Uniqueness of solutions and linearized stability for impulsive differential equations with state-dependent delay

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Abstract

We prove that under fairly natural conditions on the state space and nonlinearities, it is typical for an impulsive differential equation with state-dependent delay to exhibit non-uniqueness of solutions. On a constructive note, we show that uniqueness of solutions can be recovered using a Winston-type condition on the state-dependent delay. Irrespective of uniqueness of solutions, we prove a result on linearized stability. As a specific application, we consider a scalar equation on the positive half-line with continuous-time negative feedback, non-negative state-dependent delayed nonlinearity and impulse effect functional satisfying affine bounds.

1 Introduction

Differential equations with state-dependent delay are notorious for their lack of smoothness properties and the wealth of associated open problems pertaining to the semiflow and their invariant manifolds. See [19, 25] for background. As for the associated Cauchy problem, continuity of the initial condition is typically not enough to ensure uniqueness of solutions, as was demonstrated by the classical counterexample of Winston [42]. Under various definitions of solution, absolute continuity [15] or Lipschitz continuity [2, 33, 42] have been imposed to guarantee uniqueness of solutions.

State-dependent delay arises naturally in several areas of scientific interest, including cell biology [17, 32], structured population models [13], infectious diseases [45], electromagnetism [12] and turning processes [23]. In this setting, there is a fairly mature theory of solutions, with the most robust perhaps being the solution manifold approach originally developed by Walther [39]. General results concerning the Cauchy problem for impulsive delay differential equations have been known for some time [3, 4, 24], but they do not grant uniqueness if state-dependent delays are present. State-dependent delay is important in such control engineering problems as multi-agent consensus [14, 26], and as these protocols operate discretely in time, there is a need to understand how such systems behave in the presence of impulses. Constant discrete delays are mathematically convenient, but their use is not always justified by the physical problem being studied. The same is true for delay differential equations with impulses, of which there numerous applications in biology and control [16, 21, 30, 36, 43]. Understanding the solution sets of impulsive systems with state-dependent delay therefore has practical implications.

In another direction, stability (in the sense of Lyapunov) is a fundamental topic in dynamical systems. It is especially important in impulsive differential equations literature, due to the applications of such systems in control theory; see the 2018 survey article [44] for background. Several articles have recently considered state-dependent delayed *impulses* from the point of view of stability [27, 28, 37, 46] using Lyapunov functions/functionals. To compare, stability analysis of differential equations with state-dependent

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delay (without impulses) has been studied for several decades [11, 20, 18, 29, 31]. Stability of impulsive functional differential equations have been considered variously using Lyapunov functional-type methods [41, 40, 44, 47] and a linearized stability result has been proven [9], but these require the continuous-time functional to be at least Lipschitz continuous with domain being a phase space of discontinuous functions. As we remark in Section 1.1, this means state-dependent delays in the continuous-time dynamics can not be handled using the extant literature. As such, it seems as though stability of impulsive systems with state-dependent delay in the *continuous-time* dynamics has not been well-studied.

With this discussion in mind, in this paper we will study uniqueness of solutions and linearized stability for the impulsive differential equation with state-dependent delay

$$x'(t) = f(t, x(t), x(t - \tau_1(x(t)))), \qquad t \neq t_k \tag{1}$$
$$\Delta x(t) = g(t, x_{t^-}), \qquad t = t_k, \tag{2}$$

where $\tau_1 : \mathbb{R}^n \to [0, r]$, and the functions f and functional g will be described later (Section 1.5). The jump is defined by $\Delta x(t) = x(t) - \lim_{s \to t^-} x(s)$. For $x : \mathcal{I} \to \mathbb{R}^n$ for an interval $\mathcal{I}$, the history x_t for $[t - r, t] \subset \mathcal{I}$ is defined as usual: $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$. The left-limit x_{t^-} is defined as follows:

$$x_{t^-}(\theta) = \begin{cases} x(t+\theta), & \theta < 0 \\ x(t^-), & \theta = 0, \end{cases}$$

where $x(t^-) = \lim_{s \to t^-} x(s)$ is the usual left-limit. In this paper, the sequence of impulse times t_k is always assumed increasing, and unbounded as $k \to \infty$. It can be either finite or infinite on the left (i.e. it may be indexed by $\mathbb{Z}$ or $\mathbb{N}$), but in the case it is bi-infinite, we require $\lim_{k \to \pm\infty} |t_k| = \infty$.

1.1 The uniqueness problem

We argue that thus far, uniqueness of solutions has been an elusive topic for impulsive systems with state-dependent delay. While there are certainly contributions in the literature, we claim that many do not thoroughly address state-dependent delay. Before surveying the literature, we will illustrate the main problem. Let X denote a space of right-continuous functions mapping into (a subset of) $\mathbb{R}^d$, defined on an interval of the form $[-r, 0]$, possibly with additional structure (e.g. only finitely-many discontinuities). Consider for simplicity the impulsive differential equation with state-dependent delay

$$x'(t) = f(x(t - \tau(x_t))), \qquad t \neq t_k$$
$$\Delta x(t) = g(x(t_k^-)), \qquad t = t_k,$$

where $\tau : X \to \mathbb{R}$ is non-negative. The functional defining the right-hand side can be identified with

$$F = f \circ ev \circ (id \times (-\tau)),$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$, and $ev : X \times \mathbb{R} \to \mathbb{R}^d$ is the evaluation map defined by $ev(\phi, s) = \phi(s)$. The evaluation map is generally not locally Lipschitz continuous [25], even when X is given the structure of containing only continuous functions. Local Lipschitz conditions can be recovered if X contains only Lipschitz continuous (or higher smoothness) functions, but this excludes functions with discontinuities. Consequently, for the purposes of assuring uniqueness of solutions for impulsive systems with state-dependent delay, assuming *a priori* that $F : X \to \mathbb{R}^d$ has a local Lipschitz property, is inappropriate.

With the above discussion in mind, let us survey some classical and more recent contributions to the Cauchy problem for impulsive functional differential equations, framing them within the scope of state-dependent delay. There is the work of Ballinger and Liu [3, 4], which is stated in terms of general impulsive functional differential equations. To obtain uniqueness of solutions, Lipschitz-like conditions are assumed at the level of the functional, and this is incompatible with state-dependent delays. Ouahab

[34, 35] uses a result on contraction maps in Fréchet spaces to prove a global existence and uniqueness result for impulsive systems with multiple (fixed) delays and a general functional nonlinearity. They work in spaces of functions with at most countably-many discontinuities, but once again, a local Lipschitz condition is needed on the functional term to ensure uniqueness. In a recent paper of Chen and Ma [8], the authors aim to extend the solution manifold concept from evolution equation with state-dependent delay to the case of systems with impulses. However, the manifold the authors construct consists of continuous (in fact, C^1) functions and as such, any impulse effect will move the solution off the manifold. Similarly, discontinuous initial conditions are not permitted.

We can gain some additional insight by surveying the literature on abstract impulsive functional evolution equations. With respect to the state-dependent delay, Azevedo [1] proves local existence and uniqueness of solutions in a setting where state-dependent delay is permissible — that is, in a phase space of functions with Lipschitz conditions and some discontinuities — but the state-dependence is only in the impulse term. In the continuous-time dynamics, only a time-varying delay is permitted, and it is not clear at present how to extend this to allow state-dependent delay. There is also the work of Benchohra and Ziane [6] and Benchohra and Henderson [5] on impulsive evolution inclusions with state-dependent delay, but as this is a multivalued setting, only existence of solutions is considered. Neutral equations are considered in [22] by Hernández, Rabello and Henríquez, again with local Lipschitz conditions for uniqueness.

1.2 A simple, typical example

We claim that even the simplest impulsive equations with state-dependent delay can have multiple solutions when we allow for discontinuous initial conditions. Since the latter is strictly necessary in discussions of continuation of solutions, any pathology in this class of system should be observable from an example initial-value problem that features the following two ingredients:

- a state-dependent delay in the continuous-time dynamics;
- a discontinuous initial condition *or* a non-trivial impulse effect.

With this in mind, consider the following "trivially" impulsive differential equation with state-dependent delay and initial condition x_0 at time zero:

$$\begin{aligned} x'(t) &= x(t-\tau(x(t))), && t \neq t_k, \\ \Delta x(t) &= 0, && t = t_k, \end{aligned} \tag{3}$$

$$x_0(\theta) = \begin{cases} 1, & \theta = 0 \\ 0, & -1 \leq \theta < 0 \\ 2, & -2 \leq \theta < -1, \end{cases} \tag{4}$$

with delay function $\tau(y) = (2y)^2/(1+y^2)$. The delay has range in the interval $[0, 2]$, and τ is C^1. As the impulse effect is trivial, *the Cauchy problem with Lipschitz continuous initial data is well-posed and has a unique solution* [2, 33]. The data x_0 is not continuous, but it is piecewise smooth. We claim

$$x^{(1)}(t) = \begin{cases} 1, & t \in [0,1] \\ 0, & t \in [-1,0) \\ 2, & t \in [-2,-1), \end{cases} \qquad x^{(2)}(t) = \begin{cases} 1+\sqrt{2}, & t \in (1/\sqrt{2}, 1] \\ 1+2t, & t \in [0, 1/\sqrt{2}] \\ 0, & t \in [-1,0) \\ 2, & t \in [-2,-1), \end{cases} \tag{5}$$

are both solutions of the initial-value problem (3)–(4) defined on the common domain $[-2, 1]$. To verify this, observe that for $t \in (0, 1/\sqrt{2})$,

$$z^{(1)}(t) := t - \tau(x^{(1)}(t)) = t - 1, \qquad z^{(2)}(t) := t - \tau(x^{(2)}(t)) = t - \frac{2(1+2t)^2}{1+(1+2t)^2} < -1, \tag{6}$$

