chi_{2m-t+1} c_{m-1,t} + \Gamma_{m,t-2} \frac{1}{(t-1)t}, \quad 3 \leq t \leq 2m + 1, \]

\[ \Gamma_{m,t} = \hbar \chi_{2m-t+1} \left[ (1 - \alpha_1 - \beta_1 + i\gamma_1) C_{2m-1,t} + \left( \frac{M^2(1+i\phi)}{1+\phi^2} + iR \right) C_{m-1,t} + 2(\beta - i\gamma) \left( \alpha_{1m,t} + \alpha_{2m,t} \right) \right], \]

\[ A = (1 - \chi_m) \left( \frac{M^2(1+i\phi)}{1+\phi^2} \right) \left( Q + iP \right). \]

and the coefficients \( \alpha_{1m,t} \) and \( \alpha_{2m,t}, \) for \( m \geq 1, 0 \leq t \leq 2m + 1 \) are

\[ \alpha_{1m,t} = \sum_{k=0}^{m-1} \sum_{l=0}^{k} \min(t,2k+2) \sum_{r=\max\{0,1+t+2k-2m\}}^{C_{1m-1-k,t-r}} \sum_{p=\max\{0,r-1-2l\}}^{\bar{C}_{1l-p} C_{2l-r-p}}, \]

\[ \alpha_{2m,t} = \sum_{k=0}^{m-1} \sum_{l=0}^{k} \min(t,2k+2) \sum_{r=\max\{0,t-2k-2\}}^{C_{1m-1-k,t-r}} \sum_{p=\max\{0,r-1-2l\}}^{\bar{C}_{1l-p} \bar{C}_{2l-r-p}}, \]

\[ C_{m,t} = a_{m,t} + ib_{m,t}, \quad \bar{C}_{m,t} = a_{m,t} - ib_{m,t}, \]

\[ C_{1m,t} = (1+t)C_{m,1+t}, \quad \bar{C}_{1m,t} = (1+t)\bar{C}_{m,1+t}, \]

\[ C_{2m,t} = (1+t)C_{1m,1+t}, \quad \bar{C}_{2m,t} = (1+t)\bar{C}_{1m,1+t}. \]
The corresponding \( m \)th order approximation is of the form
\[
\sum_{m=0}^{M} F_m(z) = \sum_{m=1}^{2M} C_{m,0} + \sum_{t=0}^{2m+1} C_{m,t} z^t; \tag{37}
\]
and therefore series solution is
\[
F(z) = \lim_{M \to \infty} \left[ \sum_{m=0}^{M} F_m(z) = \sum_{m=1}^{2M} C_{m,0} + \sum_{t=0}^{2m+1} C_{m,t} z^t \right]. \tag{38}
\]

4 Heat transfer analysis

This section looks at the heat transfer analysis from the disk to fluid. Therefore the law of conservation of energy in the absence of radiation effects is
\[
\rho \frac{d}{dt} \left( \rho c \right) = T \cdot L - \text{div} \mathbf{q}, \tag{39}
\]
in which \( \mathbf{q} = -K \nabla \tau \) being the thermal conductivity is the heat flux vector, \( \tau \) the temperature and \( e = C_p \tau \) \((C_p\) being the specific heat) is the internal energy. The corresponding temperatures of the disks are
\[
\tau = \tau_1 \quad \text{at } z = h,
\]
\[
\tau = \tau_2 \quad \text{at } z = -h. \tag{40}
\]
Defining
\[
\theta = \frac{T - T_2}{T_1 - T_2} \tag{41}
\]
and making use of Eq.(21) the problem under consideration is of the form
\[
\frac{d^2 \theta}{dz^2} = -Br \left[ 1 + \frac{\beta l^2}{h^2} \left( \frac{dF}{dz} \frac{dF}{dz} \right) \right] \frac{dF}{dz} \frac{dF}{dz}, \tag{42}
\]
\[
\theta(-1) = 1, \quad \theta(1) = 0, \tag{43}
\]
where \( P_r (= aC_P/K) \) is the Prandtl number, \( Br (=E_c P_r) \) is the Brinkman number, \( E_c \)
\((= (\Omega l)^2/(\tau_1 - \tau_2)C_P) \) is the Eckertnumber and \( T_1 > T_2 \) so that \( E_c > 0 \) (i.e. heat is transferred from disk to the fluid).

