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Global Existence for Degenerate Parabolic Equations with a Non-local Forcing

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We establish local existence and comparison for a model problem which incorporates the effects of non-linear diffusion, convection and reaction. The reaction term to be considered contains a non-local dependence, and we show that local solutions can be obtained via monotone limits of solutions to appropriately regularized problems. Utilizing this construction, it is further shown that, under conditions of either 'weak reaction' or 'sufficiently small' initial mass, solutions exist for all time. Finally, we provide an alternative analysis of global existence and investigate blow up in finite time for the case of power law diffusion and convection. These results show the extent to which the assumption of weak reaction may be relaxed and still obtain global existence.

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1. Introduction

In this paper we establish the existence of non-negative solutions for reaction, diffusion, convection models of the form

$$(A_T)\qquad \begin{aligned} &u_t = (\phi(u)_x + g(u))_x + au\,\|u\|_q^{p-1} && \text{on } (0,1)\times(0,T),\\ &u(0,t) = u(1,t) = 0 && \text{on } (0,T),\\ &u(x,0) = u_0(x) && \text{on } [0\,1]. \end{aligned}$$

Of particular interest is the existence of solutions of (A_∞), i.e. global solutions. A solution of (A_∞) may be defined as a function, $u(x,t)$, which is a solution of (A_T) for every $T>0$. (The definition of a solution of (A_T), which is a rather standard weakened notion, is given in section 2.)

Herein, $u_0 \in L^\infty((0,1))$ is a prescribed non-negative function, ϕ and g are continuous on $[0,\infty)$, ϕ'' and g' are continuous on $(0,\infty)$, $\phi(0)=g(0)=0$, and $\phi'(u) \geqslant \alpha u^{m-1}$ for $u>0$ with constants α, $m>0$. The non-local dependence in the forcing term is

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governed by the $L^q((0,1))$ norm

$$\|u(\cdot,t)\|_q^q = \int_0^1 u^q(x,t)\,\mathrm{d}x \quad (q \geqslant 1),$$

and $p \geqslant 1$. As such, the above methods include the case of porous medium diffusion, $\phi(u) = u^m$, with density-dependent drift, $g(u)$. In studies of population dynamics [6] or transport through porous materials [17], it has been seen that useful mathematical models contain both these effects. Chemotaxis, the oriented migration of a species caused by the release of a chemical by others of the same species, provides yet another phenomenon where the effects of density-dependent drift and non-linear diffusion are both important [1, 16]. It has also been suggested that non-local growth terms present a more realistic model of a population which communicates through chemical means [11].

One of our main results is that if $a > 0$ and $p < m$ or if $a > 0$ is small and $p = m$, then there exists a solution of (A_∞) for any initial state u_0. (It is also true that (A_∞) is solvable for any given u_0 in the case of absorption $a \leqslant 0$.) In the remaining cases, there exists a solution of (A_∞) provided the initial data has sufficiently small $L^q((0,1))$ norm. Furthermore, if $a > 0$, we show that these solutions of (A_∞), $u(x,t)$, satisfy

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_q = \begin{cases} \left[\dfrac{4\alpha(q-1)}{a(q+m-1)^2}\right]^{1/(p-m)} & \text{if } p < m \text{ and } q > 1, \\ 0 & \text{if } p \geqslant m. \end{cases}$$

So, for example, if $q = 1$ and $p \geqslant m$, there are global solutions that decay to have zero ‘mass’.

In proving the existence of global solutions, we first develop the existence of solutions of (A_T) for small values of T. For these local solutions, a common form of continuation result holds. Namely, if $u(x,t)$ is a solution of (A_T) and $\hat{T} > T$, then either $u(x,t)$ may be continued to be a solution of $(A_{\hat{T}})$ or

$$\limsup_{t\to T^*} \|u(\cdot,t)\|_\infty = \infty$$

for some $T^* \in (T, \hat{T})$. (Here, $\|\cdot\|_\infty$ denotes the usual norm in $L^\infty((0,1))$.) In the latter case, we shall say that u blows up in finite time. Therefore, to investigate the necessity of $p < m$, $p = m$ with $a > 0$ small, or $a \leqslant 0$ to guarantee the solvability of (A_∞) for all initial states, it is useful to study blow-up in finite time.

To this end, we isolate attention to the case of diffusion and convection governed by power laws

$$(B_T) \qquad \begin{aligned} &u_t = ((u^m)_x + \varepsilon u^n)_x + au\|u\|_q^{p-1} && \text{on } (0,1)\times(0,T), \\ &u(0,t) = u(1,t) = 0 && \text{on } (0,T), \\ &u(x,0) = u_0(x) && \text{on } [0,1]. \end{aligned}$$

Here, $m, n > 0$, and, in light of the transformation $x \to 1-x$, there is no loss of generality in considering only $\varepsilon > 0$. As well, since the absorption case $a \leqslant 0$ is understood, we only consider $a > 0$. Via comparison with a suitable supersolution, we show that if $p < \max\{m, n\}$, then (B_∞) is solvable for all initial states. On the other hand, reversal of this inequality is shown to yield solutions which blow up in finite time. Hence, the condition $p < m$ is not necessary for solutions of (A_∞) to exist for any given u_0. Unfortunately, the methods used to investigate solutions of (B_T) do not apply to (A_T) in its full generality.

Our interest in the solvability of (A_∞) lies in the similarity of this model to the local problem

$$(C_T)\qquad \begin{aligned} &u_t = ((u^m)_x + \varepsilon u^n)_x + au^p && \text{on } (0,1)\times(0,T),\\ &u(0,t) = u(1,t) = 0 && \text{on } (0,T),\\ &u(x,0) = u_0(x) && \text{on } [0\,1]. \end{aligned}$$

The solvability of (C_∞) was studied in previous work [3], and results established therein are virtually identical to those proven here for (B_∞). The non-local growth term does not appear to promote or inhibit the existence of global solutions when compared to the same model with a purely local growth term. On the other hand, recent work regarding the linear diffusion model

$$u_t = u_{xx} + u^p - \int_0^1 u^p(x,\cdot)\,\mathrm{d}x; \quad 0 < x < 1,\ t > 0$$

show that such an equation has solutions which blow up in finite time for all $p > 1$ [5, 13]. Thus, in comparing (B_∞) and (C_∞), the non-local growth term behaves much like its local counterpart. However, the non-local growth term is not able to prevent blow-up in finite time in the presence of a similarly strong local growth term. Other studies regarding similar effects of non-local vs. local reaction in a linear diffusion model may be found in [7, 8, 14].

This paper is organized as follows. In section 2, the notion of a solution of (A_T) is defined. The existence of such solutions for sufficiently small values of $T > 0$ is then established via a monotone limit of solutions to regularized problems as in [2]. In section 3, uniqueness and comparison results are proven for solutions of (A_T). Although the necessary techniques are by now quite standard, see e.g. [2, 10, 12], this work also yields the fact that solutions of (A_T) as constructed in section 2 are the maximal solutions of (A_T). In section 4, the main results regarding global solutions are presented, and in section 5, results pertaining to global solutions and blow-up in finite time for (B_T) are established.

2. Local existence

As it is now well known that degenerate equations need not possess classical solutions, we begin by giving a precise definition of a solution for problem (A_T). It will also be convenient to define the notions of subsolution and supersolution at the same time. To this end, define the class of 'test functions'

$$\mathscr{T} \equiv \{\xi \in C(\overline{Q_T});\ \xi_t, \xi_{xx} \in C(Q_T) \cap L^2(Q_T);\ \xi \geqslant 0;\ \xi(0,t) = \xi(1,t) = 0\},$$

where $Q_T \equiv (0,1)\times(0,T)$, and $C(Q_T)$, $L^2(Q_T)$ denote the continuous and square integrable functions on Q_T, respectively.