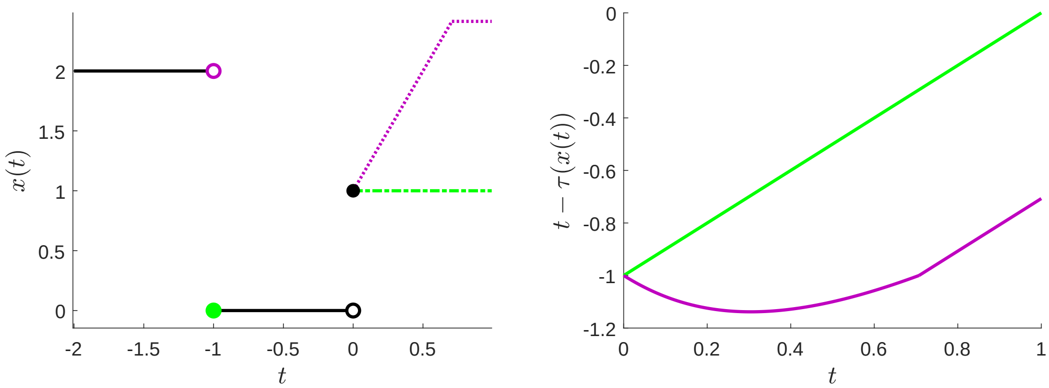

Figure 1: Two distinct solutions (left) of the initial-value problem (3)–(4) and their associated time lags (right). Discontinuities in the initial condition are indicated by solid dots (function value) and hollow dots (left-limit). The time lag for the green solution (dashed-dot line) initially flows according to the continuous history segment on $[-1, 0)$, while the time lag for the purple solution (dotted line) flows according to the one on $[-2, -1)$.

so $x^{(1)}(t - \tau(x^{(1)}(t))) = 0$ and $x^{(2)}(t - \tau(x^{(2)}(t))) = 2$ on this interval. Hence, for $t \in [0, 1/\sqrt{2}]$,

$$x_0(0) + \int_0^t x^{(1)}(s - \tau(x^{(1)}(s)))ds = 1 + \int_0^t 0ds = 1 = x^{(1)}(t)$$
$$x_0(0) + \int_0^t x^{(2)}(s - \tau(x^{(2)}(s)))ds = 1 + \int_0^t 2ds = 1 + 2t = x^{(2)}(t).$$

Also, $z^{(1)}(t) = t - 1$ and $z^{(2)}(t) = t - 1 - 1/\sqrt{2}$ for $t \in [1/\sqrt{2}, 1]$, so in particular, $x^{(1)}(t - \tau(x^{(1)}(t))) = 0$ and $x^{(1)}(t - \tau(x^{(1)}(t))) = 0$ on this interval. Taking into account,

$$x_0(0) + \int_0^{1/\sqrt{2}} x^{(2)}(s - \tau(x^{(2)}(s)))ds + \int_{1/\sqrt{2}}^t x^{(2)}(s - \tau(x^{(2)}(s)))ds$$
$$= x^{(2)}(1/\sqrt{2}) + \int_{1/\sqrt{2}}^t 0ds = x^{(2)}(t).$$

Therefore, $x^{(2)}$ is a solution (in an integrated sense) of the initial-value problem. Similarly, one can check that $x^{(1)}$ is a solution. The initial condition is piecewise-constant, so the lack of uniqueness is entirely due to the discontinuity.

A bit more analysis can give hints about why this non-uniqueness happened. First, $\tau(x_0(0)) = 1$, and -1 is a point of discontinuity of x_0. Second, we have the rather suggestive equalities

$$\frac{d}{dt}\tau(x^{(1)}(t))\Big|_{t=0^+} = 0 = \lim_{s \to -1^+} x_0(s), \qquad \frac{d}{dt}\tau(x^{(2)}(t))\Big|_{t=0^+} = 2 = \lim_{s \to -1^-} x_0(s).$$

That is, it seems as though the "differential equation for delay", $\tau(x(t))$, is itself ill-posed. The discontinuity in x_0 results in two separate directions the time lag $t - \tau(x(t))$ can flow. See Figure 1.2 for a visualization. We show in Section 3.3 that this phenomenon is fairly typical, and additional conditions on the delay τ and the functional f must generally be imposed to prevent it from occurring.

1.3 Winston's monotone lag condition

One of the earliest papers on uniqueness of solutions for differential equations with state-dependent delay is due to Elliot Winston [42] in 1974. He considers the initial-value problem

$$x'(t) = F(x(t), x(t - g(x(t)))), \qquad x_0 = \phi,$$

for ϕ continuous, and proves the following: if D is a domain in $\mathbb{R}^n$, $F : D \times D \to \mathbb{R}^n$ is locally Lipschitz continuous, $g : D \to \mathbb{R}^+$ has Lipschitz first derivative and there exists $\eta > 0$ such that $|y| \leq \eta$ implies

$$\nabla g(x) F(x, y) < 1$$

for all $x \in D$, then the above initial-value problem has a unique solution provided $||\phi|| < \eta$. His proof is based on the observation that under this condition, the lag function $t \mapsto t - g(x(t))$ is strictly increasing along any solution and that this lag function in some sense determines the solution for small time. Our observation with the present paper is that Winston's lag condition can also be exploited in the case of *discontinuous* initial functions ϕ, and the result is once again uniqueness of solutions. As a consequence, it can be adapted to equations with impulses.

1.4 Linearized stability

As mentioned in the third paragraph of Section 1, stability analysis of impulsive differential equations with state-dependent delay in the continuous-time dynamics has yet to be studied in any depth. It can be argued that the most direct approach to stability is to infer this information from linearization – that is, through a principle of *linearized stability*. This has been accomplished for impulsive functional differential equations for C^1 right-hand sides with Lipschitz derivatives [9], but of course this situation does not accommodate state-dependent delay. We will remedy this by introducing a formal linearization approach analogous to that of Cooke and Huang [11]. While that paper does indeed prove a linearized stability, the conceptual linearization done in that paper was not fully resolved until the work of Walther [38] rigorously derived the linear variational equation and interpreted it in the context of the solution manifold. In the present paper we will consider only linearized stability, and make no effort to formalize the linearization process itself.

1.5 The phase space, auxiliary assertions and definitions

Let $G^+(\mathcal{I}, \Omega)$ be the space of right-continuous regulated functions (continuous from the right with finite limits on the left at each point in the domain) defined on an interval $\mathcal{I}$ and mapping into $\Omega \subset \mathbb{R}^n$. For $\mathcal{I}$ compact, this space is complete with respect the the supremum norm (provided Ω is closed). We write $G^+(\Omega) \equiv G^+([-r, 0], \Omega)$, and when we use this symbol without any modifiers, we will be referring to the Banach space $(G^+(\Omega, || \cdot ||_\infty)$, where $\mathbb{R}^n$ is itself interpreted as the normed vector space $(\mathbb{R}^n, |\cdot|)$, with $|\cdot|$ any suitable norm on $\mathbb{R}^n$.

For a function $f : X_1 \times \cdots \times X_k \to Y$ for Banach spaces $X_1, \ldots, X_k$ and Y, the partial Fréchet derivative with respect to the jth variable is denoted $D_j f$.

Define the function space $G^{+,\mathrm{Lip}}(\Omega) = \bigcup_{k \geq 0} G^{+,\mathrm{Lip}(k)}(\Omega)$, with

$$G^{+,\mathrm{Lip}(k)}(\Omega) = \left\{ \begin{array}{l} \phi \in G^+(\Omega) : \forall x \in [-r, 0) \text{ and } y \in (-r, 0], \exists \epsilon_1, \epsilon_2 > 0 \text{ such that } \phi|_{[x, x+\epsilon_1]} \\ \text{and } \phi|_{[y-\epsilon_2, y)} \text{are Lipschitz continuous with Lipschitz constant at most } k \end{array} \right\}.$$

Define the *upper (vector) Dini derivative* of a function $\phi : \mathbb{R} \to \mathbb{R}^n$ component-wise as $D^+\phi(t) = (D^+\phi_1(t), \ldots, D^+\phi_n(t))$ whenever it exists, where

$$D^+\phi_i(t) = \limsup_{h \to 0^+} \frac{\phi(t + h) - \phi(t)}{h}.$$

Note that if ϕ is *locally Lipschitz from the right* at t — that is, $\phi|_{[t,t+\epsilon)}$ is Lipschitz continuous for some $\epsilon > 0$ — then $D^+\phi(t)$ exists.

In the following sections, we will typically assume f and g from (1)–(2) are functions of the form $f : \mathbb{R} \times \Omega \times \Omega \to \mathbb{R}^n$ and $g : \mathbb{R} \times G^+(\Omega) \to \mathbb{R}^n$, for some $\Omega \subset \mathbb{R}^n$ open. Specific conditions of regularity will be specified as needed.

Remark 1. *As we will see in Theorem 5, the functional form of g has no impact on local uniqueness of solutions. As such, we have left it very general. For example, the state-dependent delayed impulse effect of the form*

$$\Delta x(t) = \begin{cases} \mathbf{g}(t, x(t^-), x(t - \tau(x(t^-)))), & \tau(x(t^-)) \neq 0 \\ \mathbf{g}(x(t^-), x(t^-)), & \tau(x(t^-)) = 0 \end{cases}$$

can be included by imposing $g(t, \phi) = \mathbf{g}(t, \phi(0), \phi(-\tau(\phi(0))))$. The "piecewise" definition here is needed to resolve the ambiguity in the composition $x_{t^-}(-\tau(x(t^-))$ when $\tau(x(t^-)) = 0$. The impulse effect above can be equivalently written in a more functional form as

$$\Delta x(t) = \mathbf{g}(t, x_{t^-}(0), x_{t^-}(-\tau(x(t^-)))).$$

The form of g will, however, be relevant in Section 4 for linearized stability.

1.6 Structure of the paper

Section 2 is concerned with existence and continuability of solutions; the results in this section are not new, but are needed for further discussions. Uniqueness is considered in Section 3, with our converse result appearing in Section 3.3, where we show that a Winston-type lag monotonicity condition is typically necessary if one wishes to ensure uniqueness of solutions. We prove a linearized stability result in Section 4. We conclude with an application in Section 5 for a scalar equation with negative feedback, nonlinear state-dependent delays, and affine-bounded impulses on the positive half-line. Section 6 concludes with a discussion.

