Non-quadratic local stabilization for continuous-time Takagi–Sugeno models

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Abstract

This paper is concerned with non-quadratic stabilization of continuous-time Takagi–Sugeno (TS) models. The well-known problem of handling time-derivatives of membership functions (MFs) as to obtain conditions in the form of linear matrix inequalities (LMIs) is overcome by reducing global goals to the estimation of a region of attraction. Instead of parallel distributed compensation (PDC), a non-PDC control law is proposed according to the non-quadratic nature of the Lyapunov function. Examples are provided to show the advantages over the quadratic and some non-quadratic approaches.

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1. Introduction

Since they were proposed in [1] the control community has intensively studied Takagi–Sugeno (TS) models due to the fact that they can exactly represent a nonlinear model in a compact subset of the domain of the state variables. A TS model is a nonlinear blending of linear models via membership functions (MFs) which hold the convex-sum property [2]. The stabilization problem is usually addressed via the so-called parallel distributed compensation (PDC) control law [3], which is a nonlinear blending of linear-state feedbacks which uses the same MFs as the TS model.

The direct Lyapunov method altogether with quadratic Lyapunov functions has been usually employed to investigate the stability and stabilization of TS models. This method usually leads to conditions formulated in terms of linear matrix inequalities (LMIs) [4], which can be efficiently solved by convex optimization techniques. Quadratic analysis and design has produced a remarkable number of results regarding robustness, performance, observer design, output feedback and time delay systems (see e.g., [5–9] and references therein). Nonetheless, the quadratic approach presents

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serious limitations because its solutions are inherently pessimistic, i.e., there are stable or stabilizable models which do not have a quadratic solution (see [10] and references therein). Conservativeness comes from different sources: the type of TS model [11,12], the way the membership functions are dropped-off to obtain LMI expressions [13–15], the integration of membership-function information [16,17], or the choice of Lyapunov function [18,19]. This work is concerned with a relaxation in the latter sense which demands a change of perspective from global to local conditions.

Several Lyapunov functions have been proposed in the literature. Piecewise Lyapunov functions (PWLF) [18,20] have been straightforwardly applied to those TS models that induce state-space partitions from the fact that not all their linear components are simultaneously activated; unfortunately, TS models constructed via the sector nonlinearity approach lack this property. Various kinds of non-quadratic Lyapunov functions (NQLF) have been also employed; they depend on the same MFs of the model, hereby taking into account structural information otherwise ignored by the quadratic approach. However, NQLF-based results have not triggered the same developments for continuous-time TS models [21] than for discrete-time ones [22,26]. This asymmetry is explained by the difficulty of dealing with time-derivatives of the MFs that emerge while applying the direct Lyapunov method to obtain global conditions.

Some palliative solutions have been proposed to the aforementioned problem. In [21,27] the authors bound the time-derivatives of the MFs assuming that they do not depend on the input, which turns out to be very restrictive. Moreover, the proposed control law makes use of the time-derivatives of the MFs through a classical PDC scheme, thus ignoring the non-quadratic nature of the involved Lyapunov function. In [28] a line-integral Lyapunov function is proposed to circumvent the MFs’ time-derivative obstacle, though the line integral is asked to be path-independent thus significantly reducing its applicability [29]. A change of perspective for non-quadratic stability analysis of TS models has been proposed in [30]. This approach reduces global goals to less exigent conditions, thereby showing that an estimation of the region of attraction can be found (local stability); this solution parallelizes nonlinear analysis and design for models that do not admit a global solution [31]. Said in another words, classical linear parameter varying (LPV) models do not capture the nonlinear behavior of models [32] whereas TS ones, also known as quasi-LPV models, can do it exactly [33]. In the LPV case it is therefore reasonable to deal with global stability (under the constraints due to the bounds on the variables) [32], whereas obviously global stability is very often unreachable for nonlinear systems and therefore for their TS representation. This paper extends the previous results so they can be applied for controller design. A non-PDC control law is employed to fully exploit the non-quadratic properties of the Lyapunov function.

The contents are organized as follows: Section 2 introduces TS models, sector nonlinearity approach, the NQLF this work is based on and the proposed non-PDC control law; a problem statement is made. Section 3 summarizes the work presented in [30] and extends it for stabilization purposes, thus establishing new local conditions for controller design. Section 4 presents some illustrative examples to stress the fact that the few existent solutions on the subject are clearly outperformed by the new approach. Finally, Section 5 gathers some conclusions and perspectives.

2. Definitions and notation

Consider a nonlinear model of the form

\[ \dot{x}(t) = f(z(t))x(t) + g(z(t))u(t), \]  

(1)

with \( f(\cdot), g(\cdot) \) are nonlinear functions, \( x(t) \in \mathbb{R}^n \) the state vector, \( u(t) \in \mathbb{R}^m \) the input vector, and \( z(x(t)) \in \mathbb{R}^p \) the premise vector assumed to be bounded and smooth in a compact set \( C \) of the state space including the origin.

Let \( nl_j(t) \in [nl_j, \, \bar{nl}_j], \ j \in \{1, \ldots, p\} \) be the set of bounded nonlinearities in (1) belonging to \( C \). Employing the sector nonlinearity approach [5], the following weighting functions can be constructed [5],

\[ w_0^j(\cdot) = \frac{\bar{nl}_j - nl_j(\cdot)}{\bar{nl}_j - nl_j}, \quad w_1^j(\cdot) = 1 - w_0^j(\cdot), \ j \in \{1, \ldots, p\}. \]  

(2)

From the previous weights, the following MFs are defined:

\[ h_i = h_1 + i_1 + i_2 \times 2 + \cdots + i_p \times 2^{p-1} = \prod_{j=1}^{p} w_{ij}^j(z_j), \]  

(3)
with $i \in \{1, \ldots, 2^p\}$, $j \in \{0, 1\}$. These MFs satisfy the convex sum property $\sum_{i=1}^{r'} h_i(\cdot) = 1$, $h_i(\cdot) \geq 0$ in $C$. Where convenient, convex sums will be denoted as $Y_i = \sum_{i=1}^{r'} h_i(z(t)) Y_i$ and their inverse as $Y^{-1}_i = (\sum_{i=1}^{r'} h_i(z(t)))^{-1}$.