Selecting the initial guess and auxiliary linear operator of the following type
\[
\theta_0(z) = \frac{1}{2}(1 - z), \tag{44}
\]
\[
\mathcal{L}_2 \left[ \theta(z;p) \right] = \frac{\partial^2 \theta(z;p)}{\partial z^2}, \tag{45}
\]
\[
\mathcal{L}_2 [B_1 + zB_2] = 0, \tag{46}
\]
the corresponding problems at zeroth and \( m \)th orders are
\[
(1 - p)\mathcal{L}_2 \left[ \theta(z;p) - \theta_0(z) \right] = p \bar{h} \mathcal{N}_2 \left[ \hat{\theta}(z;p) \right], \tag{47}
\]
\[ \dot{\theta}(-1; p) = 1, \quad \dot{\theta} (1; p) = 0, \]  

whence

\[ N_2 [\hat{\theta}(z; p)] = \frac{\partial^2 \theta}{\partial z^2} + Br \left[ 1 + \frac{\beta l^2}{h^2} \left( \frac{\partial F}{\partial z} \frac{\partial \bar{F}}{\partial z} \right) \right] \frac{\partial F}{\partial z} \frac{\partial \bar{F}}{\partial z}, \]  

\[ L_2 [\theta_m(z) - \chi_m \theta_{m-1}(z)] = \hbar \mathcal{S}_m(z), \]

\[ \theta_m(1) = \theta_m(-1) = 0, \]

\[ \mathcal{S}_m(z) = \theta''_{m-1}(z) + Br \sum_{k=0}^{m-1} F'_{m-1-k} \bar{F}'_k + \frac{\beta l^2}{h^2} \sum_{k=0}^{m-1} F'_{m-1-k} \sum_{l=0}^k \sum_{a=0}^l \bar{F}'_{l-a} \bar{F}'_a. \]  

The solution of the mth order problem is

\[ \theta_m(z) = \sum_{t=0}^{2m} A_{m,t} z^t, \quad m \geq 0, \]  

where for \( m \geq 1 \) and \( 0 \leq n \leq 2m + 1 \), we have

\[ A_{m,0} = \chi_m \chi_{2m} A_{m-1,0} - \sum_{t=0}^{m} \frac{\Gamma_{m,2t}}{(2t + 1)(2t + 2)}, \]

\[ A_{m,1} = \chi_m \chi_{2m-1} A_{m-1,1} - \sum_{t=0}^{m} \frac{\Gamma_{m,2t+1}}{(2t + 3)(2t + 2)}, \]

\[ A_{m,2} = \chi_m \chi_{2m-2} A_{m-1,2} + \frac{1}{2} \Gamma_{m,0}, \]

\[ A_{m,t} = \chi_m \chi_{2m-t} A_{m-1,t} + \Gamma_{m,t-2} \frac{1}{(t-1)t}, \quad 3 \leq t \leq 2m + 1, \]

\[ \Gamma_{m,t} = \chi_{2m+1-t} \left( \chi_{2m-1-t} A_{m-1,t} + Br \beta_{1,m,t} \right) + Br \beta_{2,m,t} \frac{Bl^2}{h^2} \]

and the coefficients \( \beta_{1,m,t} \) and \( \beta_{2,m,t} \) for \( m \geq 1, 0 \leq n \leq 2m + 1 \) are

\[ \beta_{1,m,t} = \sum_{k=0}^{m-1} \min\{t,2k\} \sum_{q=\max\{0,1+t+2k-2m\}} \bar{C}_{1,k,q}, \]

\[ \beta_{2,m,t} = \sum_{k=0}^{m-1} \sum_{l=0}^k \sum_{a=0}^l \min\{t,2k+3\} \min\{t,2l+2\} \sum_{v=\max\{0,2l-v+1-2m\}} \sum_{u=\max\{0,2l+v-1-2k\}} \sum_{p=\max\{0,u-1-2a\}} \bar{C}_{1,t-a,p}. \]

