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TALKING AND LOOKING STRUCTURALLY AND OPERATIONALLY AS WAYS OF ACTING IN A SOCIO-POLITICAL MATHEMATICAL PRACTICE

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This paper proposes a reconceptualization of the psychological constructs of structural and operational conception. Drawing on Morgan, Moschkovich and Sfard, we present a theoretical perspective that views talking and looking operationally and structurally as ways of acting in a socio-political mathematical practice. We use transcripts of first-year undergraduate students solving a function problem to illustrate this perspective. We argue that student action is a complex interplay of the ways of talking about and looking at the mathematical objects, together with discursive, social and political ways of acting in the classroom.

INTRODUCTION

The dual ontological nature of mathematical objects as both processes and objects has been captured in psychological theories of mathematics learning such as *Action, Process, Object, Schema Theory* (Dubinsky, 1991), the theory of *reification* (Sfard, 1991), and the notion of *proceptual thinking* (Gray & Tall, 1994). It has been argued that both a *structural* and an *operational conception* of a mathematical object are necessary for mathematical understanding (Sfard, 1991), with some of these arguments focusing on the learning of functions (e.g. Breidenbach, Dubinsky, Nichols & Hawks, 1990; Moschkovich, Schoenfeld & Arcavi, 1993). These theories have dominated research on the learning of undergraduate mathematics since the early 1990s (e.g. Hazzan, 2003; Maharaj, 2010; Stewart & Thomas, 2009). This research has made a substantial contribution to our understanding of student action on mathematical objects. However, we argue that this research does not have the discursive, social and political action of students in the mathematics classroom in view and it does not take into account the macro-social issues that figure in the classroom action.

In this paper we draw on the work of Morgan (1998) as well as the later work of both Moschkovich (2004, 2007) and Sfard (2000, 2008) to present a reconceptualization, from a socio-political practice perspective, of the psychological notions of structural and operational conception. We present ways of talking and looking operationally and structurally (along with other ways of acting such as ways of endorsing arguments and of interacting socio-politically) as ways of acting in a socio-political mathematical practice. We illustrate the use of the perspective in the analysis of transcripts representing the verbal and non-verbal action of first-year undergraduate students on a function problem, and show that while the interplay between operational and structural in student action is central, it is only part of the story of the students' action.

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THEORETICAL FRAMEWORK

A socio-political perspective of mathematical practice is based on the work of Fairclough (2001, 2003) in critical linguistics. Action of undergraduate students solving a function problem is located in the socio-political practice of first-year undergraduate mathematics. This practice is *socio-political* since it is a relatively stable way of doing things in which certain material and mental activities, objects, participants, socio-political relations, beliefs, and discourse are valued. Undergraduate mathematics is networked with other practices such as school mathematics and professional research mathematics. Power relations are at work both between these practices and within the action of the students in the mathematics classroom. *Mathematical discourse* is the language aspect of a socio-political mathematical practice and gives meaning to the practice; it *represents* the practice in a certain way, it is used for *interaction*, and it *identifies* participants as particular types of people. Sfard (2008), Morgan (1998) and Moschkovich (2007) agree that there is something distinctive called *mathematical discourse*, while acknowledging that this is changing, has no fixed boundaries, and is used in a variety of mathematical practices.

Using the work of Fairclough and that of Morgan, Moschkovich, and Sfard in mathematics education, in interaction with empirical data of first-year undergraduate student action, we have identified interrelated ways of acting mathematically in discourse. These ways of acting capture the mathematical, discursive, social, and political nature of this action. We focus on those ways of acting that relate specifically to the dual nature of mathematical objects (points 1 and 5).

1. Ways of talking and writing about objects and ways of representing objects
2. Ways of making links between objects, texts, events and practices
3. Ways of endorsing arguments about mathematical objects
4. Ways of evaluating the pronouncements of other participants
5. Ways of attending (ways of looking at mathematical objects and their representations, ways of listening to talk about objects)
6. Ways of operating on mathematical objects
7. Ways of identifying oneself and others, ways of interacting socio-politically.

We use *talking operationally* and *structurally* for the talk about mathematical objects identified by Morgan (1998) and Sfard (2008). Morgan (1998) argues that such objects can be talked about as processes or objects, for example, the nominalization *rotation* represents the process of rotating as an object that can act or be acted upon. A subsequent replacement of "rotation" with the reference pronoun "it" may indicate structural talk. In her recent work on mathematical discourse, Sfard moves away from seeing *reification* as a mental shift in which a "process solidifies into object, into a static structure" (1991, p.20) to viewing this notion as involving "replacement of talk about processes with talk about objects" (2008, p.171). For example process talk about the signifier $^5/_7$ may be "I divided the whole by 7 and took 5 of the parts", whereas object talk would be "I have $^5/_7$ of the whole" (Sfard, 2008, p.171).