Based on the previous definitions, an exact representation of (1) in $C$ is given by the following TS model:

$$
\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))(A_i x(t) + B_i u(t)) = A_z x(t) + B_z u(t),
$$

with $r = 2^p \in \mathbb{N}$ representing the number of linear models and $(A_i, B_i)$, $i = 1, \ldots, r$ a set of controllable pairs of proper dimensions directly obtained from the new representation.

Instead of PDC [3], the following generalization of the non-quadratic control law in [22] is proposed:

$$
u(t) = \sum_{i=1}^{r} \sum_{j=1}^{p} h_i h_j \left( F_i + \sum_{k=1}^{r} \hat{c} w_k^0 (\hat{c} z_k(t) T A_j x(t)) \cdot G_j^{-1} \right)^{-1} x(t) = (F_z + \delta(z)) P_z^{-1} x(t),
$$

with $F_i, G_j \in \mathbb{R}^{n \times n}$ the controller gains, $P_i = P_i^T > 0$, and $\delta(z) = \sum_{k=1}^{r} (\hat{c} w_k^0 / \hat{c} z_k)((\hat{c} z_k(t) T A_z x(t)) - L G_z)$. The closed-loop TS model is thence written as

$$
\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))(A_i x(t) + B_i u(t)) = (A_z + B_z(F_z + \delta(z)) P_z^{-1}) x(t).
$$

Triple convex sums $Y_{zzz} = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z(t)) h_j(z(t)) h_l(z(t)) Y_{ijl}$ arise along matrix manipulations and their negative-definiteness is usually examined after dropping the MFs to obtain LMI conditions in terms of $Y_{ijl}$. The way the MFs are dropped from the triple sum above is called a sum-relaxation; several of them are available in the literature for double [13] and multiple sums [14]. In this work, a relaxation derived from Proposition 2 of the latter work has been adopted because it combines computational efficiency and does not need slack variables:

**Relaxation Lemma** (Sala and Ariño [14]). Let $Y_{ijl}$ be matrices of proper dimensions. Then

$$
\sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{l=1}^{r} h_i(z(t)) h_j(z(t)) h_l(z(t)) Y_{ijl} < 0
$$

holds if for a given $q \geq 3$

$$
\tilde{Y}_a = \sum_{b \in \mathfrak{H}(a)} Y_{ijl} < 0, \quad \forall a \in \mathfrak{I}^+_q,
$$

with $\mathfrak{I}_q = \{a = (a_1, \ldots, a_q) \in \mathbb{N}^q : 1 \leq a_b \leq r, \forall b = 1, \ldots, q\}$, $\mathfrak{I}^+_q = \{a \in \mathfrak{I}_q : a_b \leq a_{b+1}, \forall b = 1, \ldots, q - 1\}$, and $\mathfrak{H}(a)$ being the set of permutations (with possibly repeated elements) of multi-index $a$ in $\mathfrak{I}_q$.

Quadratic Lyapunov function $V = x(t)^T P x(t)$ has been originally employed for stabilization purposes. TS model (4) is stabilized by the PDC control law $u(t) = \sum_{i=1}^{r} h_i(z(t)) F_i x(t)$ if there exist matrices $X = P^{-1} > 0$ and $M_i = F_i X$, $i \in \{1, \ldots, r\}$ such that (8) holds with $Y_{ij} = X A_j^T + A_j X + M_j^T B_i^T + B_i M_j$ [5]. This result presents a serious inconvenient: as the number of rules increases the existence of a matrix $P$ as a common solution of the aforementioned set of LMIs gets harder to satisfy. In other words, asking for a global common Lyapunov function is a source of conservatism which can be reduced considering non-quadratic Lyapunov function (NQLF) candidates. NQLFs increase the flexibility by enlarging the choice of matrices that solve the LMI problem (8) [19,22,27]. In this framework, let us consider the following non-quadratic Lyapunov function candidate:

$$
V(x(t)) = x(t)^T \left( \sum_{i=1}^{r} h_i(z(t)) P_i \right)^{-1} x(t) = x(t)^T P_z^{-1} x(t),
$$

with $P_z = P_z^T > 0$ (therefore $P_z^{-1} > 0$).
Its time-derivative along the trajectories of TS model (6) is given by

\[ \dot{V}(x(t)) = x^T(t)(P_z^{-1}(A_z + B_z(F_z + \delta(z))P_z^{-1} + (A_z + B_z(F_z + \delta(z))P_z^{-1})^TP_z^{-1} + \dot{P}_z^{-1})x(t). \]  

(10)

Via elementary properties and taking into account that \( P_z \dot{P}_z^{-1}P_z = -\dot{P}_z \), it is verified that

\[ \dot{V}(x(t)) < 0 \iff P_z^{-1}(A_z + B_z(F_z + \delta(z))P_z^{-1} + (A_z + B_z(F_z + \delta(z))P_z^{-1})^TP_z^{-1} + \dot{P}_z^{-1} < 0 \]
\[ \iff A_zP_z + B_z(F_z + \delta(z)) + (A_zP_z + B_z(F_z + \delta(z)))^T + P_z\dot{P}_z^{-1}P_z < 0 \]
\[ \iff A_zP_z + B_z(F_z + \delta(z)) + (A_zP_z + B_z(F_z + \delta(z)))^T - \dot{P}_z < 0. \]  

(11)

Global conditions in the form of LMIs are normally derived from expressions similar to the previous one in the quadratic case. Unfortunately, obtaining non-conservative LMIs from (11) for global stabilization is no longer possible since the terms \( \dot{P}_z = \sum_{i=1}^{r} \hat{h}_i P_i \) and \( \delta(z) = \sum_{k=1}^{p} \left( \hat{c}u_k^T(\hat{c}z_k(t)/(\hat{c}x(t)))A_zx \right) \) depend on the time-derivatives of the MFs, which do not convey to readily available bounds. This situation raises some questions:

- Should the quadratic case fail, what can be done?
- Can expression (11) be handled to avoid conditions of the sort \( \| \dot{P}_z \| < \phi_z \) [19,21,27]?