2 Existence of solutions

Let $\Omega \subset \mathbb{R}^n$ be open. Let $f : \mathbb{R} \times \Omega \times \Omega \to \mathbb{R}^n$ and $g : \mathbb{R} \times G^+(\Omega) \to \mathbb{R}^n$ For some $s \in \mathbb{R}$ and $a > 0$, a function $x : [s - r, s + a] \to U$ is a *solution* of (1)–(2) if $x \in G^+([s - r, s + a], \Omega)$ and

$$x(t) = x(0) + \int_s^t f(\mu, x(\mu), x(\mu - \tau_1(x(\mu))))d\mu + \sum_{s<t_k\leq t} g\left(t_k, x_{t_k^-}\right), \quad t \in [s, s + a], \tag{7}$$

with the integral interpreted in the Lebesgue sense. We can similarly allow solutions to be defined on right-open intervals $[s - r, s + a)$. We say x *satisfies the initial condition* $(s, \phi) \in \mathbb{R} \times G^+(\Omega)$ if $x_s = \phi$.

2.1 The local existence result

The proof of the following existence result can be considered an extension of the proof of Lemma 3.3 from [4], streamlined to make use of the Schauder fixed point theorem and the assumptions H.1–H.6. As the proof is in some sense "typical", we will merely provide a brief outline.

Lemma 1. *Suppose $\tau_1 : \Omega \to \mathbb{R}^+$ is continuous and f is* composite-integrable *and* locally bounded*: for any $x, y \in G^+([s, s + a], \Omega)$ and $s \in \mathbb{R}$, $a > 0$, the function $t \mapsto f(t, x(t), y(t))$ is integrable, the image of a bounded set by f is bounded. For each $(s, \phi) \in \mathbb{R} \times G^+(\Omega)$, there exists a solution $x : [s - r, s + a] \to \Omega$ of (1)–(2) satisfying the initial condition $x_s = \phi$, for some $a > 0$. Moreover, this solution is Lipschitz continuous on $[s, s + a)$.*

Proof (Outline). Without loss of generality, let $s = 0$. Define the function space

$$\mathcal{X} = \{\psi \in C([0,a],\mathbb{R}^n) : ||\psi - \phi(0)||_\infty \leq \delta, \ \ |\psi(t_2) - \psi(t_1)| \leq k|t_2 - t_1| \ \forall t_1, t_2 \in [0,a]\}$$

parameterized by some constants a, δ and k. Define a map $j : C([0,a],\mathbb{R}^n) \to G^+([-r,a],\mathbb{R}^n)$ by

$$j\psi(t) = \begin{cases} \psi(t), & t \in (0,a] \\ \phi(t), & t \in [-r,0]. \end{cases}$$

For $\psi \in C([0,a],\mathbb{R}^n)$, we will write $\tilde{\psi} = j\psi$. Using the conditions of the lemma, one can find constants δ, k and a such that $P : \mathcal{X} \to \mathcal{X}$,

$$P\psi(t) = \phi(0) + \int_0^t f(\mu, \psi(\mu), \psi(\mu - \tau_1(\psi(\mu))))d\mu, \tag{8}$$

is well-defined and continuous. As $\mathcal{X}$ is compact, P has a unique fixed point. In particular, one can take a small enough so that $(0,a] \cap \{t_k : k \in \mathbb{Z}\} = \emptyset$, and in this way, we conclude that $x : [-r,a] \to \mathbb{R}^n$ defined by

$$x(t) = \begin{cases} z(t), & t \in [0,a] \\ \phi(t), & t \in [-r,0) \end{cases}$$

satisfies $x_0 = \phi$, the integral equation (7), and is Lipschitz continuous on $[0,a]$ with constant k. Also, $\delta > 0$ can be chosen small enough so that elements of $\mathcal{X}$ have range in U_0. □

Corollary 2. *Suppose the conditions of Lemma 1 are satisfied. The restriction of x to any interval of the form $[t_k, t_k + v] \subset [s, s+a]$ with $t_k + v < t_{k+1}$, is Lipschitz continuous.*

Proof (Outline). Since f maps bounded sets to bounded sets, one can always extract a Lipschitz constant from the integral formulation of the solution. Let $s_1, s_2 \in [t_k, t_k + v]$. Then

$$|x(s_1) - x(s_2)| \leq \int_{s_1}^{s_2} |f(\mu, x(\mu), x(\mu - \tau_1(x(\mu))))d\mu| \leq |s_2 - s_1|K$$

for some constant K that depends on x and the enclosing interval $[t_k, t_k + v]$. □

2.2 Prolongation of solutions and maximal interval of existence

For intervals $\mathcal{I}_1$ and $\mathcal{I}_2$, a *prolongation* of a solution $x : \mathcal{I}_1 \to \Omega$ of (1)–(2) with $x_s = \phi$, is a function $y \in G^+(\mathcal{I}_2, \Omega)$ that satisfies (7), such that $\mathcal{I}_1 \subset \mathcal{I}_2$ and $y|_{\mathcal{I}_1} = x$. Again, the proof of the following lemma is "typical", and we omit the proof.

Lemma 3. *Suppose the conditions of Lemma 1 are satisfied and, additionally, for all $t \in \mathbb{R}$, $\phi \in G^+(\Omega)$, we have the inclusion $\phi(0) + g(t, \phi) \in \Omega$. Let $x : \mathcal{I} \to \Omega$ be a solution of (1)–(2) with $x_s = \phi \in G^+(\Omega)$. If $\sup \mathcal{I} = b < \infty$, x admits a prolongation if and only if $\lim_{t \to b^-} x(t) \in \Omega$.*

Again a typical result, we have a statement concerning maximal prolongations of any given solution. A prolongation $y : \mathcal{I}_2 \to U_0$ of $x : \mathcal{I}_1 \to U_0$ is *maximal* if there is no prolongation $z : \mathcal{I}_3 \to U_0$ with $\mathcal{I}_3 \supset \mathcal{I}_2$. The following can be proven using the standard argument (e.g. based on Zorn's lemma), and is omitted.

Lemma 4. *Suppose the conditions of Lemma 3 are satisfied. Any solution $x : \mathcal{I} \to \Omega$ of (1)–(2) satisfying $x_s = \phi$ for some $\phi \in G^+(\Omega)$ admits a maximal prolongation.*

3 Uniqueness of solutions

In this section we will prove local and global uniqueness of solutions of (1)–(2) under a Winston-type monotone lag condition, plus some expected regularity conditions on f and τ_1.

3.1 Local uniqueness of solutions

Our first result concerns local uniqueness of solutions.

Theorem 5. *Suppose the following conditions are satisfied.*

1. *For all $U \subset \mathbb{R}$ and $K \subset \Omega$ compact, there exists $L > 0$ such that $|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq L(|x_1 - x_2| + |y_1 - y_2|)$ for $x_1, x_2, y_1, y_2 \in K$ and $t \in U$.*
2. *$\tau_1 : \Omega \to [0, r]$ is continuously differentiable and the* monotone lag condition *is satisfied:*

$$1 - \nabla\tau_1(x)f(t, x, y) \geq 0 \tag{9}$$

for all $x, y \in \Omega$ and $t \in \mathbb{R}$.

Then, for each $s \in \mathbb{R}$ and $\phi \in G^{+,\mathrm{Lip}}(\Omega)$, there exists $a > 0$ such that (1)–(2) *has a unique solution $x : [s - r, s + a] \to \Omega$ satisfying the initial condition $x_s = \phi$.*

Proof. As usual, let $s = 0$ without loss of generality. By Lemma 1, there exists a solution $x : [-r, a] \to \Omega$. We may without loss of generality choose a small enough so that $(0, a] \cap \{t_k : k \in \mathbb{Z}\} = \emptyset$. Suppose there exists another solution $y : [-r, a] \to \Omega$ and that $x \neq y$. Define $t^* = \inf\{t \in [0, a] : x(t) \neq y(t)\}$. Then $t^* \in [0, a)$, and using (7), we have that for $t \in [t^*, a]$,

$$\begin{aligned} x(t) - y(t) = &\int_{t^*}^{t} f(\mu, x(\mu), x(\mu - \tau_1(x(\mu)))) - f(\mu, x(\mu), x(\mu - \tau_1(y(\mu))))d\mu \\ &+ \int_{t^*}^{t} f(\mu, x(\mu), x(\mu - \tau_1(y(\mu)))) - f(\mu, x(\mu), y(\mu - \tau_1(y(\mu))))d\mu \\ &+ \int_{t^*}^{t} f(\mu, x(\mu), y(\mu - \tau_1(y(\mu)))) - f(\mu, y(\mu), y(\mu - \tau_1(y(\mu))))d\mu, \end{aligned} \tag{10}$$

while $x(t) = y(t)$ for $t \in [-r, t^*]$. For $t \in (t^*, a)$, we have

$$D^+(t - \tau_1(y(t))) = 1 - \nabla\tau_1(y(t))f(t, y(t), y(t - \tau_1(y(t))))$$

and by the monotone lag condition, this is non-negative. Since $t \mapsto \tau_1(y(t))$ is continuous on $[0, a)$, we conclude $t \mapsto t - \tau_1(y(t))$ is non-decreasing using (Corollary 11.4.1, [7]) for $i = 1, \ldots, \ell$. The same is true for $t \mapsto t - \tau_1(x(t))$. Define $u(t) = t - \tau(x(t))$ and $v(t) = t - \tau(y(t))$. Then each of u and v are continuous and non-decreasing on $[t^*, a]$. Let $\epsilon > 0$ be small enough so that $x|_{[u(t^*), u(t^*)+\epsilon]}$ is Lipschitz continuous with some constant $k > 0$. Note that this can always be accomplished by using either the assumption that $\phi \in G^{+,\mathrm{Lip}}(\Omega)$ (if $u(t^*) < 0$) or Corollary 2 (if $u(t^*) \geq 0$). Define

$$\delta = \sup\{s \in [t^*, a] : \max\{u(s), v(s)\} \leq u(t^*) + \epsilon\}.$$

Since $u(t^*) = v(t^*)$ and each of u and v is continuous and non-decreasing, we have $\delta > t^*$. Applying this to (10) and using condition 1. of the theorem, there is a constant $L > 0$ such that

$$|x(t) - y(t)| \leq |t - t^*|L\left(k \sup_{s\in[t^*,t]} |u(s) - v(s)| + |x(u(s)) - y(u(s))| + |x(s) - y(s)|\right)$$

for $t \in [t^*, \delta]$. Note that $\sup_{s\in[t^*,t]} |x(u(s)) - y(u(s))| \leq \sup_{s\in[0,t]} |x(s) - y(s)|$ for $t \leq \delta$. Since τ is C^1, there exists another constant $L' > 0$ such that $|u(s) - v(s)| = |\tau(y(s)) - \tau(x(s))| \leq L'|x(s) - y(s)|$ for $s \in [t^*, \delta]$. From here, we conclude that

$$|x(t) - y(t)| \leq |t - t^*|L(k + L' + 1) \sup_{s\in[0,t]} |x(s) - y(s)|, \quad t \in [t^*, \delta].$$