Definition 2.1. *A function $u(x,t)$ defined on $\overline{Q_T}$, where $T > 0$, is called a subsolution (supersolution) of (A_T) if all the following hold.*

(i) $u \in L^\infty(Q_T)$.
(ii) $u(0,t)$, $u(1,t) \leqslant (\geqslant) 0$ *for* $t \in (0,T)$, *and* $u(x,0) \leqslant (\geqslant) u_0(x)$ *for almost all* $x \in (0,1)$.
(iii) *For every* $t \in [0,T]$ *and every* $\xi \in \mathcal{T}$,

$$\int_0^1 [u(x,t)\xi(x,t) - u_0(x)\xi(x,0)]\,\mathrm{d}x$$

$$\leqslant (\geqslant) \int_0^t \int_0^1 \{u\xi_s + \phi(u)\xi_{xx} - g(u)\xi_x + au\|u\|_q^{p-1}\xi\}\,\mathrm{d}x\,\mathrm{d}s.$$

A solution (or weak solution) of (A_T) is a function which is both a subsolution and a supersolution of (A_T). A solution of (A_∞), i.e. a global solution, is a function which is a solution of (A_T) for every $T > 0$.

To prove the existence of a solution of (A_T) for some sufficiently small $T > 0$, i.e. a local solution, we introduce the sequence $\{u_k\}_{k=2}^\infty$, where u_k is the solution of

$$(A_T^k) \qquad \begin{aligned} &u_t = (\phi(u)_x + g(u))_x + au\|u_{k-1}\|_q^{p-1} && \text{on } Q_T, \\ &u(0,t) = u(1,t) = 0 && \text{on } (0,T), \\ &u(x,0) = u_0(x) && \text{on } [0,1], \end{aligned}$$

and $u_1(x,t) \equiv u_1(x)$ is chosen to satisfy $u_1 \in C([0,1])$, $u_1 \geqslant 0$, and

$$\int_0^1 u_0^q(x)\,\mathrm{d}x < \int_0^1 u_1^q(x)\,\mathrm{d}x. \tag{1}$$

Here, a solution of (A_T^k) is understood in the same manner as that for (A_T). The next lemma addresses the existence of the limit $\lim_{k\to\infty} u_k(x,t)$.

Lemma 2.1. *For each $k = 2, 3, \ldots$, there exists a solution of (A_∞^k) which is denoted by u_k. Moreover, there exists a monotone increasing sequence $\{T_k\}_{k=2}^\infty$ such that $0 \leqslant u_{k+1} \leqslant u_k$ on $(0,1)\times(0,T_k)$.*

Proof. The existence of non-negative solutions of (A_T^k) for any $T > 0$ is a consequence of results in [2]. The existence of the sequence $\{u_k\}_{k=2}^\infty$ can also be concluded from the work of Sacks [15], but the construction of these solutions as presented in the former reference shall be useful in the developments herein. A summary of this construction as it applies to (A_T^k) is now given for the convenience of the reader.

The solution of (A_T^2), u_2, is obtained as a pointwise limit of solutions to the regularized problems

$$(A_T^{2,l,i}) \qquad \begin{aligned} &v_t = (\phi(v)_x + g(v))_x + a\|u_1\|_q^{p-1}v && \text{on } Q_T, \\ &v(0,t) = v(1,t) = 1/l && \text{on } (0,T), \\ &v(x,0) = u_{0_i}(x) + 1/l && \text{on } [0\,1], \end{aligned}$$

where $l > 0$, and u_{0i} is a smooth approximation (obtained via mollification), with $\operatorname{supp} u_{0i} \subset (0, 1)$. The classical solution of $(A_T^{2,l,i})$, $v_{l,i}$, can be shown to exist, and, moreover, maximum principles, e.g. Lemma A.1 in [2], may be employed to obtain the estimates

$$1/l \leqslant v_{l,i}(x,t) \leqslant [\|u_{oi}\|_\infty + 1/l] \mathrm{e}^{a\|u_1\|_q^{p-1} t}$$

and

$$v_{l,i}(x,t) \geqslant v_{j,i}(x,t) \quad \text{if } j \geqslant l > 0$$

which are true for all $(x,t) \in \overline{Q_T}$. It follows that

$$u_2(x,t) \equiv \lim_{l \to \infty} \left[\lim_{i \to \infty} v_{l,i}(x,t) \right]$$

is a solution of (A_T^2) and such a construction is valid for any $T > 0$.

The process is now verified by induction. To this end, assume that $k \geqslant 3$ and a solution of (A_∞^{k-1}), u_{k-1}, has been constructed as above. Define $b_k(t) \equiv \|u_{k-1}(\cdot, t)\|_q^{p-1}$ and fix $T > 0$. Since $b_k \in L^\infty((0, T))$, we may suitably modify the arguments in [2] to obtain a solution of (A_T^k), u_k, according to

$$u_k \equiv \lim_{l \to \infty} \left[\lim_{i \to \infty} v_{k,l,i}(x,t) \right].$$

Here the functions $v_{k,l,i}$ are now (classical) solutions of the regularized problems

$$(A_T^{k,l,i}) \qquad \begin{aligned} & v_t = (\phi(v)_x + g(v))_x + ab_{k,i} v && \text{on } Q_T, \\ & v(0,t) = v(1,t) = 1/l && \text{on } (0, T), \\ & v(x,0) = u_{0i}(x) + 1/l && \text{on } [0,1], \end{aligned}$$

where $l > 0$, u_{0i} is as in $(A_T^{2,l,i})$, and $b_{k,i}$ is a smooth approximation of b_k. By maximum principles it can again be shown that

$$1/l \leqslant v_{k,l,i}(x,t) \leqslant [\|u_{0i}\|_\infty + 1/l] \mathrm{e}^{at\|b_{k,i}\|_\infty}$$

is valid for all $(x,t) \in \overline{Q_T}$. As $b_k \in L^\infty(0, T)$, it follows that $b_{k,i} \in L^\infty(0, T)$ for each $T > 0$. Hence, the above construction is valid for all $T > 0$, i.e. u_k is a solution of (A_∞^k).

This verifies the existence of the sequence $\{u_k\}_{k=2}^\infty$. There remains the task of proving monotonicity on Q_T for certain values of $T > 0$, which can be done by showing that the sequence $\{b_k\}$ is monotone. In this direction, let w denote the solution of $(A_T^{k,l,i})$ and fix $r \geqslant q$. Upon multiplication of the differential equation by rw^{r-1} and integration by parts, it follows that

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 w^r(x,t)\,\mathrm{d}x \\ & \quad = \int_0^1 \{-r(r-1)w^{r-2}\phi'(w)(w_x)^2 + rw^{r-1}g'(w)w_x + ab_{k,i} r w^r\}(x,t)\,\mathrm{d}x \\ & \qquad + r\left(\frac{1}{l}\right)^{r-1} [\phi(w)_x(1,t) - \phi(w)_x(0,t)] \end{aligned}$$

$$= \int_0^1 \{ -r(r-1)[\Psi_l(w)_x]^2 + H_l(w)_x + ab_{k,i} r w^r \}(x,t)\,dx$$
$$+ r\left(\frac{1}{l}\right)^{r-1} [\phi(w)_x(1,t) - \phi(w)_x(0,t)].$$