The previous works intended to derive global asymptotic conditions. In contrast, in [30] the previous questions were answered for stability analysis via a local approach. It was shown that reducing global goals respond better to stability problems, since the stability domain of a TS model can be estimated via local asymptotic conditions. This kind of estimation is customary for numerous nonlinear models whose global stability cannot be reached [31] and will be extended to the closed-loop TS model in the following section.

3. Main results

**Theorem 1** (Local stabilizability). If there exist matrices of the proper size \( P_i = P_i^T > 0, F_i, \) and \( G_i^T, i \in \{1, \ldots, r\} \), \( k \in \{1, \ldots, p\} \) such that \( A_zP_z + B_z(F_z + \delta(z)) + (A_zP_z + B_z(F_z + \delta(z)))^T < 0 \), then there exists a domain \( D, 0 \in D \), such that TS model (4) is locally asymptotically stabilizable under control law (5).

**Proof.** The NQLF candidate (9) satisfies \( V(0) = 0 \) and \( V(x) \geq 0 \) in \( \mathbb{R}^n \). Its time-derivative (10) holds \( \dot{V}(0) = 0 \). Provided that \( A_zP_z + B_z(F_z + \delta(z)) + (A_zP_z + B_z(F_z + \delta(z)))^T < 0 \) it is implied that there exists a sufficiently small \( \lambda > 0 \) such that \( A_zP_z + B_z(F_z + \delta(z)) + (A_zP_z + B_z(F_z + \delta(z)))^T + \lambda I < 0 \) which can be used to define \( D = \{ x : x \in B, \| \dot{P}_z \| < \lambda \} \). The origin belongs to domain \( D \) since

\[ \dot{P}_z = \sum_{i=1}^{r} \hat{h}_i P_i = \sum_{i=1}^{r} \left( \hat{c}h_i \right)^T \hat{c}z \frac{\partial z}{\partial x} \ddot{x}P_i = \sum_{i=1}^{r} \left[ \left( \hat{c}h_i \right)^T \hat{c}z \frac{\partial z}{\partial x} (A_z + B_z(F_z + \delta(z))P_z^{-1})x(t) \right] P_i \]

depends on the state vector \( x(t) \) in such a way that \( x(t) = 0 \) is a trivial solution of \( \| \dot{P}_z \| < \lambda \). Since \( V(x) > 0 \) and \( \dot{V}(x) < 0 \) in \( D - \{0\} \), the equilibrium point \( x = 0 \) is locally asymptotically stable, thus concluding the proof.

In [30] it has been shown that \( \dot{P}_z \) can be written as

\[ \dot{P}_z = \sum_{k=1}^{p} \hat{c}u_k^T(\hat{c}z_k(g_1(k,z) - g_2(k,z))) \hat{z}_k \]
\[ = \sum_{j=1}^{r} \left( \hat{c}z_k(t)^T \frac{\hat{c}x(t)}{\hat{c}z_k(t)} (A_zx) \right) \cdot G_z \]

with \( g_1(j,k) = [(j-1)/2^{p+1-k}] \times 2^{p+1-k} + (j-1) \mod 2^{p-k} \) and \( g_2(j,k) = g_1(j,k) + 2^{p-k} \). Substituting (12) in (11) and recalling that \( \hat{z}_k = (\hat{c}z_k(t)^T/\hat{c}x(t))(A_zx(t) + B_zu(t)) \) and
\[
A_z P_z + B_z (F_z + \delta(z)) + (A_z P_z + B_z (F_z + \delta(z)))^T - \sum_{k=1}^{p} \frac{\partial u_k^0}{\partial z_k} (P_{g1(z,k)} - P_{g2(z,k)}) \frac{\partial z_k}{\partial x(t)}
\]

\[
= A_z P_z + B_z A_z^T + B_z F_z + F_z^T B_z^T + B_z \delta(z) + \delta^T(z) B_z^T
\]

\[
- \sum_{k=1}^{p} \frac{\partial u_k^0}{\partial z_k} \frac{\partial z_k(t)}{\partial x(t)} (A_z x(t) + B_z u(t))(P_{g1(z,k)} - P_{g2(z,k)})
\]

\[
= A_z P_z + B_z F_z + P_z A_z^T + F_z^T B_z^T - \sum_{k=1}^{p} \frac{\partial u_k^0}{\partial z_k} \frac{\partial z_k(t)}{\partial x(t)} B_z u(t)(P_{g1(z,k)} - P_{g2(z,k)})
\]

\[
+ \sum_{k=1}^{p} \frac{\partial u_k^0}{\partial z_k} \frac{\partial z_k(t)}{\partial x(t)} A_z x(t) (P_{g1(z,k)} - P_{g2(z,k)}) + \sum_{k=1}^{p} \frac{\partial u_k^0}{\partial z_k} \frac{\partial z_k(t)}{\partial x(t)} A_z x(B_z G^k_z) + (B_z G^k_z)^T
\]

\[
= A_z P_z + B_z F_z + P_z A_z^T + F_z^T B_z^T - \sum_{k=1}^{p} \frac{\partial u_k^0}{\partial z_k} \frac{\partial z_k(t)}{\partial x(t)} A_z x(t)(P_{g1(z,k)} - P_{g2(z,k)}) - B_z G_z - (G_z^k)^T B_z^T
\]

\[
= A_z P_z + B_z F_z + P_z A_z^T + F_z^T B_z^T - \sum_{k=1}^{p} \frac{\partial u_k^0}{\partial z_k} \frac{\partial z_k(t)}{\partial x(t)} A_z x(t)(P_{g1(z,k)} - P_{g2(z,k)}) - B_z G_z - (G_z^k)^T B_z^T < 0.
\]

