Tackling Multiplicity of Equilibria with Gröbner Bases

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with Gröbner Bases*

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Abstract

Multiplicity of equilibria is a prevalent problem in many economic models. Often equilibria are characterized as solutions to a system of polynomial equations. This paper gives an introduction to the application of Gröbner basis methods for finding all solutions of a polynomial system. The Shape Lemma, a key result from algebraic geometry, states under mild assumptions that a given equilibrium system has the same solution set as a much simpler triangular system. Essentially the computation of all solutions then reduces to finding all roots of a single polynomial in a single unknown. The software package SINGULAR computes the equivalent simple system. If all coefficients in the original equilibrium equations are rational numbers or parameters then the Gröbner basis computations of SINGULAR are exact. This fact implies that the Gröbner basis methods cannot only be used for a numerical approximation of equilibria but in fact may allow the proof of theoretical results for the underlying economic model. Three economic applications illustrate that without much prior knowledge of algebraic geometry Gröbner basis methods can be easily applied to gain interesting insights into many modern economic models.

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1 Introduction

Multiplicity of equilibria is a prevalent problem in economics, both in equilibrium models with strategic interactions and in competitive models. While this problem has long been acknowledged in the theoretical literature it has in the past often been ignored in applied work. But finally there appears now also to be a growing interest in equilibrium multiplicity in active areas of modern applied economic analysis. For example, Bodenstein (2008) points out that multiple steady states arise for reasonable parameter values in a standard model of the international business cycle literature. Similarly, Besanko et al. (2007) show that multiple Markov perfect equilibria can easily arise in a stochastic game model of industry dynamics. This model is an example of a large class of models that has become very popular in industrial organization and marketing. And in many other applications we may often suspect that there could be multiple equilibria. But standard numerical methods only search for a single equilibrium. There is clearly a need in economics for methods that can find all equilibria for applied models.

In many economic models equilibria can be described as solutions of polynomial equations (which may also have satisfy some additional inequalities). Recent advances in the mathematical field of computational algebraic geometry have led to several powerful methods and their easy-to-use computer implementations that find all solutions to polynomial systems. Two different solution approaches stand out, all-solution homotopy methods and Gröbner basis methods. For reasons we describe below we focus on Gröbner basis methods in this paper. We provide a fairly non-technical introduction to these methods and to the computer algebra system Singular, which as of the writing of this paper is considered the best freely available software (at www.singular.uni-kl.de) for computing Gröbner bases. Three economic applications illustrate that without much prior knowledge of algebraic geometry Gröbner basis methods can be easily applied to gain interesting insights into many modern economic models.

The basic idea of the Gröbner basis methods for solving polynomial systems of equations is as follows. The Shape Lemma, a key result from algebraic geometry, states under mild assumptions that a given equilibrium system has the same solution set as a much simpler triangular system. Essentially the computation of all solutions then reduces to finding all roots of a single polynomial in a single unknown. The software package Singular computes the equivalent simple system. If all coefficients in the original equilibrium equations are rational numbers or parameters then the computations of Singular are exact. This fact implies that the Gröbner basis methods cannot only be used for a numerical approximation of equilibria but in fact may allow the proof of theoretical results for the underlying economic model.

Homotopy continuation methods provide an alternative method for finding all solutions. The basic idea is to start at a generic polynomial system \( g(x) \) whose number of roots is at least as large as the maximal number of solutions to \( f(x) = 0 \) and whose roots are all known.
Then one needs to trace out all paths (in complex space) of the homotopy \( H(x, t) = tg(x) + (1 - t)f(x) \) starting at each solution for \( t = 0 \). All solutions to \( f(x) = 0 \) can be found in this manner, see Sturmfels (2002) and Sommese and Wampler (2005). The software package PHCpack (available free of charge at http://www.math.uic.edu/~jan/) provides a fast and robust implementation of an all-solution homotopy method among its many features. The solver can be used as a black-box, entering the system of polynomial equations in a file. Homotopy methods can usually solve larger systems, with more unknowns and polynomials of higher degrees, than Gröbner basis methods. But Gröbner basis methods have the following important advantages.

1. Homotopy methods are purely numerical methods. Due to rounding errors it is sometimes difficult to determine whether a homotopy path has indeed converged or whether the final numerical solution is real or complex. On the contrary, Gröbner basis methods offer the possibility of exact calculations without any rounding errors. Therefore, such a method may allow us to prove the existence of a unique equilibrium.