Let $\epsilon' = \frac{1}{2}L^{-1}(k + L' + 1)^{-1}$. Then

$$\sup_{t\in[0,t^*+\epsilon']} |x(t) - y(t)| = \sup_{t\in[t^*,t^*+\epsilon']} |x(t) - y(t)| \leq \frac{1}{2} \sup_{s\in[0,t^*+\epsilon']} |x(s) - y(s)|,$$

which contradicts the definition of t^*. Therefore, $x = y$. □

Remark 2. *Condition 1 of Theorem 5 could be weakened from local Lipschitz continuity to a local Lipschitz-like integrability condition. For example, it is enough to require for each compact $K \subset \Omega$ and $U \subset \mathbb{R}$ the existence of an integrable function $L : U \to \mathbb{R}$ such that*

$$\Big| \int_U f(s, x_1(s), x_2(s)) - f(s, y_1(s), y_2(s))ds \Big| \leq \int_U L(s)||x(s) - y(s)||ds$$

for all $s_1 \neq s_2 \in U$, for a suitable norm $|| \cdot ||$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are $G^+(U, K \times K)$.

The following corollary can be useful in applications. Its proof is a straightforward adjustment to the previous, and is omitted.

Corollary 6. *Let D be a closed subset of Ω. Suppose condition 1. of Theorem 5 is satisfied, $\tau_1 : \Omega \to [0, r]$ is C^1, and* (9) *holds for $t \in \mathbb{R}$ and $x, y \in D$. For all $s \in \mathbb{R}$ and $(s, \phi) \in \mathbb{R} \times G^{+,\mathrm{Lip}}(D)$, there exists $a > 0$ such that there is at most one solution of* (1)–(2) *defined on the interval $[s - r, s + a]$ and having range in D.*

3.2 Prolongation and global uniqueness

Similarly to Corollary 3, one can prove the following prolongation result.

Lemma 7. *Suppose the conditions of Theorem 5 and additionally, for $t \in \mathbb{R}$, $\phi \in G^+(\Omega)$, we have the inclusion $\phi(0) + g(t, \phi) \in \Omega$. A solution $x : [s - r, s + a] \to \Omega$ of* (1)–(2) *admits a prolongation if and only if $\lim_{t\to(s+a)^-} x(s) \in \Omega$. In this case, there exists $a' > a$ such that there is a* unique *prolongation $y : [s - r, s + a'] \to \Omega$ of x.*

Subsequently, we can obtain some global uniqueness results. The proofs are straightforward and omitted.

Corollary 8. *Under the assumptions of Lemma 7, exactly one of the following occurs:*

- *the unique solution is defined on $[s - r, \infty)$, or*
- *there is a unique solution $x : [s - r, s + a) \to \Omega$ satisfying $x_s = \phi$ with $a > 0$ finite, and it admits no prolongation: that is, $x(t)$ either becomes unbounded or approaches the boundary of Ω as $t \to (s + a)^-$.*

Corollary 9. *Let the conditions of Corollary 6 hold, and additionally, for $t \in \mathbb{R}$, $\phi \in G^+(D)$ and $\tau \in [0, r]$, we have the inclusion $\phi(0) + g(t, \phi) \in D$. For any $(s, \phi) \in \mathbb{R} \times G^{+,\mathrm{Lip}}(D)$, there exists at most one solution of* (1)–(2) *satisfying $x_s = \phi$ having range in D, and if such a solution exists, exactly one of the following occurs:*

- *the unique solution is defined on $[s - r, \infty)$, or*
- *there is a unique solution $x : [s - r, s + a) \to D$ with $a > 0$ finite, and it admits no prolongation: specifically, $x(t)$ becomes unbounded or approaches the boundary of D as $t \to (s + a)^-$.*

3.3 A converse theorem

In this section, we demonstrate that a constraint akin to the monotone lag condition is necessary if one wishes to guarantee uniqueness of solutions, at least for the case of discrete state-dependent delays. First, a preparatory lemma. Its proof is straightforward.

Lemma 10. *Let $f(t,x,y) = f_0(x,y)$ for some locally Lipschitz continuous function $f_0 : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$. Suppose $(0,a]\cap\{t_k : k\in\mathbb{Z}\} = \emptyset$. If $x : [s-r, s+a]\to\mathbb{R}$ for $a>0$ satisfies $x_s = \phi$ and $t-\tau(x(t))\leq 0$ for $t\in[s,s+a]$, then x is a solution of (1)–(2) if and only*

$$x(t) = \phi(0) + \int_s^t f_0(\phi(\mu), \phi(\mu - \tau_1(x(\mu))))d\mu, \quad t\in[s,s+a].$$

With this lemma at hand, we will show that if the monotone lag condition is violated, it is generally possible to construct an initial condition in $G^{+,\mathrm{Lip}}$ such that the associated Cauchy problem is ill-posed. This is the content of the following theorem, which is in some sense "constructive". See Figure 3.3 for a visualization.

Theorem 11. *Let $f(t,x,y) = f_0(x,y)$ for some locally Lipschitz continuous function $f_0 : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$. Suppose $\tau_1 : \mathbb{R}\to[0,r]$ is C^1. Let $\phi\in G^{+,\mathrm{Lip}}(\mathbb{R})$ and denote $v = -\tau(\phi(0))$. If $v\in(-r,0)$ and*

$$1 - \nabla\tau(\phi(0))f_0(\phi(0),\phi(v^-)) < 0 < 1 - \nabla\tau(\phi(0))f(\phi(0),\phi(v)), \tag{11}$$

then there are at least two distinct solutions of (1)–(2) satisfying the initial condition (s,ϕ). The same holds if the above inequalities are reversed.

Proof. Without loss of generality, let $s=0$. Since $\phi\in G^{+,\mathrm{Lip}}(\mathbb{R})$, there exists $b_1>0$ and $b_2>0$ such that $\phi|_{[v,v+b_1]}$ and $\phi|_{[v-b_2,v)}$ are Lipschitz continuous. Denote $\phi(v^-) = \lim_{s\to v^-}$, and define

$$\phi_1(\theta) = \begin{cases} \phi(0) + \frac{\theta}{v+b_1}(\phi(v+b_1) - \phi(0)), & \theta\in[v+b_1, 0] \\ \phi(\theta), & \theta\in[v, v+b_1) \\ \phi(v), & \theta\in[-r, v) \end{cases}$$

$$\phi_2(\theta) = \begin{cases} \phi(0) + \frac{\theta}{v}(\phi(v^-) - \phi(0)), & \theta\in[v, 0] \\ \phi(\theta), & \theta\in[v-b_2, v) \\ \phi(v-b_2), & \theta\in[-r, v-b_2). \end{cases}$$

Each of these functions is Lipschitz continuous, from which it follows [2, 33] that (1)–(2) has a unique solution $x^{(i)} : [-r,a]\to\mathbb{R}$ satisfying $x_0^{(i)} = \phi_i$, for $i=1,2$, defined on a (mutual) interval $[-r,a]$ such that $(0,a]\cap\{t_k : k\in\mathbb{Z}\}\neq\emptyset$. In particular, as these solutions are classical (i.e., differentiable on $(0,a)$),

$$\lim_{t\to0^+}\frac{d}{dt}x^{(1)}(t) = f_0(\phi(0),\phi(v)), \quad \lim_{t\to0^+}\frac{d}{dt}x^{(2)}(t) = f_0(\phi(0),\phi(v^-)).$$

By (11), these derivatives are not equal and it follows that $x^{(1)}(t)\neq x^{(2)}(t)$ for $t>0$ small enough, so the solutions are distinct. Denote $z_x^{(i)}(t) = t - \tau(x^{(i)}(t))$ for $i=1,2$. These functions are continuous (in fact, differentiable), and we have

$$D^+z_x^{(i)}(t) = 1 - \nabla\tau(x^{(i)}(t))f(x^{(i)}(t), x^{(i)}(t-\tau(x^{(i)}(t)))).$$

By continuity and (11), $z_x^{(1)}$ is increasing while $z_x^{(2)}$ is decreasing on some interval $[0,a']$, for $a'\in(0,a]$. Let ϵ_1' and $\epsilon_2'>0$ be small enough so that $z_x^{(1)}(t)\in[v,v+b_1]$ for $t\in[0,\epsilon_1']$ and $z_x^{(2)}(t)\in[v-b_2,v]$ for $t\in[0,\epsilon_2']$. Then, for $t\in[0,\epsilon_i']$,

$$x^{(i)}(t) = \phi(0) + \int_0^t f(x^{(i)}(\mu), x^{(i)}(z_x^{(i)}(\mu)))d\mu = \phi(0) + \int_0^t f(x^{(i)}(\mu), \phi_i(z_x^{(i)}(\mu)))d\mu. \tag{12}$$

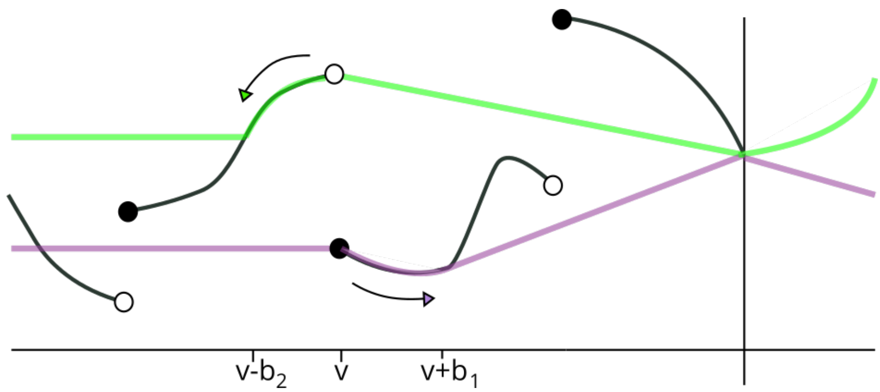

Figure 2: Visual description of the proof of Theorem 11. A discontinuity at $v \in [-r, 0]$ is used to construct two Lipschitz continuous "surrogate" initial conditions ϕ_1 (purple, bottom curve) and ϕ_2 (green, top curve) based on the true, discontinuous initial condition ϕ (black). The flow of the lagged variable $t \mapsto t - \tau(x(t))$ is initially (for $t > 0$ small) increasing for the surrogate initial condition ϕ_1, while for ϕ_2 it is decreasing. This is indicated by arrows in the figure. Since these portions of ϕ_1 and ϕ_2 track the true initial condition for $t > 0$ small, the solutions of the modified initial-value problems (for initial conditions ϕ_1 and ϕ_2) still satisfy the original initial-value problem.