Therefore, knowing the bounds $|\partial u_0^k/\partial x|_1$ and $|\partial u_0^k/\partial x|_v e_i$ allows (13) to be verified as an LMI constraint, hence guaranteeing $V(x(t)) < 0$ in the region thus defined.

To guarantee $||u||_2 < \mu$ (so as to bound $|\partial u_0^k/\partial x|_v e_i$), let us assume $|x(0)| < c$. Note that $P_z \geq c^2 I$ (or $P_{z^{-1}} \leq c^{-2} I$) altogether with the fact that $x^T(t) P_{z^{-1}} x(t) \leq x^T(0) P_{z^{-1}} x(0)$ for $t \geq 0$, means that $V(x(t)) \leq 1$; then, condition $||u||_2 \leq \mu$ is implied by

\[
\frac{1}{\mu^2} ||u||_2^2 = \frac{1}{\mu^2} x^T(t) P_{z^{-1}} (F_z + \delta(z))^T (F_z + \delta(z)) P_{z^{-1}} x(t) \leq x^T(t) P_{z^{-1}} x(t) \leq x^T(0) P_{z^{-1}} x(0) \leq 1,
\]

from which the following stems using the Schur complement and replacing $\delta(z)$ by its value

\[
x^T(t) \left( \frac{1}{\mu^2} P_{z}^{-1} (F_z + \delta(z))^T (F_z + \delta(z)) P_{z}^{-1} - P_z \right) x(t) \leq 0
\]

\[
\Leftrightarrow \frac{1}{\mu^2} (F_z + \delta(z))^T (F_z + \delta(z)) - P_z \leq 0
\]

\[
\Leftrightarrow \begin{bmatrix} P_z & (F_z + \delta(z))^T \\ F_z + \delta(z) & \mu^2 I_m \end{bmatrix} > 0 \Leftrightarrow \begin{bmatrix} P_z & F_z^T \\ F_z & \mu^2 I_m \end{bmatrix} + \begin{bmatrix} 0 & \delta^T(z) \\ \delta(z) & 0 \end{bmatrix} > 0
\]

\[
\Leftrightarrow \begin{bmatrix} P_z & F_z^T \\ F_z & \mu^2 I_m \end{bmatrix} + \sum_{k=1}^{p} \frac{\partial u_k^0}{\partial z_k} \frac{\partial z_k(t)}{\partial x(t)} A_z x \left[ \begin{array}{c} 0 \\ G_z^k \end{array} \right] > 0.
\]
Inequality (14) is therefore implied by
\[
\begin{bmatrix}
P_z & F_T^z \\
F_z & \mu^2 I_m
\end{bmatrix} + \sum_{k=1}^{p} \sum_{i=1}^{n} \sum_{s=1}^{n} (A_z)_{ys} \frac{\partial w_0^k}{\partial x_s} \begin{bmatrix} 0 & (G_z^k)^T \\
G_z^k & 0 \end{bmatrix} > 0,
\]
which as expression (13) depends on bounding terms $|\frac{\partial w_0^k}{\partial x_s} x_s|$ to yield LMI expressions.

Two choices are offered to deal with these bounds, based respectively on the following properties:
\[
X + \left(\frac{\partial w_0^k}{\partial x_s} x_s\right) Y + \left(\frac{\partial w_0^k}{\partial x_s} u_e\right) Z \leq 0 \iff X \pm \dot{\lambda}_{k_{sv}} \times Y \pm \eta_{kv} \times \mu Z \leq 0
\]
(16)
or
\[
Q = Q^T > 0, \quad S = S^T > 0,
\]
\[
X + \left(\frac{\partial w_0^k}{\partial x_s} x_s\right) Y + \left(\frac{\partial w_0^k}{\partial x_s} u_e\right) Z \leq X + \frac{1}{2}(\dot{\lambda}_{k_{sv}}^2 Q + Y Q^{-1} Y) + \frac{1}{2}(\eta_{kv}^2 S + \mu^2 Z S^{-1} Z) < 0.
\]
(17)

By means of property (16) the following result is obtained:

**Theorem 2.** If there exist matrices of proper size $P_j = P_j^T \geq c^2 I$, $F_j$, $G_j^k$, $\forall j \in \{1, \ldots, r\}$, $k \in \{1, \ldots, p\}$ such that LMIs
\[
\tilde{A}_{ijl}^2 = \sum_{h \in \mathcal{N}(a)} A_{ijl}^2 \leq 0, \quad \forall a \in \mathbb{R}^+, \quad a \in \{1, \ldots, 2^{pn(m+n)}\}
\]
(18)
hold for a given $q \geq 3$ with
\[
\tilde{A}_{ijl}^2 = \text{block-diag}[\tilde{Y}_{ijl}^2, \tilde{Q}_{ijl}^2],
\]
\[
\tilde{Y}_{ijl}^2 = A_i P_j + B_i F_j + P_j A_i^T + F_j B_i^T - \sum_{k=1}^{p} \sum_{i=1}^{n} \sum_{e=1}^{m} (-1)^{d_{k_{sv}(e)n)}(\eta_{kv} t(B_i)_{ve}(P_{g_1(j,k)} - P_{g_2(j,k)}))
\]
\[
- \sum_{k=1}^{p} \sum_{i=1}^{n} \sum_{s=1}^{n} (-1)^{d_{k_{vs}(n)s}} \dot{\lambda}_{k_{vs}} (A_i)_{vs} (P_{g_1(j,k)} - P_{g_2(j,k)} - B_i G_j^k - (G_j^k)^T B_i^T),
\]
\[
\tilde{Q}_{ijl}^2 = \begin{bmatrix}
P_j & F_j^T \\
F_j & \mu^2 I_m
\end{bmatrix} - \sum_{k=1}^{p} \sum_{i=1}^{n} \sum_{s=1}^{n} (-1)^{d_{k_{vs}(n)s}} \dot{\lambda}_{k_{vs}} (A_i)_{vs} \begin{bmatrix} 0 & (G_j^k)^T \\
G_j^k & 0 \end{bmatrix},
\]
defs $d_{k_{vs}(n)s}$, $d_{k_{vs}(n)s}$ defined from $\alpha - 1 = d_{p(n+m)}^z + d_{p(n+m-1)}^z \times 2 + \cdots + d_{1}^z_{11} \times 2^{p(n(m-1))}$, and $g_1(j,k)$, $g_2(j,k)$ defined as in (12), then $x(t), |x(0)| = c$ tends to zero exponentially for any trajectory satisfying (6) in the outermost Lyapunov level contained in $R_0 \subseteq (R \cup C)$ with
\[
R = \bigcap_{e,k,s,v} \left\{ x : \frac{\partial w_0^k}{\partial x_s} \leq \mu \eta_{kv}, \frac{\partial w_0^k}{\partial x_v} \leq \dot{\lambda}_{k_{sv}} \right\}.
\]

**Proof.** From relaxation Lemma in (8), LMIs in (18) imply $A_{zzzz}^2 = \text{block-diag}[\tilde{Y}_{zzzz}^2, \tilde{Q}_{zzzz}^2] < 0$, or equivalently,
\[
\tilde{Y}_{zzzz}^2 = A_z P_z + B_z F_z + P_z A_z^T + F_z B_z^T - \sum_{k=1}^{p} \sum_{i=1}^{n} \sum_{s=1}^{n} (-1)^{d_{k_{sv}(e)n)}(\eta_{kv} t(B_z)_{ve}(P_{g_1(z,k)} - P_{g_2(z,k)}))
\]
\[
- \sum_{k=1}^{p} \sum_{i=1}^{n} \sum_{s=1}^{n} (-1)^{d_{k_{vs}(n)s}} \dot{\lambda}_{k_{vs}} (A_z)_{vs} (P_{g_1(z,k)} - P_{g_2(z,k)} - B_z G_z^k - (G_z^k)^T B_z^T) < 0,$
Given that all the possible sign combinations of the terms involving $\eta_{kv}$ and $\lambda_{kvs}$ are taken into account in the previous expressions by means of parameter $\alpha$, and provided that $|\langle \hat{e}w_{k}^{T}/\partial x_{v}\rangle u_{e}| \leq \alpha \eta_{kv}, |\langle \hat{e}w_{k}^{T}/\partial x_{v}\rangle x_{v}| \leq \lambda_{kvs}$ in $R_{p}$, it follows that:

$$
A_{z}P_{z} + B_{z}F_{z} + P_{z}A_{z}^{T} + F_{z}^{T}B_{z} - \frac{p}{\mu^{2}} \sum_{k=1}^{n} \sum_{v=1}^{n} \sum_{s=1}^{n} (-1)^{(k, v, s)} \lambda_{kvs} (A_{z})_{vs} \left[ \begin{array}{c}
0 \\
G_{z}^{k}
\end{array} \right] < 0.
$$

The time-derivative (10) of the non-quadratic Lyapunov function candidate (9) is negative if (11) holds, which is implied by condition (19). Moreover, $\|u\|_{2} < \mu$ holds if (15) does so, which is guaranteed by condition (20), thus concluding the proof. □

A similar result is now presented via property (17):

**Theorem 3.** If there exist matrices of proper size $P_{j} = P_{j}^{T} \geq c^{2}I$, $F_{j}$, $G_{j}^{k}$, $Q_{jij}^{kvs} > 0$, $S_{jij}^{kve} > 0$, and $T_{ij}^{kvs} > 0$, $\forall j, l, i \in \{1, \ldots, r\}, k \in \{1, \ldots, p\}$, $s, v \in \{1, \ldots, n\}$, $e \in \{1, \ldots, m\}$, such that LMIs

$$
A_{i} = \sum_{b \in A(a)} A_{ijl} < 0, \quad \forall a \in \mathbb{R}_{+}^{q},
$$

hold for a given $q \geq 3$ with $A_{ijl} = \text{block-diag} \{Y_{ijl}, \Omega_{ijl}\}$,
From (17) it is clear that
\[
\begin{align*}
\Omega_{ij} &= \begin{bmatrix}
-\left[ P_j F_j^T \right] & + \frac{1}{2} \sum_{k=1}^{p} \sum_{i=1}^{n} \sum_{s=1}^{n} \theta_{k^{ijs}}^2 T_{ij} \kappa^{(k^ijs)} & (*) & \cdots & (*) & \cdots & (*) \\
F_j^T \mu_i I_m & & & & & & \\
\end{bmatrix} \\
(\forall_{ij})_{11} \begin{bmatrix}
0 & -(G_j^p)^T \\
-G_j^p & 0 \\
\end{bmatrix} & -2T_{ij}^{111} & 0 & \cdots & 0 \\
(\forall_{ij})_{1n} \begin{bmatrix}
0 & -(G_j^p)^T \\
-G_j^p & 0 \\
\end{bmatrix} & 0 & \cdots & -2T_{ij}^{p11} & 0 \\
(\forall_{ij})_{nn} \begin{bmatrix}
0 & -(G_j^p)^T \\
-G_j^p & 0 \\
\end{bmatrix} & 0 & \cdots & 0 & -2T_{ij}^{pnn}
\end{align*}
\]