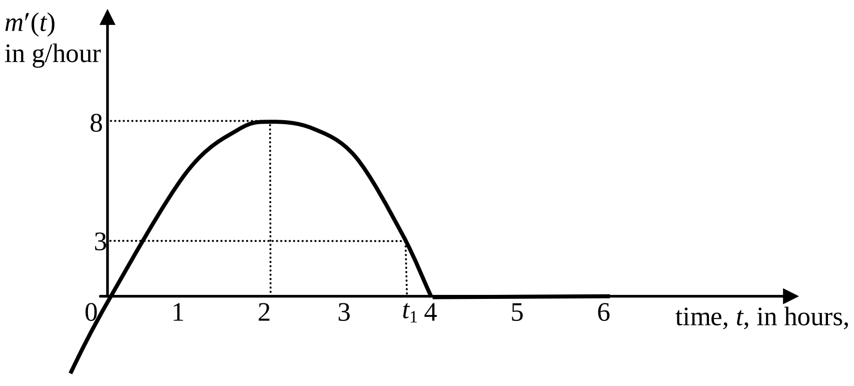
Sfard and Moschkovich propose a concept of mathematical discourse that includes “what we see” (Sfard, 2008, p.146) or *perspectives/ways of seeing* (Moschkovich, 2004, p.50). Moschkovich (2004) argues, from a socio-cultural perspective, that viewing a linear function as a process or as an object is part of expert mathematical practice for working with functions, and does not refer to “an individual competency” (p.57) as in the psychological perspective. The notions of *looking structurally* and *operationally* as ways of attending in mathematical discourse are productive in our analysis of non-verbal student action. For example, substituting the value $t = 2$ to check the accuracy of the formula $y = -2t^2 + 8t$ as a representation of a parabola graph requires an operational view of the quadratic function, while using transformations to identify the formula for a parabola graph suggests that a student is looking structurally at the quadratic function.

THE STUDY

The transcripts presented in this paper are taken from a wider study (Le Roux, 2011) that investigates innovation in the form of practical problems and a learner-centred pedagogy in a first-year undergraduate mathematics course at a South African university. The course is offered to students identified as disadvantaged by enduring inequities in the schooling system, and is designed to facilitate students’ transition from school mathematics to advanced mathematics.

In this paper we present, in brief, the action of four students, Kelsa, Lwazi, Ndumiso and Thokozile (pseudonyms), as they work in a small group (with the support from a tutor) to answer question (d) of the Chemical Reaction Problem (see Figure 1). This data has been chosen as it represents the complex interplay of the different ways of acting mathematically in discourse.

Quantities of two chemicals A and B are mixed together in a reaction chamber, and they react to form a new product. The **rate** at which the produce is formed is given by $m'(t)$, where m is the mass of the product formed, in grams, and the time t from the start of the reaction is measures in hours. The graph of $m'(t)$ is a parabola graph until time $t = 4$ hours, after which it is zero. It is also given that, from the start of the reaction, some of the product X is removed from the reaction chamber at a constant rate of 3g/hour.



d) Find the equation of the parabola graph - it will express $m'(t)$ as a quadratic function of t.

Figure 1: Chemical Reaction Problem, question (d)

Answering question (d) in Figure 1 involves the student making a link to school mathematics and viewing the quadratic function structurally by identifying one of a number of possible general formulae ($y = at^2 + bt + c$ or $y = a(t - r_1)(t - r_2)$ or $y = a(t - p)^2 + q$) as representing a family of functions and hence an appropriate representation of a parabola graph. The student makes links between certain points on the given parabola graph and particular symbols in the selected formula. Then, viewing the function operationally, the student acts operationally by substituting and rearranging the subject of the formula. Acting in this way, the student arrives at the formula $y = -2t^2 + 8t$ for the function. In this discussion we do not focus on the nature of this quadratic function as the derivative function $m'(t)$; this, together with additional data from the study, is the subject of a further paper.