Now define $\tilde{x}^{(i)} : [-r, \min\{\epsilon_1', \epsilon_2'\}] \to \mathbb{R}$ by $\tilde{x}^{(i)}(t) = x^{(i)}(t)$ for $t \geq 0$, and $\tilde{x}^{(i)}(t) = \phi(t)$ for $t < 0$. By definition of ϵ_1', ϵ_2' and the respective ϕ_1 and ϕ_2, these "modified" solutions still satisfy (12). By Lemma 10, each of the $\tilde{x}^{(i)}$ are solutions of (1)–(2), satisfy $\tilde{x}_0^{(i)} = \phi$, and are not equal for $t > 0$ small. □

4 Linearized stability

In this section we will establish a linearized stability result that holds regardless of whether we have uniqueness of solutions. The functional form of g will be relevant in this instance, and we will assume throughout that

$$g(t, \phi, \tau) = \tilde{g}(t, \phi(0), \phi(-\tau_2(\phi(0))))$$

for a suitable $\tilde{g} : \mathbb{R} \times \Omega \times \Omega \to \mathbb{R}^n$ and $\tau_2 : \Omega \to [0.r]$. In what follows we will abuse notation and drop the tilde, writing formally $g(t, \phi, \tau) = g(t, \phi(0), \phi(-\tau))$. This should not cause confusion.

In this section, we will assume $0 \in \Omega$ and the following baseline hypotheses on f and g.

- H.1 For all $t \in \mathbb{R}$, $f(t, 0, 0) = 0$ $f(t, \cdot, \cdot) : \Omega \times \Omega \to \mathbb{R}^n$ is C^1, and $Df(t, \cdot, \cdot)$ is locally Lipschitz continuous uniformly in $t \in \mathbb{R}$.
- H.2 For all $t = t_k$, $g(t, 0, 0) = 0$, $g(t, \cdot, \cdot) : \Omega \times \Omega \to \mathbb{R}^n$ is C^1, and $Dg_0(t, \cdot, \cdot)$ is locally Lipschitz continuous uniformly in t, and $x + g(t, x, y) \in \Omega$ whenever $x, y \in \Omega$.
- H.3 There exists $\xi > 0$ such that $t_{k+1} - t_k \geq \xi$ for all $k \in \mathbb{Z}$ (respectively, $k \in \mathbb{N}$).

Definition 1. *Suppose $f(t, 0, 0) = 0$ and $g(t, 0, 0) = 0$ for all $t \in \mathbb{R}$. The solution $x = 0$ of the impulsive differential equation with state-dependent delay (1)–(2) is* exponentially stable *if there exist constants $K \geq 1, \alpha > 0$ and $\eta > 0$ such that for all $(s, \phi) \in \mathbb{R} \times G^+(\mathbb{R}^n)$, any solution $x : \mathcal{I} \to \mathbb{R}^n$ of (1)–(2) satisfying the initial condition $x_s = \phi$ admits the exponential bound $||x_t|| \leq Ke^{-\alpha(t-s)}||\phi||$ whenever $||\phi|| \leq \eta$.*

Note that assuming $x = 0$ is an equilibrium solution (in H.1 and H.2) and defining stability relative to this solution is not a great restriction. If $\tilde{x}$ is a bounded solution, one can perform a time-dependent affine change of coordinates to translate it to zero.

The result we will prove is the following.

Theorem 12. *Suppose each of τ_1 and τ_2 are continuously differentiable and conditions H.1–H.3 are satisfied. If the linear impulsive delay differential equation*

$$y'(t) = D_2 f(t,0,0)y(t) + D_3 f(t,0,0)y(t-\tau_1(0)), \qquad t \neq t_k \tag{13}$$

$$\Delta y(t) = D_2 g(t,0,0)y(t^-) + D_3 g(t,0,0)y_{t^-}(-\tau_2(0)), \qquad t = t_k \tag{14}$$

is exponentially stable — that is, there exists $K_0 \geq 1$ and $\alpha_0 > 0$ such that for $(s,\phi) \in \mathbb{R} \times G^+(\mathbb{R}^n)$ all solutions $y : [s-r,\infty) \to \mathbb{R}^n$ satisfy the exponential bound $||y_t|| \leq K_0 e^{-\alpha_0(t-s)}||\phi||$ for $t \geq s$ — then the solution $x = 0$ of (1)–(2) is also exponentially stable.

The proof appears in Section 4.2. A brief remark: exponential stability of the linear system in Theorem 12 is equivalent to the associated evolution family $U(t,s)$ satisfying $||U(t,s)|| \leq K_0 e^{-\alpha_0(t-s)}$. See later Proposition 20. We do not prove the natural converse of this theorem, which is that if the formal linearized system is strongly unstable (the unstable fibre bundle is non-empty; see [10]), then the trivial solution is unstable in (1)–(2). The proof of this result is rather more technical and is postponed to future research.

Before we continue, we need to define a few extra pieces of notation. If S is a set, we denote its cardinality by $\#S$. If $x, y \in \mathbb{R}$, we define the convex hull $H(x,y) = [\min\{x,y\}, \max\{x,y\}]$.

4.1 Preparatory results

Proposition 13. *Let $x \in G^{+,\mathrm{Lip}(k)}(\mathbb{R}^n)$ and $D = \{u \in [-r,0] : x(u) \neq x(u^-)\}$. If D is finite, then*

$$|x(t) - x(s)| \leq k|t-s| + \sum_{u \in D} |x(u) - x(u^-)|. \tag{15}$$

Proof. Let $s \leq t$ and $[s,t] \cap D = \{d_1, \ldots, d_N\}$ with $d_1 < d_2 < \cdots < d_N$. Denote $\Delta x(t) = x(t) - x(t^-)$. Without loss of generality, assume $s < d_1 < d_N < t$. Then

$$\begin{aligned}
|x(t) - x(s)| &= \left| x(t) - x(d_N) + \sum_{n=1}^{N-1}(x(d_{n+1}) - x(d_n)) + x(d_1) - x(s) \right| \\
&= \left| x(t) - x(d_N) + \sum_{n=1}^{N-1}(x(d_{n+1}^-) + \Delta x(d_{n+1}) - x(d_n)) + x(d_1) - x(s) \right| \\
&\leq \left| x(t) - x(d_N) + \sum_{n=1}^{N-1}(x(d_{n+1}^-) - x(d_n)) + x(d_1^-) - x(s) \right| + \left| \sum_{n=1}^{N-1} \Delta x(d_{n+1}) + \Delta x(d_1) \right| \\
&\leq |x(t) - x(d_N)| + \sum_{n=1}^{N-1} |x(d_{n+1}^-) - x(d_n)| + |x(d_1^-) - x(s)| + \sum_{m=1}^{N} |\Delta x(d_m)|,
\end{aligned}$$

and the result follows because x is k-Lipschitz continuous on the intervals $[s,d_1)$, $[d_1,d_2), \ldots, [d_{N-1},d_N)$, $[d_N,t]$. ☐

Proposition 14. *Let $\tau_1 : \Omega \to [0,r]$ be C^1. For any $\epsilon > 0$, there exists $\delta > 0$ such that for $s \in \mathbb{R}$, $a > 0$ and $x \in G^+([s-r,s+a],\Omega)$ the function*

$$F(t) = f(t,x(t),x(t-\tau_1(x(t)))) - D_2 f(s,0,0)x(t) - D_3 f(s,0,0)x(t-\tau_1(0))$$

defined for $t \in [s, s+a]$ satisfies

$$|F(t)| \leq \epsilon ||x_t|| + ||D_3 f(t,0,0)|| \cdot |x(t-\tau_1(x(t))) - x(t-\tau_1(0))| \tag{16}$$

provided $\sup_{v \in [s-r, s+a]} |x(v)| \leq \delta$.