The mth order approximation is

\[ A_{2,m,t} = (1 + t)A_{m,t+1}, \quad A_{1,m,t} = (1 + t)A_{m,t+1}, \]

\[ \sum_{m=0}^{M} \theta_m(z) = \sum_{m=1}^{2M} A_{m,0} + \sum_{t=0}^{2m+1} A_{m,t} z^t, \]

and the expression of the temperature is

\[ \theta(z) = \lim_{M \to \infty} \left[ \sum_{m=0}^{M} \theta_m(z) = \sum_{m=1}^{2M} \left\{ A_{m,0} + \sum_{t=0}^{2m+1} A_{m,t} z^t \right\} \right]. \]
5 Convergence of the HAM solution

The explicit expressions of the series (38) and (54) are the solutions of the considered problem if one guarantees the convergence of these series. As suggested by Liao the convergence of the series strongly depends upon $h_1$ and $h_2$. To ensure the convergence the $h_1$ and $h_2$ curves are plotted. Computations are made for 15th order of approximation by fixing $M, b, V_1, V_2, R$ and $\phi$. Figs. 1 and 2 show that the range for the admissible values of $h_1$, $h_2$ and $\phi$ are $-1.85 \leq h_1 \leq 0$, $-2.25 \leq h_2 \leq 0$ and $-3.5 \leq h \leq 1$ respectively.

Fig.1 The $h_1$ and $h_2$ curve for the 15th order approximation of velocity

Fig.2 The $\phi$ curve for the 10th order approximation of the temperature.

Numerical results and discussion

The analytic solution of the differential equations (23) and (48) subject to the boundary condition (24) and (49) are obtained in the form of series. A comparison of the velocity profiles and temperature is also given. Special attention has been focused to analyze the effects of various physical parameters like, $M, \beta, \gamma, \alpha_1, \beta_1$ and $\gamma_1$. Further, the effect of Hall
parameter $\phi$ on the velocity and temperature is also shown in Table 1.

<table>
<thead>
<tr>
<th>Types</th>
<th>$f/\Omega$</th>
<th>$g/\Omega$</th>
<th>$\theta$</th>
<th>$f/\Omega$</th>
<th>$g/\Omega$</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fourth grade fluid</td>
<td>-0.0335821</td>
<td>-0.0480789</td>
<td>0.506262</td>
<td>-0.0319473</td>
<td>-0.0515259</td>
<td>0.506263</td>
</tr>
<tr>
<td>Third grade fluid</td>
<td>-0.0224848</td>
<td>-0.0411789</td>
<td>0.506195</td>
<td>-0.0196857</td>
<td>-0.0451585</td>
<td>0.506195</td>
</tr>
<tr>
<td>Second grade fluid</td>
<td>-0.0265531</td>
<td>-0.0291432</td>
<td>0.505573</td>
<td>-0.0233888</td>
<td>-0.0314923</td>
<td>0.505573</td>
</tr>
<tr>
<td>Viscous fluid</td>
<td>-0.0240132</td>
<td>-0.0261104</td>
<td>0.505572</td>
<td>-0.021174</td>
<td>-0.0282365</td>
<td>0.505572</td>
</tr>
</tbody>
</table>

Fig. 3. The variation of velocity profile for various values of the Hartmann number $M$ when $V_1 = 1/20, V_2 = 1/10 = R$ and $\phi = 1/10$ are fixed.

Fig. 4. The variation of velocity profile for various values of the Hartmann number $M$ when $e_1 = \beta_1 = \gamma_1 = \beta = 0.1, \phi = 0, R = 0.1$.
$V_1 = 1/20, V_2 = 1/10 = R$ and $\phi = 0$ are fixed.

Fig. 5. The variation of velocity profile for various values of $\beta$ when $V_1 = 1/20, V_2 = 1/10 = R$ and $\phi = 1/10$ are fixed.

Fig. 6. The variation of velocity profile for various values of $\beta$ when $V_1 = 1/20, V_2 = 1/10 = R$ and $\phi = 0$ are fixed.

Fig. 7. The variation of velocity profile for various values of $\gamma$ when $V_1 = 1/20, V_2 = 1/10 = R$
and $\phi = 0$ are fixed.

Fig.8. The variation of velocity profile for various values of $\gamma$ when $V_1 = 1/20$, $V_2 = 1/10 = R$ and $\phi = 1/10$ are fixed.