Here, the functions Ψ_l and H_l are defined by

$$\Psi_l(u) \equiv \int_{1/l}^{u} \sqrt{s^{r-2}\phi'(s)}\,ds$$

and

$$H_l(u) \equiv \int_{1/l}^{u} r s^{r-1} g'(s)\,ds.$$

Since $\phi(w)_x(1,t) - \phi(w)_x(0,t) \leqslant 0$ and $\| b_{k,i} \|_\infty \leqslant \| b_k \|_\infty$, there follows

$$\frac{d}{dt}\int_0^1 w^r(x,t)\,dx \leqslant \int_0^1 \{ -r(r-1)[\Psi_l(w)_x]^2 + ab_{k,i} r w^r \}(x,t)\,dx$$
$$\leqslant ar \| b_k \|_\infty \int_0^1 w^r(x,t)\,dx. \tag{2}$$

Subsequently, with $r = q$, the solutions of $(A_T^{k,l,i})$ must satisfy

$$\int_0^1 v_{k,l,i}^q(x,t)\,dx \leqslant e^{aqt\|b_k\|_\infty} \int_0^1 \left[u_{0i}(x) + \frac{1}{l} \right]^q dx,$$

which upon passing to the limit $i \to \infty$, $l \to \infty$, yields

$$\int_0^1 u_k^q(x,t)\,dx \leqslant e^{aqt\|b_k\|_\infty} \int_0^1 u_0^q(x)\,dx.$$

Select $T_2 > 0$ so that

$$e^{aqb_2T_2} \int_0^1 u_0^q(x)\,dx \leqslant \int_0^1 u_1^q(x)\,dx.$$

Such a value T_2 exists due to the choice of u_1 according to (1). It follows $b_3 \leqslant b_2$ on $[0, T_2]$. Utilizing non-negative, symmetric mollifiers in the construction of $b_{3,i}$ and $b_{2,i}$ ensures that $b_{3,i} \leqslant b_{2,i}$ is also true on $[0, T_2]$. Hence the maximum principle [2, Lemma A.1] may be applied to compare solutions of $(A_{T_2}^{2,l,i})$ and $(A_{T_2}^{3,l,i})$ and obtain $v_{3,l,i} \leqslant v_{2,l,i}$ on $\overline{Q_{T_2}}$. Passing to the limits $i \to \infty$, $l \to \infty$ gives $u_3 \leqslant u_2$ on $\overline{Q_{T_2}}$.

In order to construct T_3, observe that now

$$b_4(t) = \int_0^1 u_3^q(x,t)\,dx \leqslant \int_0^1 u_2^q(x,t)\,dx = b_3(t)$$

for all $t \in [0, T_2]$. Subsequently, $T_3 > 0$ may be chosen in such a way that $T_3 \geqslant T_2$ and $b_4 \leqslant b_3$ on $[0, T_3]$. Using the maximum principle as above yields $u_4 \leqslant u_3$ on $\overline{Q_{T_3}}$. Continuing in this manner, the sequence $T_2 \leqslant T_3 \leqslant T_4 \leqslant \cdots$ is constructed. □

Due to the above lemma, the limit

$$T^* \equiv \lim_{k \to \infty} T_k$$

exists, and, as well, the pointwise limit

$$u(x,t) \equiv \lim_{k\to\infty} u_k(x,t)$$

exists for $(x,t) \in Q_{T^*}$. Furthermore, as the convergence of the sequence is monotone, passage to the limit $k \to \infty$ in the identity

$$\int_0^1 [u_k(x,t)\xi(x,t) - u_0(x)\xi(x,0)]\,\mathrm{d}x$$

$$= \int_0^t \int_0^1 \{u_k \xi_s + \phi(u_k)\xi_{xx} - g(u_k)\xi_x + a u_k \| u_{k-1} \|_q^{p-1} \xi\}\,\mathrm{d}x\,\mathrm{d}s$$

is justified by monotone and dominated convergence theorems for any $\xi \in \mathscr{T}$ and $t \in [0, T^*)$. The following theorem is thus established.

Theorem 2.1 (Local existence and continuation). *Given $u_0 \geqslant 0$, $u_0 \in L^\infty((0,1))$, there is some $T^* = T^*(u_0) > 0$ such that there exists a non-negative solution, $u(x,t) = u(x,t;u_0)$, of (A_T) for each $T < T^*$. Furthermore, either $T^* = \infty$ or*

$$\limsup_{t\to T^*} \| u(\cdot,t)\|_\infty = \infty.$$

Before leaving this section it will be useful in subsequent work to recognize an alternative approach to solutions of (A_T). Recall that the sequence $\{v_{k,l,i}\}$, which contains the solutions of $(A_T^{k,l,i})$, is monotone decreasing in both subscripts k and l. Therefore, the solution, u, of (A_T), which is constructed above, has

$$u = \lim_{k\to\infty} \lim_{l\to\infty} \lim_{i\to\infty} v_{k,l,i}$$

$$= \lim_{l\to\infty} u^l,$$

where $u^l \equiv \lim_{k\to\infty} \lim_{i\to\infty} v_{k,l,i}$. So u can also be developed as the pointwise limit of solutions of

$$({}^l A_T) \qquad \begin{aligned} &u_t = (\phi(u)_x + g(u))_x + a u \| u \|_q^{p-1} && \text{on } Q_T,\\ &u(0,t) = u(1,t) = 1/l && \text{on } (0,T),\\ &u(x,0) = u_0(x) + 1/l && \text{on } [0,1]. \end{aligned}$$

Solutions of problem $({}^l A_T)$ are defined in a manner similar to that for (A_T).

Because of the monotonicity of the sequence $\{u_k\}_{k=2}^\infty$, it is plausible that we may refer to $u \equiv \lim u_k$ as the 'limit solution' of (A_T) and, hence, discuss a theory for the limit solutions of (A_T) even in the absence of a uniqueness theory for weak solutions. This is, in fact, true once it has been shown that the construction above is independent of the choice of $u_1(x)$, which will follow from the verification that solutions of $({}^l A_T)$ are unique.

In the next section, we take up the question of the uniqueness of weak solutions for problems (A_T) and $({}^l A_T)$ at the same time. It is more efficient to proceed in such a manner due to the nearly identical nature of all necessary calculations. From these calculations it will also be seen that the limit solution of (A_T) is actually the maximal weak solution.

3. Uniqueness and comparison

The technique for proving uniqueness and comparison for problems $({}^{l}A_T)$ and (A_T) is quite standard. For example, see [2, 4, 9, 10]. Here, we shall sketch the argument for the convenience of the reader, pointing out those items which require special care.

To begin, let u and v denote a non-negative subsolution and a non-negative supersolution of $({}^{l}A_T)$, respectively. (Subsolutions/supersolutions of $({}^{l}A_T)$ are defined exactly as for (A_T) except for the addition of a boundary integral resulting from formal integration by parts.) Subtracting the integral inequalities for u and v, yields

$$\begin{aligned}\int_0^1 [u(x,t)-v(x,t)]\,\xi(x,t)\,\mathrm{d}x &\\ \leqslant \int_0^1 [u(x,0)-v(x,0)]\,\xi(x,0)\,\mathrm{d}x &\\ + \int_0^t\int_0^1 (u-v)\{\xi_s+\Phi\xi_{xx}-G\xi_x\}\,\mathrm{d}x\,\mathrm{d}s &\\ + \int_0^t\int_0^1 a\{u\|u\|_q^{p-1}-v\|v\|_q^{p-1}\}\,\mathrm{d}x\,\mathrm{d}s &\\ + \int_0^t \{-[\phi(u(1,s))-\phi(v(1,s))]\,\xi_x(1,s)+[\phi(u(0,s)) &\\ -\phi(v(0,s))]\,\xi_x(0,s)\}\end{aligned}$$