Proof. We can write

$$\begin{aligned}
F(t) = & \int_0^1 D_2 f(t, \mu x(t), \mu x(t-\tau_1(x(t))))x(t) + D_3 f(t, \mu x(t), \mu x(t-\tau_1(x(t))))x(t-\tau_1(x(t)))d\mu \\
& - D_2 f(t,0,0)x(t) - D_3 f(t,0,0)x(t-\tau_1(0)) \\
= & \int_0^1 [D_2 f(t, \mu x(t), \mu x(t-\tau_1(x(t)))) - D_2 f(t,0,0)]x(t) d\mu \\
& + \int_0^1 [D_3 f(t, \mu x(t), \mu x(t-\tau_1(x(t)))) - D_3 f(t,0,0)]x(t-\tau_1(x(t)))d\mu \\
& + \int_0^1 D_3 f(t,0,0)[x(t-\tau_1(x(t))) - x(t-\tau_1(0))]d\mu
\end{aligned}$$

Using the uniform local Lipschitz continuity of Df (and hence the partial derivatives $D_2 f$ and $D_3 f$), we get the claimed result. □

Lemma 15. *Assume the hypotheses of Proposition 14 and that $\tau_1 : \Omega \to [0, r]$ is C^1. For all $\epsilon > 0$, there exist $\delta, N, M > 0$ such that if $x \in G^+([s-r, s+a], \Omega)$ has only finitely-many discontinuities, the bound* (16) *can be refined further to*

$$|F(t)| \leq \epsilon ||x_t|| + N \left(Mk||x_t|| + \sum_{u \in D_t(x)} |x_t(u) - x_t(u^-)| \right), \tag{17}$$

provided $x_t \in G^{+,\mathrm{Lip}(k)}(\Omega)$ and $\sup_{v \in [s-r,s+a]} |x(v)| \leq \delta$, where

$$D_t(x) = \{v \in H(-\tau_1(x(t)), -\tau_1(0)) : x_t(v) \neq x_t(v^-)\}.$$

Also, δ, N, M can be chosen such that they remain bounded as $\epsilon \to 0$.

Proof. Let $N = \sup_{v \in \mathbb{R}} ||D_3 f(v,0,0)||$. Without loss of generality, assume $\tau_1(x(t)) \geq \tau_1(0)$. We can bound $|x(t-\tau_1(x(t))) - x(t-\tau_1(0))|$ using Proposition 13, taking into account that the discontinuities of $x_t(\cdot)$ are precisely the elements of $D_t(x)$. With $\sup_{|y| \leq \delta} |\nabla \tau(y)| \leq M$, we get (17) by applying Proposition 14. □

Remark 3. *The length of the interval $H(-\tau_1(x(t)), -\tau_1(0))$ is bounded above by $M|x(t)| \leq M||x_t||$. Consequently, if $x : [s-r, s+a] \to \Omega$ is a solution and $t \geq s+r$, then the discontinuities of $x_t : [-r, 0] \to \mathbb{R}^n$ are due only to the impulses. By assumption H.3, for such $t \geq s+r$, the interval $H(-\tau_1(x(t)), -\tau_1(0))$ contains at most (the integer floor of) $\xi^{-1}M||x_t||$ discontinuities. Therefore $\#(D_t(x)) \leq \xi^{-1}M||x_t||$. This will be incredibly important later.*

In an analogous fashion, one can obtain a bound for g.

Lemma 16. *Let $\tau_2 : \Omega \to [0, r]$ be C^1. For all $\epsilon > 0$, there exist $\nu, \mathcal{N}, \mathcal{M} > 0$ such that if $x \in G^+([s-r, s+a], \Omega)$ has only finitely-many discontinuities, the function*

$$G(k) = g(t_k, x_{t_k^-}(0), x_{t_k^-}(-\tau(x(t_k^-)))) - D_2 g(t_k, 0, 0)x_{t_k^-}(0) - D_3 g(t_k, 0, 0)x_{t_k^-}(-\tau_2(0))$$

defined for $k \in \mathbb{Z}$ (or $k \in \mathbb{N}$, for one-sided indexed impulses) satisfies

$$|G(k)| \leq \epsilon||x_{t_k^-}|| + \mathcal{N}\left(\mathcal{M}k||x_{t_k^-}|| + \sum_{u\in D_{t_k}^-(x)} |x_{t_k^-}(u) - x_{t_k^-}(u^-)|\right), \tag{18}$$

provided $x_{t_k^-} \in G^{+,\mathrm{Lip}(k)}(\Omega)$ and $\sup_{v\in[s-r,s+a]} |x(v)| \leq \nu$, where

$$D_t^-(x) = \{v \in H(-\tau_2(x(t)), \tau_2(0)) : x_t(v) \neq x_t(v^-)\}.$$

Also, $\nu, \mathcal{N}, \mathcal{M}$ can be chosen such that they remain bounded as $\epsilon \to 0$.

Remark 4. *Analogous way to Remark 3, for $x : [s-r, s+a] \to \Omega$ a solution, we have $\#(D_t^-(x)) \leq \xi^{-1}\mathcal{M}||x_{t^-}||$ whenever $t \geq s+r$.*

Proposition 17. *There exists $J > 0$ and $\rho > 0$ such that $|f(t,x,y)| \leq J\max\{|x|,|y|\}$ and $|g(t,x,y)| \leq J\max\{|x|,|y|\}$ for $|x|,|y| \leq \rho$.*

Lemma 18. *If $||\phi||\exp\left(aJ(1+\xi^{-1})\right) \leq \rho$ for some $a > 0$, for the constants J and ρ from Proposition 17, then any solution x of (1)–(2) defined on $[s-r, s+a]$ and satisfying $x_s = \phi$ is uniformly bounded, with $||x_t|| \leq ||\phi||e^{(t-s)J(1+\xi^{-1})}$ for $t \in [s, s+a]$.*

Proof. Define the "non-uniform" left-limit $x_t^-(\theta) = \lim_{s\to(t+\theta)^-} x(s)$. We can easily get the bound

$$||x_t|| \leq ||\phi|| + \int_s^t J||x_\mu||d\mu + \sum_{s<t_k\leq t} J||x_{t_k}^-||$$

for $t \in [s, s+a]$ from the integral equation (7), for any solution defined on the interval $[s-r, s+a]$ that remains bounded by ρ. Define $X(t) = ||x_t||$. Then $t \mapsto X(t)$ is right-continuous with limits on the left (Lemma 3.1.1, [10]), and a straightforward verification shows that $\lim_{t\to t_k^-} X(t) = ||x_{t_k}^-||$. Therefore,

$$X(t) \leq ||\phi|| + \int_s^t J|X(\mu)|d\mu + \sum_{s<t_k\leq t} JX(t_k^-).$$

By the impulsive Gronwall inequality (Lemma 3.2.1, [10]),

$$X(t)| \leq ||\phi||e^{(t-s)J}(1+J)^{\#\{k:t_k\in(s,t]\}} \leq ||\phi||e^{(t-s)J(1+\xi^{-1})}.$$

Suppose that there exists $\bar{t} \in (s, s+a]$ such that $|x(t)| > \rho$. Then $t^* = \inf\{t > s : |x(t)| \geq \rho\}$ exists, and $t^* > s$. By the above argument, $|x(t)| \leq \rho$ for $t \in [s-r, t^*)$, so $x(t) \leq ||\phi||e^{(t-s)J(1+\xi^{-1})}$ for $t \in [s, t^*)$. If $t^* \notin \{t_k : k \in \mathbb{Z}\}$ and $t^* < s+a$, then x is continuous at t^*, and the condition on $||\phi||$ implies that $|x(t^*)| < \rho$. This contradicts the definition of t^*. Conversely, if $t^* = t_k$ for some k, then

$$|x(t_k)| \leq (1+J)X(t_k^-) \leq (1+J)||\phi||e^{(t_k-s)J}(1+J)^{\#\{j:t_j\in(s,t_k)\}} \leq ||\phi||e^{(t_k-s)J(1+\xi^{-1})} < \rho,$$

which by continuity of x on the right, contradicts the definition of t^*. □

To make the exponential bound of Lemma 18 useful, we need to ensure that all solutions from the relevant initial condition are continuable to time $s+a$. This can be done by suitably restricting to a smaller closed neighbourhood of zero for initial conditions ϕ. The proof is straightforward and therefore omitted.

Corollary 19. *If Ω is open, $0 \in \Omega$, there exists $\rho' \in (0,\rho]$ such that if $||\phi|| \exp\left(aJ(1+\xi^{-1})\right) \leq \rho'$ for some $a > 0$, every solution x of (1)–(2) satisfying $x_s = \phi$ is continuable to $[x-s, x+a]$ and satisfies $||x_t|| \leq ||\phi|| e^{(t-s)J(1+\xi^{-1})}$ for $t \in [s, s+a]$.*

Finally, we will require a result concerning the variation-of-constants representation of solutions for impulsive delay differential equations in the phase space of right-continuous regulated functions. The following is a consequence of the theory in [10].

Proposition 20. *If $\mathbf{f} : [s, s+a] \times G^+(\Omega) \to \mathbb{R}^n$ has the property that $t \mapsto \mathbf{f}(t, x_t)$ is regulated for any $x \in G^+([s-r, s+a], \Omega)$, then any solution $z : [s-r, s+a] \to \mathbb{R}^n$ of the semilinear impulsive functional differential equation*

$$\begin{aligned} z'(t) &= Az_t + \mathbf{f}(t, z_t), && t \neq t_k \\ \Delta z(t) &= Bz_{t^-} + \mathbf{g}(k, z_{t^-}), && t = t_k, \end{aligned}$$

satisfies the variation-of-constants formula

$$z_t = U(t,s)z_s + \int_s^t U(t,\mu)\chi_0 \mathbf{f}(\mu, z_\mu) d\mu + \sum_{s < t_k \leq t} U(t, t_k)\chi_0 \mathbf{g}(k, z_{t_k^-}), \quad t \geq s,$$

with the integral interpreted in the Gelfand-Pettis sense, $\chi_0 h(\theta) = h$ for $\theta = 0$ and $\chi_0 h(\theta) = 0$ for $\theta < 0$ and $h \in \mathbb{R}^n$, and $U(t,s) : G^+(\mathbb{R}^n) \to G^+(\mathbb{R}^n)$ the evolution family associated to the linear system

$$\begin{aligned} u'(t) &= Au_t, && t \neq t_k \\ \Delta u(t) &= Bu_{t^-}, && t = t_k. \end{aligned}$$

4.2 Proof of Theorem 12

Recall the constants δ, N, M introduced in Lemma 15, the constants $\nu, \mathcal{N}, \mathcal{M}$ of Lemma 16, and ρ' from Corollary 19. Choose $\zeta \in (0, \rho']$ and $\epsilon > 0$ small enough so that

$$\epsilon + \max\{NM, \mathcal{N}\mathcal{M}\} J\zeta(1+\xi^{-1}) < K_0^{-1}(1+\xi^{-1})^{-1}\alpha_0. \tag{19}$$

Let $||\phi|| \leq \eta$, where $\eta > 0$ is chosen so that

$$\eta e^{rJ(1+\xi^{-1})} K_0 < \min\{\zeta, \delta, \nu\}. \tag{20}$$

Without loss of generality, we may take $s = 0$. First, suppose there exists some time $\bar{t} > 0$ such that a solution x satisfies $|x(t)| > \min\{\zeta, \delta, \nu\}$. Define $t^* = \inf\{t > 0 : |x(t)| \geq \min\{\zeta, \delta, \nu\}\}$. Then $t^* \in (0, \bar{t}]$. We first prove that $t^* > r$. By Corollary 19, $||x_t|| \leq ||\phi|| e^{tJ(1+\xi^{-1})}$ for $t \in [0, t^*)$, and by definition of η, since $K_0 \geq 1$, it follows that $||x_t|| < \min\{\zeta, \delta, \nu\}$ for $t \in [0, r)$. By a similar argument to one appearing near the end of the proof of Lemma 18, we can conclude that in fact, $r < t^*$.