Fig.9. The variation of velocity profile for various values of $\alpha_1$ when $V_1 = 1/20$, $V_2 = 1/10 = R$ and $\phi = 0$ are fixed.

Fig.10. The variation of velocity profile for various values of $\alpha_1$ when $V_1 = 1/20$, $V_2 = 1/10 = R$
$R$ and $\phi = 1/10$ are fixed.

Fig.11. The variation of velocity profile for various values of $\beta_1$ when $V_1 = 1/20, V_2 = 1/10 = R$ and $\phi = 0$ are fixed.

Fig.12. The variation of velocity profile for various values of $\beta_1$ when $V_1 = 1/20, V_2 = 1/10 = R$ and $\phi = 1/10$ are fixed.

Fig.13. The variation of velocity profile for various values of $\gamma_1$ when $V_1 = 1/20, V_2 = 1/10 =$
\( R \) and \( \phi = 0 \) are fixed.

![Graph 1](image1)
![Graph 2](image2)

Fig. 14. The variation of velocity profile for various values of \( \gamma_1 \) when \( V_1 = 1/20 \), \( V_2 = 1/10 = R \) and \( \phi = 1/10 \) are fixed.

![Graph 3](image3)
![Graph 4](image4)

Fig. 15. The variation of velocity profile of viscous fluid for various values of \( M \) when \( V_1 = 1/20 \), \( V_2 = 1/10 = R \), and \( \phi = 0.1 \) are fixed.

![Graph 5](image5)
![Graph 6](image6)

Fig. 16. The variation of velocity profile of viscous fluid for various values of \( M \) when \( V_1 = \)}
$1/20, V_2 = 1/10 = R$ and $\phi = 0$ are fixed.

Fig. 17 The variation of temperature for various values of $Br = \Pr.E_c$ when $V_1 = 1/20, V_2 = R = M = 1/10$ and $\phi = 0$ are fixed.

Fig.18. The variation of temperature for various values of $Br = \Pr.E_c$ when $V_1 = 1/20, V_2 = R = M = 1/10$ and $\phi = 1/10$ are fixed.

Fig.19. The variation of temperature for various values of $\beta$ when $V_1 = 1/20, V_2 = R = M = 1/10$ and $Br = 1/100$ are fixed.
Firstly, numerical results corresponding to the fourth-grade fluid, third grade fluid, second grade fluid and viscous fluid are listed in the Table1. From this Table it can be seen that velocity and temperature in the fourth-grade fluid are much greater than the other types of fluid in the presence as well as in the absence of Hall parameter. The real component of the velocity is greater in the absence of Hall parameter in all types of the fluid, whereas the $y-$ component of the velocity is smaller in the absence of Hall parameter and increases in the presence of Hall parameter. The temperature in the fourth-grade fluid increases due to the presence of Hall parameter but there is no effect of Hall parameter on the temperature profile in other fluids.

Figs. 1-19 show the components of the velocity profile and temperature for different values of $M, \beta, \gamma, \alpha_1, \beta_1, \gamma_1$ and $Br$ in the presence and absence of Hall parameter. Figs. 3 and 4 show that velocities decrease rapidly in the absence of Hall parameter as compared with the presence of Hall parameter. Figs. 5 and 6 show that velocity profile decreases with the increase of parameter $\beta$. Figs. 7 and 8 show that $y-$ component of velocity decreases with an increase in $\gamma$ but the $x-$ component of velocity profile is not effected by increasing $\gamma$ in the absence as well as in the presence of Hall parameter. Figs. 9-14 depict that the $y-$ component of the velocity decreases much in comparison to the $x-$ component of velocity when the fourth grade parameters $\alpha_1, \beta_1$ and $\gamma_1$ are increased. Figs. 15-16 elucidate that the velocity in viscous fluid decreases in the absence as well as in the presence of Hall parameter when $M$ increases. Figs. 17 and 18 display that temperature profile increases in the case of fourth grade and viscous fluids when Brinkman number $Br$ increases. Fig. 19 shows that temperature profile does not change through the variation of parameter $\beta$.

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27. T. Hayat and Z. Abbas, Int J. Heat Mass Transfer. 51 (2008),
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