where

$$\Phi(x,t)\equiv\int_0^1 \phi'[\theta u(x,t)+(1-\theta)v(x,t)]\,\mathrm{d}\theta$$

and

$$G(x,t)\equiv\int_0^1 g'[\theta u(x,t)+(1-\theta)v(x,t)]\,\mathrm{d}\theta.$$

As $u(0,\cdot), u(1,\cdot)\leqslant 1/l\leqslant v(0,\cdot), v(1,\cdot)$, and the test function $\xi\in\mathscr{T}$ can be easily seen to satisfy $-\xi_x(1,\cdot), \xi_x(0,\cdot)\geqslant 0$, it follows that

$$\begin{aligned}\int_0^1 [u(x,t)-v(x,t)]\,\xi(x,t)\,\mathrm{d}x &\\ \leqslant \int_0^1 [u(x,0)-v(x,0)]\,\xi(x,0)\,\mathrm{d}x &\\ + \int_0^t\int_0^1 (u-v)\{\xi_s+\Phi\xi_{xx}-G\xi_x+a\|u\|_q^{p-1}\xi\}\,\mathrm{d}x\,\mathrm{d}s &\\ + \int_0^t\int_0^1 av[\|u\|_q^{p-1}-\|v\|_q^{p-1}]\,\xi\,\mathrm{d}x\,\mathrm{d}s. &\end{aligned}$$

To handle the final integral of this inequality, observe that, as in [9, p. 194],

$$\| u \|_q^{p-1} - \| v \|_q^{p-1} = \left[\int_0^1 u^q(y,s)\,\mathrm{d}y \right]^{(p-1)/q} - \left[\int_0^1 v^q(y,s)\,\mathrm{d}y \right]^{(p-1)/q}$$
$$= D(s) \int_0^1 E(y,s)(u(y,s) - v(y,s))\,\mathrm{d}y,$$

where

$$D(s) \equiv \int_0^1 \frac{p-1}{q} \left[\theta \int_0^1 u^q(y,s)\,\mathrm{d}y + (1-\theta) \int_0^1 v^q(y,s)\,\mathrm{d}y \right]^{[(p-1)/q]-1} \mathrm{d}\theta$$

and

$$E(y,s) \equiv \int_0^1 q[\theta u(y,s) + (1-\theta) v(y,s)]^{q-1}\,\mathrm{d}\theta.$$

Hence,

$$\int_0^1 [u(x,t) - v(x,t)]\xi(x,t)\,\mathrm{d}x$$
$$\leqslant \int_0^1 [u(x,0) - v(x,0)]\xi(x,0)\,\mathrm{d}x$$
$$+ \int_0^t \int_0^1 (u-v)\{\xi_s + \Phi\xi_{xx} - G\xi_x + a\|u\|_q^{p-1}\xi\}\,\mathrm{d}x\,\mathrm{d}s$$
$$+ \int_0^t \left\{ \int_0^1 av\xi\,\mathrm{d}x \right\} \left\{ D \int_0^1 E(u-v)\,\mathrm{d}y \right\} \mathrm{d}s.$$

Provided that Φ and G^2/Φ are bounded, appropriate test functions ξ may be chosen exactly as in [2, pp. 118–123] to obtain

$$\int_0^1 [u(x,t) - v(x,t)]^+\,\mathrm{d}x$$
$$\leqslant \|\xi\|_\infty \int_0^1 [u(x,0) - v(x,0)]^+\,\mathrm{d}x$$
$$+ a\|\xi\|_\infty \|v\|_\infty \int_0^t \int_0^1 D(s)E(x,s)[u(x,s) - v(x,s)]^+\,\mathrm{d}x\,\mathrm{d}s$$

with $[u-v]^+ \equiv \max\{u-v, 0\}$. Now if D and E are bounded, then $u(\cdot,0) \leqslant v(\cdot,0)$ implies $u \leqslant v$ on Q_T. Since $q \geqslant 1$, a bound on E may be obtained for bounds on u and v. Thus, there is only the issue of developing estimates of Φ, G^2/Φ, and D in order to establish the comparison of subsolutions and supersolutions of $({}^lA_T)$. Note that all of the above arguments also hold in regard to a comparison theory for (A_T).

Theorem 3.1. (Comparison for $({}^lA_T)$). *Let u and v be a non-negative sub-solution and a non-negative supersolution of $({}^lA_T)$, respectively, with $l > 0$. If $u(\cdot,0) \leqslant v(\cdot,0)$, then $u \leqslant v$ on Q_T.*

Proof. Let $U(x,t)$ denote a solution of $({}^lA_{T^*})$, which was constructed in the previous section, such that $u(\cdot,0) \leqslant U(\cdot,0) \leqslant v(\cdot,0)$. Recall that $U \geqslant 1/l > 0$ on Q_{T^*}. If, in the above calculations, u is replaced with U, then a comparison result for U and v may be concluded from bounds on $\tilde{\Phi}$, $\tilde{G}^2/\tilde{\Phi}$, and $\tilde{D}$. Here,

$$\tilde{\Phi}(x,t) \equiv \int_0^1 \phi'(\theta U(x,t) + (1-\theta)v(x,t))\,d\theta,$$

and $\tilde{G}, \tilde{D}$ are defined by modifying G, D, respectively, in exactly the same way.

Fixing $t \in [0, \hat{T}]$, where $\hat{T} < \min\{T, T^*\}$, and $x \in [0,1]$, it must be the case that either $v(x,t) \geqslant 1/2l$ or $0 \leqslant v(x,t) < 1/2l$. In the first case

$$\begin{aligned}\min&\left\{\phi'(z): \frac{1}{2l} \leqslant z \leqslant U(x,t) + v(x,t)\right\}\\ &\leqslant \tilde{\Phi}(x,t)\\ &\leqslant \max\left\{\phi'(z): \frac{1}{2l} \leqslant z \leqslant U(x,t) + v(x,t)\right\},\end{aligned}$$

while in the second case

$$\tilde{\Phi}(x,t) = \frac{\phi(U(x,t)) - \phi(v(x,t))}{U(x,t) - v(x,t)},$$

so

$$\frac{\phi(U(x,t))}{U(x,t)} \leqslant \tilde{\Phi}(x,t) \leqslant \frac{\phi(U(x,t)) + \phi(v(x,t))}{(1/2l)}.$$

Therefore, there exist constants c_1 and c_2 such that

$$0 < c_1 \leqslant \tilde{\Phi}(x,t) \leqslant c_2$$

is satisfied on $\overline{Q_{\hat{T}}}$. In similar fashion, an estimate of $|\tilde{G}|$ is obtained.

In order to show that $\tilde{D}$ is also bounded, note that $\|U(\cdot,t)\|_q \geqslant 1/l$ for all $t \in [0, \hat{T}]$. Consideration of the cases $\|v(\cdot,t)\|_q \geqslant 1/2l$ and $\|v(\cdot,t)\|_q < 1/2l$ as done above yields the desired bound on $\tilde{D}$. Since $\tilde{\Phi}$, $\tilde{G}^2/\tilde{\Phi}$, and $\tilde{D}$ have all been shown to be bounded on $\overline{Q_{\hat{T}}}$, it follows that $U \leqslant v$ on $\overline{Q_{\hat{T}}}$. By the continuation theorem, $T^* > T$, and so $U \leqslant v$ on Q_T.