Then $x(t)$ tends to zero exponentially for any trajectory satisfying (6) in the outermost Lyapunov level $\mathcal{R}_0 \subseteq (R \cap C)$ with
\[
R = \bigcap_{e,k,s,v} \left\{ x : \lvert \frac{\dot{w}^k}{\dot{x}_v} u_e \rvert \leq \mu_{k^{ijs}} \left| \frac{\dot{w}^k}{\dot{x}_v} x_s \right| \leq \kappa^{(k^ijs)} \right\}
\]

Proof. From (17) it is clear that
\[
\begin{align*}
A_z P_z + B_z F_Z + P_z A_z^T + F_z^T B_z^T & - \sum_{k=1}^{p} \sum_{i=1}^{n} \sum_{s=1}^{n} (A_z)_{k^{ijs}} \frac{\dot{w}^k}{\dot{x}_v} x_s (P_{g_1(z,k)} - P_{g_2(z,k)}) - B_z G_z^k - (G_z^k)^T B_z^T \\
& = A_z P_z + B_z F_Z + P_z A_z^T + F_z^T B_z^T + \sum_{k=1}^{p} \sum_{i=1}^{n} \sum_{s=1}^{n} \frac{\dot{w}^k}{\dot{x}_v} u_e (B_z)_{k^{ijs}} (P_{g_2(z,k)} - P_{g_1(z,k)}) \\
& \quad + \sum_{k=1}^{p} \sum_{i=1}^{n} \sum_{s=1}^{n} (A_z)_{k^{ijs}} \frac{\dot{w}^k}{\dot{x}_v} x_s (P_{g_2(z,k)} - P_{g_1(z,k)}) + B_z G_z^k + (G_z^k)^T B_z^T \\
& \leq A_z P_z + B_z F_Z + P_z A_z^T + F_z^T B_z^T \\
& \quad + \frac{1}{2} \sum_{k=1}^{p} \sum_{i=1}^{n} \sum_{s=1}^{m} \left[ \mu_i^2 \eta_{k^{ijs}}^2 S_{k^{ijs}}^{kve} + (B_z)_{k^{ijs}}^2 (P_{g_2(z,k)} - P_{g_1(z,k)}) (S_{k^{ijs}}^{kve})^{-1} (P_{g_2(z,k)} - P_{g_1(z,k)}) \right] \\
& \quad + \frac{1}{2} \sum_{k=1}^{p} \sum_{i=1}^{n} \sum_{s=1}^{m} \left[ \theta_{k^{ijs}}^2 Q_{k^{ijs}}^{kve} + (A_z)_{k^{ijs}}^2 (P_{g_2(z,k)} - P_{g_1(z,k)}) + B_z G_z^k + (G_z^k)^T B_z^T \right] (Q_{k^{ijs}}^{kve})^{-1} \\
& \quad \times (P_{g_1(z,k)} - P_{g_1(z,k)} + B_z G_z^k + (G_z^k)^T B_z^T) < 0.
\end{align*}
\]
Successively applying the Schur complement to the previous expression leads to

$$
\begin{bmatrix}
A_z P_z + B_z F_z + P_z A_z^T + F_z B_z^T \\
+ \frac{1}{2} \sum_{k=1}^{p} \sum_{s=1}^{n} \sum_{l=1}^{n} \sigma^2_{k \ell} Q_{zzzz} \\
+ \frac{1}{2} \sum_{k=1}^{p} \sum_{n=1}^{m} \sum_{m=1}^{n} \nu^2_{k \ell \ell} \delta_{k \ell} \\
(A_z)_{11} \left(F_{g2(z1,1)} - P_{g1(z1,1)} + B_z G_z^T + (G_z^T) B_z^T \right) \\
-2Q_{zzzz}^{zzzz} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0
\end{bmatrix} = 0
$$

A similar procedure shows that applying Schur complement to the last inequality in

$$
- \begin{bmatrix}
P_z F_z^T \\
F_z \\
\mu^2 I_m
\end{bmatrix}
- \sum_{k=1}^{p} \sum_{s=1}^{n} \sum_{l=1}^{n} (A_z)_{vs} \frac{\partial w^k}{\partial x_v} x_s \begin{bmatrix}
0 \\
(G_z^k)^T \\
G_z^k \end{bmatrix}
= - \begin{bmatrix}
P_z F_z^T \\
F_z \\
\mu^2 I_m
\end{bmatrix}
+ \sum_{k=1}^{p} \sum_{s=1}^{n} \sum_{l=1}^{n} (A_z)_{vs} \frac{\partial w^k}{\partial x_v} x_s \begin{bmatrix}
0 \\
-(G_z^k)^T \\
-G_z^k \end{bmatrix}
\leq - \begin{bmatrix}
P_z F_z^T \\
F_z \\
\mu^2 I_m
\end{bmatrix}
+ \frac{1}{2} \sum_{k=1}^{p} \sum_{s=1}^{n} \sum_{l=1}^{n} \sigma^2_{k \ell} T_{zzz}^{kvs} + (A_z)_{vs}^2 \begin{bmatrix}
0 \\
-(G_z^k)^T \\
-G_z^k \end{bmatrix}
[T_{zzz}^{kvs}]^{-1} \begin{bmatrix}
0 \\
-(G_z^k)^T \\
-G_z^k \end{bmatrix}
$$

which allows defining $A_{zzz} = \text{block-diag}[Y_{zzz}, \Omega_{zzz}]$.