ANALYTIC TOOLS

The transcripts were analysed using Fairclough's three-stage method of critical discourse analysis. We focus here on the first two stages; the detailed textual analysis of the transcripts and the use of these textual clues to identify the ways of acting mathematically in discourse. Fairclough's method was supplemented with Sfard's (2000) method of focal analysis so as to bring the specificity of mathematical action into focus. We illustrate with action on question (d) in Figure 1:

After Thokozile has proposed "x^2" as a formula for the graph, Ndumiso responds:

261 Ndumiso: That's a that's a <u>general</u> parabola↑ that's the <u>easy</u> one … this has <u>moved</u> ((*Showing shift with his fingers over the graph, looking at Thokozile as he speaks*))

The pronounced focus is what Ndumiso says in line 261. The attended focus is what Ndumiso is "looking at, listening to" when speaking (Sfard, 2000, p.304); he looks at the given parabola graph and listens to Thokozile's proposed answer of "x^2". We then perform critical discourse analysis on the pronounced focus. In his use of the demonstrative pronouns "that" (for Thokozile's "x^2") and "this" (for the parabola graph), Ndumiso is talking structurally about the graphs as objects that can move. This analysis is supported by Ndumiso's description of the graph as having "moved" and his gesturing of a shift in the graph; here the material process of moving suggests that Ndumiso is viewing the function as an object that can be acted on. In addition, through the emphasis on the adjective "<u>easy</u>", Ndumiso identifies himself as attending to a more complicated graph than Thokozile.

We use this method of analysis to now describe the four students' ways of acting mathematically in discourse as they answer question (d) in Figure 1.

THE STUDENTS' WAYS OF ACTING MATHEMATICALLY

The students begin by making links to school mathematics and they recruit three possible general formulae for the quadratic function. Ndumiso pronounces "that equation" $y = a(x - x_1)(x - x_2)$, by verbally naming the symbols from left to right. Kelsa's responds negatively by verbally stating an alternative general quadratic formula from school ($y = ax^2 + bx + c$) and stating, "That's the equation for parabola↑". Her emphasis on the reference pronoun "that's" represents her formula as the only one. Ndumiso is persuaded that his formula ("this") represents something else; "What's this equation for?" These references to formulae suggest that the students do not identify the two general formulae as equivalent, suggesting that they are looking operationally at the formulae as representing different processes. This argument is supported by Ndumiso's discussion with Lwazi. Here Lwazi is promoting a third general formula, $y = a(x - p)^2 + q$, although this is not pronounced verbally:

248	Lwazi:	There is an equation like that
249	Ndumiso:	For what though? ((*Looking at Lwazi*))
250	Lwazi:	But that's not it though
251	((*Ndumiso and Kelsa laugh, Thokozile is looking at Lwazi*))	
252	Lwazi:	No really↑ the *a* is right at the beginning↑ but the stuff in the middle isn't ((*Pointing to Ndumiso's equation* $y = a(x - x_1)(x - x_2)$))
253	Ndumiso:	It IS
254	Lwazi:	It's not
255	Ndumiso:	I know it is
256	Lwazi:	It's not … I'm telling you it's not

In this action, the use of the demonstrative pronoun "that" and the reference pronoun "it" suggests that the students are looking operationally and identifying the different formulae as representing different graphs.

Other ways of acting mathematically in discourse can be identified in the transcript lines presented so far. The students' endorsements are in the form of personal opinion ("I know", line 255), statements of fact with no supporting evidence ("That's the equation for parabola↑"), and alternative forms of the general quadratic formula. There is an absence of endorsements that make use of the properties of the quadratic function and links between the formulae and the graph are not given significance. Furthermore, the socio-political interaction between the students is competitive as they claim personal ownership ("I") of the different versions of the quadratic formula, action that may prevent identification of the three formulae as equivalent.

Unable to resolve whose quadratic formula is "it" (line 250), the students then pursue a link, made initially by Thokozile, to a method for finding the equation of a parabola used elsewhere in the undergraduate mathematics course. This method, named "the whole movement" thing by Thokozile, involves finding quadratic functions of the form $y = a(x - p)^2 + q$ (where a is ±1 only) using translations and reflections. The analysis (see for example Ndumiso's focus in line 261**)** suggests that the students view the quadratic function structurally as they consider how the graphs "shift" up and down. However, the students' ways of acting mathematically constrain the productive use of this method. Firstly, in making a link to other parts of her course, Thokozile is constrained by not identifying the course method as only applicable to a certain class of functions of the form $y = a(x - p)^2 + q$ and hence assumes that $a = \pm 1$ in her formula. Secondly, the students pronounce possible quadratic formulae verbally, from left to right, as in Thokozile's "minus *x* squared plus 4". Such verbal pronouncements constrain the students' attention to the appropriate use of symbols; in this case the students interpret Thokozile's pronouncement as $-x^2 + 4$, when it could be written as $-(x^2 + 4)$. Thirdly, Kelsa's evaluation of Thokozile's expression "$-x^2 + 4$" involves her substituting the point $x = 2$ into the graph to test whether a *y*-value of 8 is obtained, as shown in line 276 (for ease of reading we use symbols in the brackets {…} to represent verbal pronouncements):