In a similar fashion to the work above, comparison of u and U can also be established. Hence, $u \leqslant U \leqslant v$ on Q_T. □

As a result of this comparison theorem, the construction utilized in developing the existence of solutions of (A_T) is independent of the choice of $u_1(x)$. Subsequently, it makes sense to refer to the 'limit solution' of (A_T) as being that solution constructed in the previous section, and limit solutions of (A_T) are unique even in the absence of a uniqueness result for weak solutions of (A_T). Because the limit solution is independent of the choice of $u_1(x)$, it might be asked if there is some prefered method for choosing u_1. Recall that a limit solution, u, of (A_T) exists on Q_{T^*} where $T^* = \lim T_n$, $\{T_n\}$ is a monotone decreasing sequence, and T_2 is chosen to satisfy

$$e^{a(p-1)b_2T_2}\|u_0\|_q^{p-1} = b_2,$$

where $b_2 = \| u_1 \|_q^{p-1}$. T_2 can be maximized by selecting $b_2 = \mathrm{e}\| u_0 \|_q^{p-1}$, and, for such choice of u_1, there follows

$$T_2 = \frac{1}{a(p-1)\mathrm{e}} \| u_0 \|_q^{1-p}.$$

Since T_2 might potentially be made larger and still preserve $u_3 \leqslant u_2$on Q_{T_2}, we have

$$T^* \geqslant T_2 \geqslant \frac{1}{a(p-1)\mathrm{e}} \| u_0 \|_q^{1-p}.$$

It also follows from the above theorem that the limit solution of (A_T) is the maximal of all weak solutions of (A_T). To see this, let U denote the limit solution of (A_T) and suppose u is a weak solution of (A_T) such that $U(\cdot,0) = u(\cdot,0)$. Then $U = \lim u^l$ where u^l is the solution of $({}^lA_T)$ having $u_l(\cdot,0) = U(\cdot,0) + 1/l$. It is clear that u is a subsolution of $({}^lA_T)$ and $u(\cdot,0) \leqslant U(\cdot,0) + 1/l$. Hence $u \leqslant u^l$ on Q_T for each $l > 0$ from which it follows that $u \leqslant U$. The same argument may be used to show $u \leqslant U$ for any non-negative subsolution of (A_T) with $u(\cdot,0) \leqslant U(\cdot,0)$.

In order to prove the uniqueness of solutions of (A_T), as well as comparison result involving supersolutions of (A_T), a slightly more delicate argument is required. Let v and U denote a supersolution and limit solution of (A_T), respectively. Then $U = \lim u^l$, where u^l is the solution of $({}^lA_T)$ such that $u^l(\cdot,0) = U(\cdot,0) + 1/l$. Subtracting the integral inequalities in the formulation of weak solutions of (A_T) and $({}^lA_T)$ yields

$$\begin{aligned}
&\int_0^1 [u^l(x,t) - v(x,t)]\xi(x,t)\,\mathrm{d}x \\
&\quad \leqslant \int_0^1 [u^l(x,0) - v(x,0)]\xi(x,0)\,\mathrm{d}x \\
&\qquad + \int_0^t \int_0^1 (u^l - v)\{\xi_s + \Phi_l \xi_{xx} - G_l \xi_x\}\,\mathrm{d}x\,\mathrm{d}s \\
&\qquad + \int_0^t \left\{\int_0^1 av\xi\,\mathrm{d}x\right\}\left\{D_l \int_0^1 E_l(u^l - v)\,\mathrm{d}y\right\}\mathrm{d}s \\
&\qquad + \int_0^t \{-[\phi(1/l) - \phi(v(1,s))]\xi_x(1,s) + [\phi(1/l) - \phi(v(0,s))]\xi_x(0,s)\}\,\mathrm{d}s.
\end{aligned}$$

Here, Φ_l, G_l, D_l, and E_l are obtained as in the definitions of Φ, G, D, and E, respectively, upon replacing u with u^l. This is the same inequality considered above in proving comparison for $({}^lA_T)$ except here the boundary integral cannot be so easily estimated. Since all that is known is $v(0,s)$, $v(1,s) \geqslant 0$, the terms $\phi(1/l) - \phi(v(0,s))$ and $\phi(1/l) - \phi(v(1,s))$ become non-positive only upon passage to the limit $l \to \infty$. Hence, the boundary derivatives $\xi_x(0,s)$ and $\xi_x(1,s)$ must now be considered in the analysis. Ultimately, this necessitates a different method for selecting the test functions ξ.

First, consider the case where v is a solution of (A_T) with $v(x,0) = U(x,0)$. As a result of the work above, $u \leqslant U$ on $\overline{Q_T}$, so only the reverse inequality must be shown in order to prove the uniqueness of solutions of (A_T). Choose $\lambda, \delta > 0$ so that $\delta \leqslant \Phi_l$ and $\lambda \geqslant |G_l|$, and let $\chi \in C^\infty((0,1))$ with $0 \leqslant \chi \leqslant 1$, $\chi(x) = \chi(1-x)$, $\chi(0) = \chi'(\frac{1}{2}) = 0$, $\chi' \geqslant 0$ on $[0, 1/2]$, and $\chi'' \leqslant 0$ on $[0, 1]$. If the test function ξ is chosen to satisfy

$$\begin{aligned} &\xi_s + \delta\xi_{xx} + \lambda\xi_x = 0 && \text{on } (0,1/2)\times(0,t),\\ &\xi(0,s) = \xi_x(1/2,s) = 0 && \text{on } (0,t),\\ &\xi(x,t) = \chi(x) && \text{on } [0,1/2], \end{aligned}$$

and $\xi(x,s) = \xi(1-x,s)$ for $1/2 \leqslant x \leqslant 1$, then

$$\begin{aligned} &\int_0^1 [u^l(x,t) - v(x,t)]\chi(x)\,\mathrm{d}x\\ &\quad\leqslant \int_0^1 [u^l(x,0) - v(x,0)]\,\xi(x,0)\,\mathrm{d}x\\ &\qquad + \int_0^t \int_0^1 (u^l - v)\{[\Phi_l - \delta]\,\xi_{xx} - G_l\xi_x - \lambda|\xi_x|\}\,\mathrm{d}x\,\mathrm{d}s\\ &\qquad + \int_0^t \left\{\int_0^1 av\xi\,\mathrm{d}x\right\}\left\{D_l \int_0^1 E_l(u^l - v)\,\mathrm{d}y\right\}\mathrm{d}s\\ &\qquad + \phi(1/l)\int_0^1 \{-\xi_x(1,s) + \xi_x(0,s)\}\,\mathrm{d}s. \end{aligned} \tag{3}$$

From basic maximum principles, e.g. [2, Lemma A.1], it follows that $0 \leqslant \xi \leqslant 1$, $\xi_x \geqslant 0$ on $[0,1/2]\times[0,t)$, and $\xi_{xx} \leqslant 0$ on Q_t. So $\xi_s + \delta\xi_{xx} + \lambda|\xi_x| = 0$ on Q_t, a fact which has been used above. Integrating the differential equation, we also have

$$\delta \int_0^t \{-\xi_x(1,s) + \xi_x(0,s)\}\,\mathrm{d}s = \int_0^1 [\chi(x) - \xi(x,0)]\,\mathrm{d}x + 2\lambda \int_0^t \xi(1/2,s)\,\mathrm{d}s.$$

Before passing to the limit $l \to \infty$ in inequality (3), it is necessary to also have estimates of D_l and E_l which are independent of l. As $q \geqslant 1$ in E_l, the only potential difficult here is in estimating D_l even in the case $(p-1)/q < 1$. For this purpose, observe from the definition of a solution of (A_T) that $\int_0^1 U(x,\cdot)\xi(x,\cdot)\,\mathrm{d}x$ is a continuous function, and, moreover, if $\|U(\cdot,\hat{t})\|_q = 0$ for some $\hat{t} < T^*$, then $U \equiv 0$ on $[0,1]\times[\hat{t}, T^*)$ follows from (2). Hence, there exists $\hat{t} \leqslant T^*$ such that

$$\|U(\cdot,s)\|_q \begin{cases} >0 & \text{if } 0 \leqslant s < \hat{t},\\ =0 & \text{if } \hat{t} \leqslant s < T^*, \end{cases}$$

and, for $t < \hat{t}$, there exists $\lambda > 0$ such that

$$\int_0^1 (u^l)^q(x,s)\,\mathrm{d}x \geqslant \int_0^1 U^q(x,s)\,\mathrm{d}x \geqslant \lambda,$$

for all $0 \leqslant s \leqslant t$. The desired bound of D_l now may be obtained exactly as done previously for D.