Applying Relaxation Lemma (8) to $A_{zzz}$ leads to LMIs (21), thus concluding the proof. □
Remark 1. Theorems 2 and 3 provide LMI conditions for local non-quadratic controller design of continuous-time TS models. The control law thus designed is guaranteed to stabilize the TS model in the outermost Lyapunov level of region \( R \), whose bounds depend on those of the compact set \( C \), the designed bound \( \mu \) of the input, and the inherited bounds of the partial derivatives \( \dot{w}_i^k / \dot{x}_k \). Let us recall that, in many previous studies, see e.g., [13,19,21–25,27], controller design conditions have been obtained by making very restrictive assumptions of the sort \( \| P_z \| < \phi_c \). It means that these bounds cannot be known in advance since they depend on the time-derivative of the state, which cannot be assured to be a priori stable. Therefore, all these previous methods fail to give a solution to this problem whereas the parameters \( |(\dot{w}_i^k / \dot{x}_k)| \) and \( |(\dot{w}_0^k / \dot{x}_k)| \) are always known. Indeed, it is well-known that the quasi-LPV (or TS) description is only validated in a compact set of the state space, therefore \( x \) is known and the functions \( w_i^k \) being smooth, their derivative \( \dot{w}_i^k / \dot{x}_k \) limits are easy to compute. Moreover, considering any practical application, conditions (15) allow to guarantee \( \| u \|_2 < \mu \) so as to bound \( |(\dot{w}_0^k / \dot{x}_k)| \). The parameter \( \mu \) (which can be obtained from the actuators technical characteristics) ensures that, with the obtained solution to LMIs, the control signal will always be bounded while the initial conditions belong inside the domain of attraction \( D \). Therefore, the proposed study provides an answer to the previous non-quadratic approaches where unknown parameters had to be assumed for the LMI computation. At last, it is important to point out that most of the previous studies dealing with TS systems tend to explore global stability regardless of their local nature. In contrast, local analysis and controller design for nonlinear systems are widely used; in this sense the proposed approach better fits the nonlinear spirit by reducing global objectives to local stability conditions for TS models.

Remark 2. Conditions in Theorems 2 and 3 reduce to those of the quadratic global controller design if \( P_i = P \), \( G_i^k = 0 \), \( i \in \{1, \ldots , r\}, k \in \{1, \ldots , p\} \). For Theorem 2 it cancels the last two triple sums of \( \Upsilon_{ij} \) and the last triple sum of \( \Omega_{ij} \) in (18), allowing \( \lambda_{kuv} \) and \( \eta_{kuv} \) to grow arbitrarily large (global controller). As for Theorem 3, it makes zero for all the non-diagonal terms of \( \Upsilon_{ij} \) and \( \Omega_{ij} \) in (21), so matrices \( Q_{ij}^{kuv} > 0 \), \( S_{ij}^{kuv} > 0 \), and \( T_{ij}^{kuv} > 0 \) can always be found to fulfill conditions (21) despite the values of \( \lambda_{kuv} \) and \( \eta_{kuv} \).

Remark 3. Even if conditions in Theorem 2 or 3 fail to be fulfilled for a given set of constants \( \lambda_{kuv}, \eta_{kuv} > 0 \) (i.e., for a given region), the largest region of attraction can still be found via linear programming by testing conditions in these theorems for successively larger values of a common factor \( \lambda > 0 \) multiplying both sets of constants, assuming that conditions in Theorem 1 hold.

Remark 4. Notice that compared with the quadratic case, Theorem 2 increases the number of LMI constraints from \( r + 1 \) to \( r + r^2 \times 2^{m-1} \) whereas Theorem 3 increases them from \( r + 1 \) to \( r + r^2 \) while increasing the size of each LMI from \( \mathbb{R}^{n \times n} \) to \( \mathbb{R}^{(n \times k \times u) \times (n \times k \times u)} \).

4. Examples

Example 1. Consider the following TS model [28]:

\[
\dot{x} = \sum_{i=1}^{2} h_i(x_1)(A_i x + B_i u)
\]

(22)

with

\[
h_1(x_1) = \frac{1 - \sin(x_1)}{2}, \quad h_2(x_1) = \frac{1 + \sin(x_1)}{2}, \quad A_1 = \begin{bmatrix} 2 & -10 \\ 2 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} a & -5 \\ 1 & 2 \end{bmatrix},
\]

\[
B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} b \\ 2 \end{bmatrix}, \quad |x_1| < \frac{\pi}{2}, \quad -20 \leq a \leq 10, \quad 0 \leq b \leq 25.
\]
Fig. 1. Stabilized domains: with $\mu = 30$ (dashed closed curve); with $\mu = 100$ (solid closed curve).

Fig. 2. Stabilized domains: without gains $G_k^i$ (dashed closed curve); with gains $G_k^i$ (solid closed curve).

Fig. 1 shows the effects of parameter $\mu$ in a single stabilization case ($a = 4.6$, $b = 0$). Conditions in Theorem 2 and linear programming have been combined to find local controllers maximizing the region of attraction for $\mu = 30$ (bounded by the dashed closed curve) and $\mu = 100$ (bounded by the solid closed curve). As expected, the latter provides the biggest region and includes the former one. The bounds of

$$R = \bigcap_{\epsilon, k, s, v} \left\{ x : \left| \frac{\hat{u}_k^i}{\partial x_v} u_\epsilon \right| \leq \mu \eta_{kv}, \left| \frac{\hat{u}_k^i}{\partial x_v} x_3 \right| \leq \dot{s}_{kv} \right\}$$

for $\mu = 100$ are shown in dotted lines.

Fig. 2 shows the advantages of using gains $G_k^i$ for the previous case ($a = 4.6$, $b = 0$). Theorem 2 and linear programming have been used again to find local controllers maximizing the region of attraction without $G_k^i$ (bounded by a dashed closed curve) and including $G_k^i$ (bounded by the solid closed curve). It is clear that the new control law (5) outperforms the existing non-quadratic ones [22]. As in Fig. 1, the bounds of region R for the case including $G_k^i$ are shown in dotted lines.
Comparison between Theorems 2 and 3.