276 Kelsa: No it doesn't↑ … minus 2 squared plus 4 gives you zero $\{-2^2 + 4 = 0\}$

277 Thokozile: Minus 2 squared $\{-2^2\}$ ((*Looking across briefly at Kelsa*)) the *x* is ... ... this minus is here ((*Pointing to her equation*)) ja minus *x* squared $\{-x^2\}$

278 Kelsa: So why don't you just say *x* squared plus ... ... plus 4 $\{x^2 + 4\}$? … cause your minus putting your minus there means negative 1 times *x* squared $\{-1 \times x^2\}$ … … so you must just make it *x* squared plus 4 $\{x^2 + 4\}$↑

By providing a way of evaluating formulae, Kelsa identifies herself and is identified by the other students as an authority in first-year undergraduate mathematics. Her way of evaluating becomes the valued way in the group, as used by Thokozile in line 277. Kelsa also uses her method of substitution to adapt the proposed formulae, suggested by her repeated use of "just" in line 278. Kelsa's use of substitution for both evaluation and construction of formulae points to an operational view of the quadratic function.

After an ongoing struggle with the question, Ndumiso and Lwazi enlist the help of the tutor, who begins by re-establishing the link to school mathematics ("something uhm like … you did at school"). Lwazi and Ndumiso pronounce their versions of the school formulae simultaneously in lines 477 and 478 (overlapping text in square brackets):

477 Lwazi: Yes we did but I forgot the formula there… it's got an *a* it's got an *l* … [you know what I'm talking about it's got a bracket ((*Gesturing in the air, looking at the Tutor as he speaks*))]

478 Ndumiso: [It's *y* equals to *a* *x* minus x_1 *x* minus x_2 $\{y = a(x - x_1)(x - x_2)\}$↑ ((*Nodding his head in time as he says each term*))]

479 Lwazi: No ((*Shaking his* head)) it's not [[it's]]

480 Tutor: [[YES ((*Looking at Ndumiso*))]]

481 Ndumiso: Exactly

482 Tutor: What is x_1? ((*Pointing at Ndumiso*))

483 Ndumiso: x_1 is going to be your first intercept↑ ((*Pointing to something on his graph*))

The Tutor's validation of Ndumiso's formula $y = a(x - x_1)(x - x_2)$ in line 480, an evaluation that resides in his personal authority rather than a mathematical argument, together with the absence of further attention to Lwazi's formula, constrains any exploration of links between the two formulae. Having endorsed one formula from school mathematics, the tutor has a further role to play in making links between the symbols in the formula $y = a(x - x_1)(x - x_2)$, different points on the graph and different sketches of parabola graphs, as he begins to do in discussion with Ndumiso in line 482. This intervention by the tutor enables Ndumiso to produce a correct solution, who in turn assists Kelsa on her request; she only identifies Ndumiso as an authority after validation from the tutor. However, Lwazi and Thokozile do not identify with Ndumiso's formula. Lwazi starts to use it, but skips steps to the correct answer and continues debating possible formulae with Ndumiso. Thokozile does not produce any written work, but proceeds to use the correct formula in the next question.

DISCUSSION

This analysis points to the interplay between students' ways of talking about and looking operationally and structurally at the quadratic function, in particular how these ways may enable or constrain their action and how they adopt different ways of talking and looking within the same problem. This interplay talks to the psychological research on structural and operational conceptions. For example, initially, the students' identification of general formulae suggests a structural view of these formulae as representing families of functions (Sfard & Linchevski, 1994). Yet they may not be invoking a structural view (Moschkovich, Schoenfeld & Arcavi, 1993), suggested by their not identifying the three school formulae as equivalent and looking operationally at the formulae as representing different computational processes (Sfard, 1992). Yet their subsequent drawing and gesturing of transformations of the parabola graph suggests a structural view of the quadratic function (Moschkovich, Schoenfeld & Arcavi, 1993). Throughout the students' way of evaluating their proposed formulae using substitution suggests an operational view of the function (Sfard, 1991).

The analysis suggests, however, that the socio-political perspective of mathematical practice allows the researcher to view student action at the micro-level of the classroom as more than "an intricate interplay between operational and structural conceptions" (Sfard, 1991, p.1) that are located in the mind of the individual student. Rather, it is a complex interplay of mathematical, discursive, social and political ways of acting mathematically. The students' ways of talking and looking operationally and structurally interact with their ways of making links (between practices, within the course, and between mathematical representations), their ways of endorsing arguments and evaluating, their ways of talking, their ways of identifying themselves and others, and their ways of interacting socio-politically. The framework we have presented also has the potential to link the classroom action to the wider socio-political space in which it is located, but this is the topic of a further paper.

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