Observe that

$$\Phi_l(x,t) \geqslant \int_0^1 \alpha(\theta u^l(x,t) + (1-\theta)v(x,t))^{m-1}\,\mathrm{d}\theta$$

$$\geqslant \begin{cases} \alpha(\|u^l\|_\infty + \|v\|_\infty)^{m-1}, & \text{if } m \leqslant 1 \\ \dfrac{\alpha}{m}\left(\dfrac{1}{l}\right)^{m-1}, & \text{if } m > 1, \end{cases}$$

so δ is independent of l if $m \leqslant 1$. In the case $m > 1$, we must employ the assumption that

$$\frac{\phi(1/l)}{\delta} = \frac{m}{\alpha} l^{m-1}\phi(1/l) \to 0$$

as $l \to \infty$. It is now possible to pass to the limit $l \to \infty$ in (3) and find that

$$\int_0^1 [U(x,t) - v(x,t)]\chi(x)\,\mathrm{d}x$$

$$\leqslant \int_0^1 [U(x,0) - v(x,0)]^+\,\mathrm{d}x$$

$$+ \{1 + a\|v\|_\infty \|D_l\|_\infty \|E_l\|_\infty\} \int_0^t \int_0^1 [U - v]^+\,\mathrm{d}x\,\mathrm{d}s$$

for $t \in [0, \hat{t})$. Proper choice of χ now yields the result $U \leqslant v$. The comparison theorem for solutions of (A_T) now follows from the same result for solutions of $({}^lA_T)$.

The above argument, unfortunately, does not extend to the case where v is a supersolution of (A_T) because it is not known at the outset (nor is it to be expected) that $U \geqslant v$. We simply note here that supersolution comparison can be proven by the same methods used for $({}^lA_T)$ in situations enumerated below.

Theorem 3.2 (Comparison for (A_T)). *Let u and v denote a non-negative subsolution and supersolution of* (A_T). *Assume that either*

(i) $\phi', g' \in C([0,\infty))$ *and* $(g')^2/\phi' \in L^\infty([0,\delta))$ *for* $\delta > 0$, *or*
(ii) $v \geqslant \Delta > 0$ *on* $\overline{Q_T}$ *for some* $\Delta > 0$.

If $u(\cdot,0) \leqslant v(\cdot,0)$, *then* $u \leqslant v$.

Alternatively, if v is a solution of (A_T) *with* $\phi(u) = o(u^{m-1})$ *as* $u \to 0^+$ *or if v is a supersolution of* $({}^lA_T)$ *for some* $l > 0$, *then the comparison result is still true without assumption* (i) *or* (ii).

Regarding assumption (i) above, Gilding has shown this to be unnecessary if $a = 0$, and it is assumed that either $\phi(U)_x$ is continuous or $\phi(s)|g'(s)| = o(\phi'(s))$ as $s \to 0^+$ [12]. As the results above shall be adequate for the analysis of solutions to follow, we will not explore this issue further in the present work.

4. Existence and decay of solutions of (A_∞)

The main result of this section is the following.

Theorem 4.1. *Under any of the conditions*

(i) $a \leqslant 0$,
(ii) $a > 0$ *and* $p < m$, *or*
(iii) $a > 0$ *it sufficiently small and* $p = m$, *there exists a solution of* (A_∞) *for any initial state* $u_0 \in L^\infty((0,1))$, $u_0 \geqslant 0$.

In order to prove Theorem 4.1, it may be first noted that an estimate of $\| u(\cdot, t) \|_q$ which is independent of t is sufficient for obtaining a similar estimate of $\| u(\cdot, t) \|_\infty$. The verification of this lies in the fact that if $\| u(\cdot, t) \|_\infty$ is bounded, then the differential equation in (A_T) has a sublinear forcing. Thus, a supersolution of the form Ke^{ct} may be used to bound u on Q_T for any $T > 0$. The estimate of $\| u(\cdot, t) \|_q$ is a consequence of the next lemma, which also contains information regarding the eventual decay of $\| u(\cdot, t) \|_r$ for $r \geqslant q$.

Lemma 4.1. *Let* u *denote the solution of* (A_T) *with* $a > 0$. *For* $r \geqslant q$, *define* $F(t) \equiv \|u(\cdot, t)\|_r$ *and, in case* $r > 1$,

$$C \equiv \left[\frac{4\alpha(r-1)}{a(r+m-1)^2} \right]^{1/(p-m)}.$$

Here, $\alpha, m > 0$ *are the constants with* $\phi'(u) \geqslant \alpha u^{m-1}$ *for all* $u > 0$.

(i) *If* $p < m$, *then* $T^* = \infty$ *and* $\limsup_{t\to\infty} F(t) \leqslant C$.
(ii) *If* $p = m$ *and* $a \leqslant 4\alpha(r-1)/(r+m-1)^2$, *then* F *is monotone decreasing on* $[0, T]$.
(iii) *If* $p > m$ *and* $F(t_0) < C$ *for some* $t_0 < T$, *then* F *is monotone decreasing on* $[t_0, T]$.

Proof. Recall that inequality (2), developed in section 2, contained

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 w^r(x,t)\,\mathrm{d}x \leqslant \int_0^1 \{-r(r-1)[\Psi_l(w)_x]^2 + ab_{k,i} r w^r\}(x,t)\,\mathrm{d}x,$$

where w is the classical solution of $(A_T^{k,l,i})$. (Note that in the cases $r = 1$ or $a \leqslant 0$, it is a simple matter to obtain an estimate of $\| u(\cdot, t) \|_r$ directly from this inequality.) In previous work, the gradient term above was discarded in the process of proving existence of solutions for (A_T). Here, this term is handled more carefully, and the lemma follows.

Towards estimating the gradient term, observe

$$\begin{aligned} w^r(x,t) &= \{\Psi_l^{-1}(\Psi_l(w(x,t)))\}^r \\ &= \left\{ \Psi_l^{-1} \left[\int_0^x \Psi_l(w)_x \right] \right\}^r. \end{aligned}$$

Now, by Hölder's inequality,

$$\int_0^x \Psi_l(w)_x \leqslant \left\{ \int_0^x [\Psi_l(w)_x]^2 \right\}^{1/2}.$$

So $\hat{F}(t) \equiv \| w(\cdot, t) \|_r$ has

$$\hat{F}^r(t) \leqslant \left\{ \Psi_l^{-1} \left[\left(\int_0^1 (\Psi_l(w)_x)^2 \right)^{1/2} \right] \right\}^r,$$

and, hence,

$$[\Psi_l(\hat{F}(t))]^2 \leqslant \int_0^1 (\Psi_l(w)_x)^2.$$

The assumption $\phi'(u) \geqslant \alpha u^{m-1}$ for $u > 0$ is for the purpose of obtaining

$$\Psi_l(u) \geqslant \int_{1/l}^{u} \sqrt{s^{r-2}(\alpha s^{-1})}\,\mathrm{d}s$$
$$= \frac{2\sqrt{\alpha}}{r+m-1}\left[u^{(r+m-1)/2} - \left(\frac{1}{l}\right)^{(r+m-1)/2}\right].$$