2. We can calculate Gröbner bases for parameterized polynomials. This fact implies that we can establish bounds on the number of equilibria for entire classes of economic models.

3. Parameterized Gröbner bases enable us to search for specific parameter values for which there are multiple equilibria or to prove that equilibria are unique for all parameter values in a given set.

We illustrate these points with three economic examples. While these are small examples which are chosen to illustrate the advantages of Gröbner bases we feel that they also provide some interesting economic insights. Hopefully they serve as a motivation for the reader to apply the methods presented in this paper to other and larger models.

We first consider a game under incomplete information with cheap talk (the arms race game of Baliga and Sjöström (2004)) and show how multiplicity of perfect Bayesian equilibria in such a game can be addressed with Gröbner bases. Agents’ types are i.i.d. with a cumulative distribution function \( F \). Under the assumption that \( F \) is polynomial, the solutions of a system of polynomial equations constitute cut-off values in type space which in turn determine the cheap-talk message agents send in equilibrium. We compute the Gröbner basis for the polynomials appearing in the equations and then compute all solutions. We show how the two equilibria in the game change as some key parameter in the model changes.

Secondly we consider a strategic market game (a variation of Shapley and Shubik (1977)) with a large but finite number of players and show how all competitive equilibria in the underlying economy are approximated by all Nash equilibria of the game as the number of agents becomes large. In our example there are two types of agents with heterogeneous CES utility and heterogeneous endowments and finitely many identical individuals within
in each type. If the total number of players is small there is a unique Nash equilibrium. We show how the number of Nash equilibria increases as the number of players goes up and how finally all Nash equilibria approximate the three competitive equilibria of the economy. While there is a large theoretical literature on strategic market games and the convergence of Nash equilibria to competitive equilibria in these games (see e.g. Postlewaite and Schmeidler (1978)), little work has been done on the computation of all equilibria in these games. Our example illustrates that interesting insights can be obtained from relatively simple computations in these games.

Finally we give an example of multiplicity of steady states in a general equilibrium model with overlapping generations and individuals that live for more than two periods. For a class of models, we show that there can never be more than three steady states and give examples of models where this bound is actually attained. Models of overlapping generations are routinely used in applied policy analysis (see e.g. Auerbach and Kotlikoff (1987)). The multiplicity of steady states in these models potentially casts doubts on the validity of this analysis. Our analysis illustrates that examples of multiplicity can easily be constructed in standard models but that it is also true that steady states are unique for a large range of parameter values.

There is, of course, a growing literature on the computation of all equilibria in normal form games, see Dutta (2007) for an excellent recent survey and also Sturmfels (2002). In this paper, we do not address this problem because we consider it somewhat less important in applied economic modeling.

This paper is organized as follows. In Section 2 we give a simple non-technical introduction to Gröbner bases. Section 3 describes how to use SINGULAR to compute Gröbner bases and all solutions to polynomial equations. In Sections 4 – 6 we provide examples of simple but interesting economic applications. The Appendix provides more formal results on Gröbner bases.

## 2 Some Background on Polynomials and Gröbner Bases

In this section we summarize some basic definitions and concepts from the field of algebraic geometry that are fundamental to our analysis in this paper. We refer the interested reader to the textbooks by Cox et al. (1997, 1998). The treatment in this section is deliberately simple. In the Appendix we provide a more detailed and formal description of the main concepts.

### 2.1 Polynomials

For the description of a polynomial $f$ in the $n$ variables $x_1, x_2, \ldots, x_n$ we first need to define monomials. A monomial in $x_1, x_2, \ldots, x_n$ is a product $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ where all exponents $\alpha_i, i = 1, 2, \ldots, n$, are nonnegative integers. It will be convenient to write a monomial as
\(x^\alpha \equiv x_1^{\alpha_1} \cdot x_2^{\alpha_2} \ldots x_n^{\alpha_n}\) with \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_+^n\), the set of nonnegative integer vectors of dimension \(n\). A polynomial is a linear combination of finitely many monomials with coefficients in a field \(K\). In this paper we not need to consider arbitrary fields of coefficients but instead we can focus on three commonly used fields. These are the field of rational numbers \(\mathbb{Q}\), the field of real numbers \(\mathbb{R}\), and the field of complex numbers \(\mathbb{C}\). Polynomials over the field of rational numbers are computationally convenient since modern computer algebra systems (such as SINGULAR which is described below) perform exact computations over the field \(\mathbb{Q}\). Economic parameters are typically real numbers and thus force us to consider \(\mathbb{R}\[x]\).