<table>
<thead>
<tr>
<th>Approach</th>
<th>$\mu = 0.01$, $c = \pi/2$</th>
<th>$\mu = 0.01$, $c = \sqrt{2}\pi/2$</th>
<th>$\mu = 0.1$, $c = \pi/2$</th>
<th>$\mu = 0.1$, $c = \sqrt{2}\pi/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theorem 2</td>
<td>$\dot{\lambda}_{k,v} = 0.8836$</td>
<td>$\dot{\lambda}_{k,v} = 0.8713$</td>
<td>$\dot{\lambda}_{k,v} = 1.4603$</td>
<td>$\dot{\lambda}_{k,v} = 1.2149$</td>
</tr>
<tr>
<td></td>
<td>$\eta_{k,v} = 0.5625$</td>
<td>$\eta_{k,v} = 0.5547$</td>
<td>$\eta_{k,v} = 0.9297$</td>
<td>$\eta_{k,v} = 0.7734$</td>
</tr>
<tr>
<td>Theorem 3</td>
<td>$\dot{\lambda}_{k,v} = 0.8836$</td>
<td>$\dot{\lambda}_{k,v} = 0.8713$</td>
<td>$\dot{\lambda}_{k,v} = 1.4481$</td>
<td>$\dot{\lambda}_{k,v} = 1.2026$</td>
</tr>
<tr>
<td></td>
<td>$\eta_{k,v} = 0.5625$</td>
<td>$\eta_{k,v} = 0.5547$</td>
<td>$\eta_{k,v} = 0.9219$</td>
<td>$\eta_{k,v} = 0.7656$</td>
</tr>
</tbody>
</table>

Conditions in [21] are unfeasible for this model in any point of the considered grid. The proposed approach is therefore compared with that in [28], for which the following pole placement LMI s for a D-region must be included:

\[
\begin{bmatrix}
-\bar{T} P_i & \tilde{G}_{iij}^2 \\
(\ast) & -\bar{T} P_i
\end{bmatrix} < 0, \quad \begin{bmatrix}
\sin \theta (\tilde{G}_{iij}^2 + (\tilde{G}_{iij}^2)^T) & \cos \theta (\tilde{G}_{iij}^2 - (\tilde{G}_{iij}^2)^T) \\
\cos \theta (\tilde{G}_{iij}^2 - (\tilde{G}_{iij}^2)^T) & \sin \theta (\tilde{G}_{iij}^2 + (\tilde{G}_{iij}^2)^T)
\end{bmatrix} < 0,
\]

\[
\tilde{G}_{iij}^2 + (\tilde{G}_{iij}^2)^T + \beta P_i < 0,
\]

with $\tilde{G}_{iij}^2 = A_j P_i + B_i (F_i + \sum_{k=1}^{p} (A_{ij})_{ik} (A_{ij})_{kj} (-1)^{\delta_{ikj}} \dot{\lambda}_{k,ij} G_i^2)^2 < 0$, $i, j \in \{1, \ldots, r\}$, $\alpha \in \{1, \ldots, 2^{pn^2}\}$, and $\bar{T}$, $\theta$ and $\beta$ as parameters defining the D-region.

The proposed approach is $\mu$-dependent ($|u| < \mu$). Fig. 3 shows the effect of this parameter on the stabilizability region size for $-20 \leq a \leq 10$, $0 \leq b \leq 25$. As $\mu$ increases, more feasible points (“+” marks) are found via conditions in Theorem 1 and (23); these points are then compared with those of the quadratic approach (“•” marks) and those of the line integral approach in [28] (“x” marks). Note that with $\mu = 500$, the new approach includes any feasible point of the other approaches considered.
Example 2. Consider the following TS model [21]:

\[
\dot{x} = \sum_{i=1}^{2} h_i(x_1)(A_i x + B_i u),
\]

with

\[
h_1(x_1) = \frac{1 + \sin(x_1)}{2}, \quad h_2(x_1) = \frac{1 - \sin(x_1)}{2}, \quad A_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix},
\]

\[
B_1 = \begin{bmatrix} 0 \\ 10 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad |x_i| < \frac{\pi}{2}.
\]
The approach in [21] has several drawbacks, none of which is shared by the proposed one: it only applies for TS models whose membership functions have strictly state-dependent time-derivatives (like (24)); its conditions are not LMIs unless some parameters are given; its proposed control law includes time-derivatives of the membership functions which are not always available. For the sake of comparison between Theorems 2 and 3, they have been applied for stabilization of TS model (24) for different values of $\mu$ and $c$, with $|u| < \mu$, $|x(0)| < c$; the resulting maxima $\eta_{k_{10}}$ and $\eta_{k_{12}}$ are summarized in Table 1. Fig. 4 illustrates the states evolution of TS model (24) under a control law (5) whose gains have been obtained via Theorem 3 with $\mu = 0.1$, $c = \sqrt{2}/2$; correspondingly, Fig. 5 shows the control law evolution corresponding to this case.

5. Conclusions and perspectives

A novel approach for local non-quadratic stabilization of continuous-time TS models has been presented. Thanks to the information provided by the MFs and a proper manipulation of their time-derivatives, a new non-PDC control law has been proposed which locally stabilizes a continuous-time TS model through straightforward LMI conditions. It has been shown that reducing global goals to the estimation of a region of attraction constitutes a good way-out from the quadratic framework, since it provides new basis to overcome old issues on stabilization of continuous-time TS models while incorporating former results as particular cases. Two examples have been provided that clearly show that the proposed approach outperforms the palliative solutions of the previous non-quadratic studies.

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