Utilization of inequality (2) thus yields

$$\frac{\mathrm{d}}{\mathrm{d}t}[\hat{F}^r(t)] \leqslant \frac{-4\alpha r(r-1)}{(r+m-1)^2}\left[\hat{F}(t)^{(r+m-1)/2} - \left(\frac{1}{l}\right)^{(r+m-1)/2}\right]^2 + arb_{k,i}(t)\hat{F}^r(t),$$

and, upon passing to the limits $k \to \infty$ and $i \to \infty$, there follows

$$F_l^r(t) - F_l^r(s) \leqslant \int_s^t R_l(F_l) \tag{4}$$

for all $0 \leqslant s \leqslant t \leqslant T$. Here $F_l(t) \equiv \| u^l(\cdot, t)\|_r$, recalling that the solution of $({}^lA_T)$ is u^l, and

$$R_l(\lambda) \equiv \frac{-4\alpha r(r-1)}{(r+m-1)^2}\lambda^{r+m-1} + ar\lambda^{r+p-1} + \frac{4\alpha r(r-1)}{(r+m-1)^2}\left(\frac{1}{l}\right)^{(r+m-1)/2}\left[2\lambda^{(r+m-1)/2} - \left(\frac{1}{l}\right)^{(r+m-1)/2}\right].$$

If $p < m$, then $R_l(\lambda) \to -\infty$ as $\lambda \to \infty$ and $R_l(1/l) > 0$. Hence, for each $l > 0$, there exists C_l such that $R_l(C_l) = 0$ and $R_l < 0$ on (C_l, ∞). Furthermore, for $l \geqslant 1$,

$$R_l(\lambda) \leqslant \frac{-4\alpha r(r-1)}{(r+m-1)^2}\lambda^{r+m-1} + ar\lambda^{r+p-1} + \frac{8\alpha r(r-1)}{(r+m-1)^2}\lambda^{(r+m-1)/2}$$

and, for $l \geqslant 1/2C$, $R_l(C) > 0$. So there exists $\Lambda > C$ with the property $C_l \in (C, \Lambda)$ and $R_l(C_l) = 0$.

For $\Delta > 0$ and $l \geqslant \max\{1, 1/2C\}$. As u^l is continuous on Q_T [15], inequality (4) may be used to conclude that $F_l \leqslant C_l + \Delta$ on $[\hat{t}, \infty)$ for some $\hat{t} \geqslant 0$. So $\limsup_{t\to\infty} F_l(t) \leqslant C_l + \Delta$, and, as this is true for all $\Delta > 0$, there follows $\limsup_{t\to\infty} F_l(t) \leqslant C_l$.

Extracting a convergent subsequence, $\{C_{l_n}\}_{n=1}^{\infty}$, from $\{C_l\}$, it can be seen that $C_{l_n} \to C$ as $n \to \infty$. Now $F \leqslant F_{l_n}$ which implies

$$\limsup_{t\to\infty} F(t) \leqslant \limsup_{t\to\infty} F_{l_n}(t) \leqslant C_{l_n}$$

for all n, and, hence, $\limsup_{t\to\infty} F(t) \leqslant C$.

If $p = m$ and $a \leqslant 4\alpha(r-1)/(r+m-1)^2$, then letting $l \to \infty$ in (4) yields

$$F^r(t) - F^r(s) \leqslant \int_s^t r\left[\frac{-4\alpha(r-1)}{(r+m-1)^2} + a\right] F^{r+m-1} \leqslant 0$$

for all $0 \leqslant s \leqslant t \leqslant T$. So it is immediate that F is monotone decreasing on $[0, T]$.

If $p > m$, then $R_l(\lambda) \to \infty$ as $\lambda \to 0$. Furthermore, for fixed $\Delta > 0$ and $F(t_0) < C$, there exists $l > 0$ so that $F_l(t_0) < C - \Delta$ and $R_l(\lambda) < 0$ for $\Delta < \lambda < C - \Delta$. Enlisting inequality (4) again, it follows that $F_l < C - \Delta$ and, consequently, $F \leqslant F_l < C$ on $[t_0, T]$. Now, similar to the previous case,

$$F^r(t) - F^r(s) \leqslant \int_s^t \left[\frac{-4\alpha r(r-1)}{(r+m-1)^2} F^{r+m-1} + arF^{r+p-1}\right] \leqslant 0$$

for $t_0 \leqslant s \leqslant t \leqslant T$, and the monotonicity is established. □

It should also be noted that global existence and decay of $F(t)$ can be similarly developed from (4) in the cases $r = 1$ or $a \leqslant 0$. Furthermore, in (ii) and (iii) above, it follows that $T^* = \infty$ and $\lim_{t \to \infty} F(t) = 0$.

5. Global existence and blow-up in finite time for problem (B_T)

The purpose of this section is to investigate the necessity of the conditions $p < m$ or $p = m$ and $a > 0$ sufficiently small in yielding the solvability of (A_∞). It turns out that $p \leqslant m$ may be weakened somewhat regarding the existence of solutions of (B_∞). The appropriate condition involves the ordering of p and $\max\{m, n\}$, and the degree to which such an ordering is necessary and sufficient is addressed in the following results.

Theorem 5.1 (i) *If $p < \max\{m, n\}$ or if $p = \max\{m, n\}$ and $a > 0$ is sufficiently small, then there exists a solution of (B_∞) for any non-negative initial state $u_0 \in L^\infty((0, 1))$.*

(ii) *Assume $n > 1$ and $m \geqslant 1$. If $p > \max\{m, n\}$ or if $p = \max\{m, n\}$ and a is sufficiently large, then there exists a constant $c_0 \equiv c_0(a, \varepsilon, m, n, p, q) > 0$ such that $T^*(u_0) < \infty$ whenever $u_0(x) \geqslant c_0 \sin \pi x$. That is, the solution having initial state u_0 blows up in finite time.*

Proof. (i) Due to the results proven for (A_∞), it is only necessary to consider the case of $\max\{m, n\} = n$. To this end, assume $n > m$ and $p \leqslant n$. The function $v(x) = M(2 - x)$ is a supersolution of (B_T) for each $T > 0$ provided M is chosen so that

$$m(m-1)M^m(2-x)^{m-2} - \varepsilon n M^n (2-x)^{n-1} + aA_1 M^p (2-x) \leqslant 0, \tag{5}$$

where

$$A_1 \equiv \left[\int_0^1 (2-x)^q \, dx\right]^{(p-1)/q}, \quad \text{and } M \geqslant \|u_0\|_\infty.$$

Upon selecting $M \geqslant \|u_0\|_\infty + 1$ large enough to guarantee

$$m(m-1)M^{-(n-m)} < \varepsilon n \tag{6}$$

and

$$bM^{n-p}[\varepsilon n - m(m-1)M^{-(n-m)}] - 2a(2^{p-1}) \geqslant 0, \tag{7}$$

with $b = 1$ for $n \geqslant 2$ and $b = 2^{n-2}$ if $n < 2$, it can be seen that (5) is satisfied.

It is possible to choose such an M to satisfy (6) due to the assumption $n > m$. Inequality (7) may be verified for large M in the cases $p < n$ or $p = n$ with $a > 0$ sufficiently small. As $v(x)$ is a strictly positive supersolution, the comparison theorem implies that $u \leqslant v$ on $\overline{Q_T}$ for the solution of (B_T) having $u(\cdot, 0) = u_0$. The continuation theorem now guarantees $T^*(u_0) = \infty$.