By (7), for $t \in [0, t^*)$ and $c > 0$ small,

$$|x(s_1) - x(s_2)| \leq \int_{s_1}^{s_2} |f(\mu, x(\mu), x(\mu - \tau_1(x(\mu))))| ds \leq |s_2 - s_1| J\zeta$$

for any $s_1, s_2 \in [t, t+c]$. Similarly, for $t \in (0, t^*]$ and $c > 0$ small enough, $|x(s_1) - x(s_2)| \leq |s_2 - s_1| J\zeta$ for $s_1, s_2 \in [t-c, t)$. Therefore, $x_t \in G^{+,\mathrm{Lip}(J\zeta)}(\Omega)$ for $t \in [r, t^*)$, and by (7) has finitely-many discontinuities.

By Proposition 20, we can write

$$x_t = U(t,r)x_r + \int_r^t U(t,\mu)\chi_0 F(\mu) d\mu + \sum_{r < t_k \leq t} U(t, t_k) G(k),$$

for the functions F and G of Lemma 15 and Lemma 16. For $t \in [r, t^*)$, we have $||x_t|| < \min\{\zeta, \delta, \nu\}$, which means by those previous lemmas we can majorize the above as follows:

$$||x_t|| \le K_0 e^{-\alpha_0(t-r)}||x_r|| + \int_r^t K_0 e^{-\alpha_0(t-\mu)} \left(\epsilon ||x_\mu|| + N \left(MJ\zeta ||x_\mu|| + \sum_{u \in D_\mu(x)} |x_\mu(u) - x_\mu(u^-)| \right) \right) d\mu$$
$$+ \sum_{r < t_k \le t} K_0 e^{-\alpha_0(t-t_k)} \left(\epsilon ||x_{t_k^-}|| + \mathcal{N} \left(\mathcal{M} J\zeta ||x_{t_k^-}|| + \sum_{u \in D_{t_k}^-} |x_{t_k^-}(u) - x_{t_k^-}(u^-)| \right) \right)$$

where the Lipschitz constant k has been replaced by ζJ due to the above discussion. Note that for $\mu + u = t_k < t^*$ for some $k \in \mathbb{Z}$, we have

$$|x_\mu(u) - x_\mu(u^-)| \le |g(t_k, x(t_k^-), x_{t_k^-}(-\tau_2(x(t_k^-))))| \le J ||x_{t_k^-}|| \le J\zeta.$$

Now we make use the above and the observations of Remark 3 and Remark 4 to obtain the further bound

$$||x_t|| \le K_0 e^{-\alpha_0(t-r)}||x_r|| + \int_r^t K_0 e^{-\alpha_0(t-\mu)} \left(\epsilon + NMJ\zeta(1+\xi^{-1}) \right) ||x_\mu|| d\mu$$
$$+ \sum_{t < t_k \le t} K_0 e^{-\alpha_0(t-t_k)} \left(\epsilon + \mathcal{N}\mathcal{M} J\zeta(1+\xi^{-1}) \right) ||x_{t_k^-}||.$$

Applying the impulsive Gronwall inequality (Lemma 3.2.1, [10]) and our previous bound for $||x_r||$, it follows that $||x_t|| \le ||\phi|| e^{rJ(1+\xi^{-1})} K_0 e^{\beta(t-r)}$, where

$$\begin{aligned}\beta &= -\alpha_0 + K_0(\epsilon + NMJ\zeta(1+\xi^{-1})) + \xi^{-1} K_0(\epsilon + \mathcal{N}\mathcal{M} J\zeta(1+\xi^{-1})) \\ &\le -\alpha_0 + (1+\xi^{-1}) K_0 \left(\epsilon + \max\{NM, \mathcal{N}\mathcal{M}\} J\zeta(1+\xi^{-1}) \right) \\ &< 0\end{aligned}$$

by (19). Since $||\phi|| e^{rJ(1+\xi^{-1})} K_0 < \min\{\zeta, \delta, \nu\}$ by (20), it follows that $||x_t|| < \min\{\zeta, \delta, \nu\}$. By the same lines as the proof of Lemma 18, we get a contradiction to the definition of t^*. Therefore, $||x_t|| < \min\{\zeta, \delta, \nu\}$ for all $t \ge 0$. The bound $||x_t|| \le ||\phi|| e^{rJ(1+\xi^{-1})} K_0 e^{\beta(t-r)}$ can then be easily shown to be satisfied for all $t \ge r$, and combining this with the previous exponential bound on $[0, r]$, we get the required exponential stability.

5 Application: negative feedback and state-dependent nonlinearity with impulses

With the previous theorems and lemmas in place we will consider a specific application to a scalar problem with negative feedback and bounded nonlinearity, proving a global existence and uniqueness and the existence of a compact, attracting invariant set.

Proposition 21. *Consider the following scalar nonlinear impulsive differential equation with state-dependent delay and negative feedback:*

$$x'(t) = -\gamma x(t) + \mu F(x(t - h(x(t)))), \qquad t \ne t_k \tag{21}$$
$$\Delta x(t) = g(x(t^-)), \qquad t = t_k, \tag{22}$$

with $\gamma > 0$ *and* $\mu > 0$. *Let* $F : \mathbb{R} \to [0, 1]$ *be locally Lipschitz continuous. Assume there exists* $p > 0$ *such that* $p \le t_{k+1} - t_k$ *for all* $k \ge 0$, *and* $g : G^+([-r, 0], \mathbb{R}) \to \mathbb{R}$ *is functional with the following properties.*

- *g is non-negative: for $\phi \geq 0$, $\phi(0^-) + g(\phi^-) \geq 0$, where $\phi^-(\theta) = \phi(\theta)$ for $\theta < 0$ and $\phi^-(0) = \phi(0^-)$.*
- *g maps bounded sets to bounded sets, and $g(x) \leq \alpha + \beta x$ for some $\alpha \geq 0$ and $\beta > -1$.*

If $h : \mathbb{R} \to [0, r]$ is continuously differentiable with $0 \leq h'(x) \leq \mu^{-1}$ for $x \geq 0$, then for any $\phi \in G^{+,\mathrm{Lip}}(\mathbb{R})$ with $\phi \geq 0$, there exists a unique solution $x : [-r, \infty) \to \mathbb{R}^+$ of (21)–(22) satisfying $x_0 = \phi$. If additionally

$$-\gamma + \frac{1}{p}\log(1+\beta) < 0, \tag{23}$$

the following assertions hold.

1. *There exists a compact interval $\Omega_0 \subset [0, \infty)$ that is attracting for nonnegative initial conditions: if $\phi \geq 0$, then $\lim_{t\to\infty} d_H(x(t), \Omega_0) = 0$ for x the solution from 1 and d_H the Hausdorff distance $d_H(x, B) = \inf\{|x - b| : b \in B\}$.*
2. *The semiflow on $X = G^{+,\mathrm{Lip}}(\Omega_0)$ is well-defined, in the sense that to any $\phi \in X$ there is a unique solution $x : [-r, \infty) \to \Omega_0$ with $x_0 = \phi$ and $x_t \in X$ for $t \geq 0$.*

Finally, if F, g and h are continuously differentiable and $g(0) = F(0) = 0$, the solution $x = 0$ is exponentially stable provided the same is true of the linear system

$$\begin{aligned} y'(t) &= -\gamma y(t) + \mu F'(0) y(t - h(0)), && t \neq t_k \\ \Delta y(t) &= g'(0) y(t^-), && t = t_k, \end{aligned}$$

and this holds regardless of whether (23) is satisfied.

Proof. Let $\Omega = \mathbb{R}$, $D = [0, \infty)$. It is straightforward to check the conditions of Corollary 9. Indeed, the monotone lag condition is satisfied: $1 - h'(x)(-\gamma x + \mu F(y)) \geq 0$ for all $x, y \in D$. By Corollary 9, for each $s \in \mathbb{R}$ and $\phi \in G^{+,\mathrm{Lip}}(D)$, there exists at most one non-negative solution x of (21)–(22) satisfying $x_s = \phi$.

By Lemma 1, one can show that at least one solution exists for any $\phi \in G^+(D)$. Let $b = \inf\{t \geq s : x(t) < 0\}$. Since g is non-negative, b can not be an impulse time. If $b < \infty$, then $x(b) = 0$ and by appealing to the integral equation, we have

$$x(t) = \int_b^t -\gamma x(s) + \mu F(x(s - h(x(s))))ds$$

for $t \in [b, b + \epsilon]$ and $\epsilon > 0$ small enough. x is differentiable from the right and continuous in this interval, so

$$x'(b) = -\gamma(b) + \mu F(x(b - h(x(b)))) = \mu F(x(b - h(0))) \geq 0$$

by definition of F. This contradicts the definition of b. Therefore, any solution with non-negative initial condition $\phi \in G^{+,\mathrm{Lip}}$ remains in D. By Corollary 9, such a solution must either be globally defined, or becomes unbounded in finite time.