(ii) Letting $\sigma \geqslant \max\{1/m, 1/(n-1)\}$ and $h(t) > 0$, the function $w(x,t) \equiv h(t)[\sin \pi x]^{\sigma}$ is a subsolution of (B_T) provided

$$\begin{aligned} h'(t) \leqslant{}& m\sigma(m\sigma - 1)\pi^2 h^m(t)(\sin \pi x)^{(m-1)\sigma - 2}(\cos \pi x)^2 \\ & - m\sigma\pi^2 h^m(t)(\sin \pi x)^{(m-1)\sigma} \\ & + \varepsilon n\sigma\pi h^n(t)\sin \pi x)^{(n-1)\sigma - 1}\cos \pi x \\ & + ah^p(t)A_2, \end{aligned} \tag{8}$$

where $A_2 \equiv [\int_0^1 (\sin \pi x)^{\sigma q}\, \mathrm{d}x]^{(p-1)/q}$. Since $\sin \pi x \leqslant 1$ and $\cos \pi x \geqslant -1$, inequality (8) will hold if $h(t)$ is chosen so that

$$h'(t) \leqslant -m\sigma\pi^2 h^m(t) - \varepsilon n\sigma\pi h^n(t) + aA_2 h^p(t). \tag{9}$$

Define

$$Q(s) = \begin{cases} aA_2 s^{p-m} - m\sigma\pi^2 - \varepsilon n\sigma\pi s^{-(m-n)} & \text{if } p \geqslant m \geqslant n, \\ aA_2 s^{p-n} - \varepsilon n\sigma\pi - m\sigma\pi^2 s^{-(n-m)} & \text{if } p \geqslant n > m, \end{cases}$$

and let s_0 denote the positive root of Q. As $p > \max\{m,n\}$ or $p = \max\{m,n\}$ with a sufficiently large, it follows that $Q(s)$, $Q'(s) > 0$ for $s > s_0$. Selecting $h(t)$ to be the solution of

$$h'(t) = Q(h)h^k(t), \quad t > 0,$$
$$h(0) = s_0 + 1,$$

where $k = \max\{m,n\}$, it can be observed that (9) is satisfied.

Let $c_0 = s_0 + 1$. Then $w(x,t)$ is a subsolution of (B_T) such that

$$w(x,0) = c_0[\sin \pi x]^{\sigma} \leqslant u_0(x)$$

on $[0,1]$. Hence, $w(x,t) \leqslant u(x,t)$, where u is the solution of (B_T) with $u(\cdot,0) = u_0$. Finally, since $k > 1$, $\lim_{t \to \hat{T}} h(t) = \infty$ for some $\hat{T} < \infty$ which implies $T^*(u_0) \leqslant \hat{T} < \infty$. □

In the direction of relaxing the restrictions $m \geqslant 1$ and $n > 1$ in the blow-up result above, a different technique allows these to be dropped if, instead, p and q are assumed to be large enough. It is worth noting that the assumptions regarding initial states which give rise to non-existence of solutions to (B_∞) may also be relaxed significantly for such p and q. The precise result is as follows.

Theorem 5.2. *Assume* $p > \max\{m, n\} + 1$ *and* $q \geqslant \max\{m, n\}$. *There exists* $c_1 = c_1(a, \varepsilon, m, n, p) > 0$ *such that if*

$$\int_0^1 u_0(x) \sin \pi x \, \mathrm{d}x > c_1,$$

then $T^*(u_0) < \infty$. *That is, there is no solution of* (B_∞) *with initial state* u_0.

Proof. Let $\xi(x) = \sin \pi x$. In the definition of a solution for (B_T), put $\xi(x, t) \equiv \xi(x)$ to get

$$\int_0^1 u(x,t)\xi(x)\,\mathrm{d}x = \int_0^1 u_0(x)\xi(x)\,\mathrm{d}x - \pi^2 \int_0^t \int_0^1 u^m(x,\tau)\xi(x)\,\mathrm{d}x\,\mathrm{d}\tau$$
$$- \varepsilon \int_0^t \int_0^1 u^n(x,\tau)\xi'(x)\,\mathrm{d}x\,\mathrm{d}\tau$$
$$+ a \int_0^t \| u(\cdot, \tau) \|_q^{p-1} \int_0^1 u(x,\tau)\xi(x)\,\mathrm{d}x\,\mathrm{d}\tau.$$

Setting $J(t) \equiv \int_0^1 u(x,t)\xi(x)\,\mathrm{d}x$, a formal differentiation with respect to t in the above equation yields

$$J'(t) = -\pi^2 \int_0^1 u^m(x,t)\xi(x)\,\mathrm{d}x - \varepsilon \int_0^1 u^n(x,t)\xi'(x)\,\mathrm{d}x$$
$$+ a \left[\int_0^1 u^q(x,t)\,\mathrm{d}x \right]^{(p-1)/q} \int_0^1 u(x,t)\xi(x)\,\mathrm{d}x.$$

Since $q \geqslant \max\{m, n\}$, Hölder's inequality now implies

$$\int_0^1 u^m(x,t)\xi(x)\,\mathrm{d}x \leqslant \| u(\cdot, t) \|_q^m$$

and

$$\int_0^1 u^n(x,t)\xi'(x)\,\mathrm{d}x \leqslant \pi \| u(\cdot, t) \|_q^n.$$

Define

$$R(s) \equiv \frac{a}{2} s^{(p-1)} - \pi^2 s^m - \varepsilon \pi s^n,$$

and let $s_1 > 0$ denote the root of R. As $p - 1 > \max\{m, n\}$, it is the case that $R(s)$, $R'(s) > 0$ for all $s > s_1$. For $s \equiv s(t) \equiv \| u(\cdot, t) \|_q$, the above calculations imply

$$J'(t) \geqslant \frac{a}{2} J(t) s^{p-1} + \frac{a}{2} (J(t) - 1) s^{p-1} + R(s). \tag{10}$$

So, if $c_1 \equiv \max\{s_1, 1\}$, then $J(t) \leqslant s(t)$ (which follows by Hölder's inequality) gives $s(0) \geqslant J(0) > c_1$. Hence, $J'(0) > 0$.

Suppose $\hat{t} \in (0, T^*(u_0))$ is such that $J' > 0$ on $[0, \hat{t})$ and $J'(\hat{t}) = 0$. Then, using (10), either $J(\hat{t}) < 1/2$ or $s(\hat{t}) < s_1$. Because J is strictly increasing on $[0, \hat{t})$ and

$J(0) > c_1 \geqslant 1$, the first of these alternatives is impossible. On the other hand,

$$s(\hat{t}) \geqslant J(\hat{t}) > J(0) > c_1 \geqslant s_1,$$

so the second alternative is also impossible. Thus, $J' > 0$ on the entire interval $[0, T^*(u_0))$.

Now $s(t) \geqslant J(t) > c_1$ and (10) yield

$$J'(t) \geqslant \frac{a}{2} J(t) s^{p-1} \geqslant \frac{a}{2} J^p(t).$$

Since $p > 1$, it follows that

$$J(t) \geqslant \left\{ J^{(1-p)}(0) - \frac{a}{2}(p-1)t \right\}^{-1/(p-1)}, \tag{11}$$

so $\lim_{t \to \hat{T}} J(t) = \infty$ for some $\hat{T} < \infty$.

Upon regularization of (B_T), i.e. $({}^l B_T)$, the formal calculations may be justified for solutions of the regularized problems. Then, passing to the limit $l \to \infty$ in (11) proves $T^*(u_0) \leqslant \hat{T} < \infty$. □

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