We will now show that solutions remain bounded for all time. Let $U(t, s)$ be the fundamental solution of the linear equation

$$\begin{aligned} u'(t) &= -\gamma u(t), && t \neq t_k \\ \Delta u(t) &= \beta u(t^-), && t \neq t_k. \end{aligned}$$

Let $-\psi = -\gamma + \frac{1}{p}\log(1+\beta) < 0$. It follows that for any $t \geq s \geq 0$,

$$U(t, s) = e^{-\gamma(t-s)} \prod_{s<t_k\leq t} (1 + \beta).$$

Then $U(t,s) \geq 0$. If $\beta \leq 0$, then $U(t,s) \leq e^{-\gamma(t-s)} \leq e^{-\psi(t-s)}$. If $\beta > 0$, then we have

$$\prod_{s<t_k\leq t}(1+\beta) = (1+\beta)^{\#\{t_k: s<t_k\leq t\}} < (1+\beta)^{\frac{t-s}{p}} = \exp\left(\frac{t-s}{p}\log(1+\beta)\right),$$

from which we can obtain the bound $U(t,s) \leq e^{-\psi(t-s)}$. We conclude that $0 \leq U(t,s) \leq e^{-\psi(t-s)}$. Any solution x of (21)–(22) with $x_0 = \phi \geq 0$ satisfies the variation-of-constants formula

$$x(t) = U(t,0)\phi(0) + \mu\int_0^t U(t,s)F(x(s-\tau(x(s))))ds + \sum_{s<t_k\leq t} U(t,t_k)[g(x_{t_k^-}) - \beta x(t_k^-)].$$

By consequence of the conditions of the corollary – specifically, $f \geq 0$ on $\mathbb{R}^+$ and the properties of g – one can verify that $x \geq 0$ as long as the solution exists. On the other hand, we also have

$$\begin{aligned}
x(t) &\leq U(t,0)\phi(0) + \mu\int_0^t U(t,s)ds + \alpha\sum_{s<t_k\leq t} U(t,t_k)\\
&\leq e^{-\psi t}\phi(0) + \mu\int_0^t e^{-\psi(t-s)}ds + \alpha\sum_{s<t_k\leq t} e^{-\psi(t-t_k)}\\
&\leq e^{-\psi t}\phi(0) + \frac{\mu}{\psi}(1-e^{-\psi t}) + \frac{\alpha}{p}\int_0^{t+p} e^{-\psi(t-s)}ds\\
&= e^{-\psi t}\phi(0) + \frac{\mu}{\psi}(1-e^{-\psi t}) + \frac{\alpha e^{p\psi}}{p\psi}\left(1-e^{-\psi(t-p)}\right).
\end{aligned}$$

Regardless the sign of ψ, it follows that solutions remain exponentially bounded for all time, which concludes the proof of the global existence and uniqueness assertion.

Suppose now that $\psi > 0$. That is, (23) is satisfied. We directly get from the previous bound for $x(t)$ that solutions exist and are bounded for all time, and that $\limsup_{t\to\infty} x(t) \leq \sup(\Omega_0)$, with

$$\Omega_0 = \left[0, \frac{\mu}{\psi} + \frac{\alpha e^{p\psi}}{p\psi}\right].$$

This proves assertion 1. Similarly, for $\phi \in G^{+,\mathrm{Lip}}(\Omega_0)$, the same analysis demonstrates that $x_t(\theta) \in \Omega_0$. By our previous verification of uniqueness of solutions, it follows that the semiflow on X is well-defined, proving assertion 2. The assertion concerning stability follows directly from Theorem 12. □

As a specific instance of the system from (21)–(22), consider

$$x'(t) = -\gamma x(t) + F(x(t-h(x(t)))), \qquad t \neq k \tag{24}$$
$$\Delta x(t) = (\beta x - (\beta+1)x^2)\theta(1-x), \qquad t = k, \tag{25}$$

for $\beta \geq 0$, $h(x) = x^2/(1+x^2)$, F piecewise-defined by

$$F(x) = \begin{cases} \frac{x}{1+x}, & x \geq 0\\ 0 & x < 0,\end{cases}$$

and θ the Heaviside step function. In (25), each instance of x on the right-hand side should be interpreted as $x(t^-)$. In the language of Proposition 21, we have $\mu = 1$. The function F is nonnegative on $[0,\infty)$, Lipschitz continuous, and $|F| \leq 1$. As for the jump map $g(x) = (\beta x - (\beta+1)x^2)\theta(1-x)$, one can verify that $0 \leq x + g(x)$ for $x \geq 0$ and $g(x) \leq \beta x$. We have $h'(x) \leq 3\sqrt{3}/8 < 1$, so the Cauchy problem is well-posed and all solutions are globally defined and unique for $t \geq 0$ by Proposition 21.

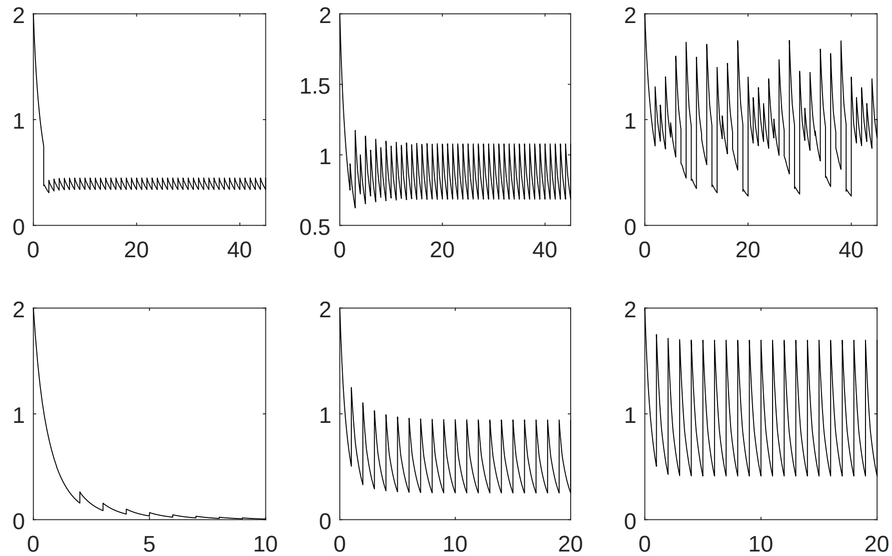

Figure 3: Simulations of (24)–(25) for $\beta = 1, 4, 6$ (left to right) from the constant initial condition $\phi = 2$. Top row: $\gamma = 1$. Bottom row: $\gamma = 2$. Time t on the horizontal axis, with $x(t)$ on the vertical.

For stability, observe that we can smoothly extend F to $(-1, \infty]$ by instead defining it by $F(x) = x/(1+x)$. Then, as $h(0) = 0$ the stability condition is very easy to derive. Since $F'(0) = 1$, the formal linearization is

$$\begin{aligned} y'(t) &= (1-\gamma)y(t), && t \neq k \\ \Delta y(t) &= \beta y(t^-), && t = k. \end{aligned}$$

We will have exponential stability of the solution $x = 0$ provided $(1+\beta)e^{1-\gamma} < 1$. The condition (23) for the attracting invariant set will be satisfied if and only if $\beta < e^{\gamma} - 1$. In this case, the interval $\Omega_0 = \left[0, \psi^{-1}\right]$ with $\psi = \gamma - \log(1+\beta)$, will be attracting.

Figure 3 provides simulations from the constant initial condition $\phi = 2$ for $\beta = 1, 4, 6$ and $\gamma = 1, 2$. In the case $\gamma = 1$, for $\beta = 1$ the region Ω_0 is attracting and appears to contain a periodic solution. With $\beta = 4$, the condition (23) is violated and the solution seems to converge to a periodic solution. At $\beta = 6$ the dynamics may be chaotic. This makes some sense, since the impulse effect is essentially a constrained logistic update. In all cases, the trivial solution appears to be unstable, which is consistent with (but is not proven by) the stability condition. To compare, when $\gamma = 2$ and $\beta = 1$, we have $(1+\beta)e^{1-\gamma} \approx 0.735 < 1$, and the trivial solution is exponentially stable, as expected. This indeed appears to be the case from the figure. Since $e^{\gamma} - 1 \approx 6.389 > 6$ for $\gamma = 2$, the simulations in the $\gamma = 2$ case all feature attractivity of the region Ω_0, and for $\beta = 4, 6$ it appears to contain a periodic solution.

6 Discussion

We have presented in Section 3.3 an argument that, absent any Winston-type constraints on the state-dependent delay, the Cauchy problem for impulsive delay differential equations is fundamentally ill-posed. With such lag monotonicity conditions present, however, uniqueness of solutions can be saved. We have focused on the case of a single discrete delay in the continuous-time dynamics, but of course this could be readily extended to multiple discrete state-dependent delays, or to other classes of functional dependence. Our proof of linearized stability crucially uses the assumption that the time between successive impulses is bounded below by a constant $\xi > 0$. The reason this is needed is because we wanted to ensure that the number of discontinuities in any interval of the form $[t - r, t]$ remains finite and, in particular, bounded by some global constant.

A natural direction of further research could be to extend our linearized stability result to the case where the maximum delay (in this paper, r) is not known a priori. To accomplish this, it would be necessary to ensure that $\tau(x(t))$ remains uniformly bounded along solutions for sufficiently small initial conditions, so that the previous argument concerning the number of impulses in intervals such as $[t - \tau(x(t)), t]$ can be controlled. We do not foresee this being incredibly difficulty, and expect it to be more of a technical exercise. However, the variation-of-constants formula of Proposition 20 has not been extended to the case of unbounded delay, and this was used to initiate the Gronwall inequality argument that ultimately provides stability. As such, it might be necessary to adjust the argument somewhat and use a Euclidean space version of the variation-of-constants formula, rather than the one in the infinite-dimensional phase space that was used here.

Another question is whether the natural converse of Theorem 12 holds. That is, does the instability of the formal linearization (13)–(14) imply the instability of the trivial solution in (1)–(2)? Such a result was proven for impulsive functional differential equations [9] with strong instabilities (i.e. non-trivial unstable fibre bundle) in the case of differentiable functional (i.e. no state-dependent delay) by exhibiting a solution on the unstable manifold, but this machinery is not available in the case of state-dependent delays.

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