We can write a polynomial \(f\) as

\[f(x) = \sum_{\alpha \in S} a_\alpha x^\alpha, \quad a_\alpha \in K, \quad S \subset \mathbb{Z}_+^n \text{ finite}.
\]

We denote the collection of all polynomials in the variables \(x_1, x_2, \ldots, x_n\) with coefficients in the field \(K\) by \(K[x_1, \ldots, x_n]\), or, when the dimension is clear from the context, by \(K[x]\). The set \(K[x]\) is called ‘a polynomial ring’ (it satisfies the properties of a so-called commutative ring but this is irrelevant for our purposes).

We are interested in the set of real solutions to a system of polynomial equations, that is, given \(f_1, \ldots, f_k \in K[x_1, \ldots, x_n]\) we want to find all elements in (the hopefully finite) set

\[
\{x \in \mathbb{R}^n : f_1(x) = \ldots = f_k(x) = 0\}.
\]

The study of solution sets of polynomial equations requires the variables \(x\) to range over an algebraically closed field. Unlike the fields \(\mathbb{Q}\) and \(\mathbb{R}\) the field of complex numbers \(\mathbb{C}\) is algebraically closed. Therefore, our objective is to find all elements in the set

\[
V = \{x \in \mathbb{C}^n : f_1(x) = \ldots = f_k(x) = 0\}.
\]

This solution set is called the complex variety defined by \(f_1, \ldots, f_k\). The key observation in finding all solutions is that we can multiply each of the polynomials \(f_i\) by another non-zero polynomial and add any polynomials \(f_i\) and \(f_j\) without eliminating any of the original solutions and without introducing additional solutions. It turns out that Gaussian elimination in linear algebra has a close analogue for polynomial equations. In order to make this intuition more formal, we need an additional definition. For given polynomials \(f_1, \ldots, f_k\), the set

\[
I = \{\sum_{i=1}^k h_i f_i : h_i \in K[x]\} = \langle f_1, \ldots, f_k \rangle,
\]

is called the ideal generated by \(f_1, \ldots, f_k\). The ideal \(\langle f_1, \ldots, f_k \rangle\) is the set of all linear combinations of the polynomials \(f_1, \ldots, f_k\), where the ‘coefficients’ in each linear combination are themselves polynomials in the polynomial ring \(K[x]\). There two aspects about ideals that are crucial for our analysis. First note that

\[
\{x \in \mathbb{C}^n : f_1(x) = \ldots = f_k(x) = 0\} = \{x \in \mathbb{C}^n : g(x) = 0 \text{ for all } g \in \langle f_1, \ldots, f_k \rangle\}.
\]
In other words, the set of solutions to a polynomial system of equations is identical to the set of solutions to all (infinitely many!) polynomials in the ideal generated by the system. Therefore, we can call the solution set \( V \) the complex variety of the ideal \( \langle f_1, \ldots, f_k \rangle \). Secondly note that we can find other polynomials, \( g_1, \ldots, g_r \) such that \( \langle g_1, \ldots, g_r \rangle = \langle f_1, \ldots, f_k \rangle \) and

\[
\{ x \in \mathbb{C}^n : f_1(x) = \ldots = f_k(x) = 0 \} = \{ x \in \mathbb{C}^n : g_1(x) = \ldots = g_r(x) = 0 \}.
\]

The sets of polynomials \( g_1, \ldots, g_r \) and \( f_1, \ldots, f_k \) are called bases of the ideal \( \langle f_1, \ldots, f_k \rangle \).

The idea is then to find an alternative basis for the ideal generated by \( f_1, \ldots, f_k \) that is easy to solve.

Consider the example of two polynomials \( f_1 \) and \( f_2 \) in the two unknowns \( x_1 \) and \( x_2 \),

\[
f_1 = 2x_1^2 + 3x_2^2 - 11 \quad \text{and} \quad f_2 = x_1^2 - x_2^2 - 3.
\]

What can we say about the set of common roots of these two polynomials? Note that for any ideal \( I \), simply by definition, if \( f_1, \ldots, f_k \in I \), then \( \langle f_1, \ldots, f_k \rangle \subset I \). Thus, showing \( f_1, f_2 \in \langle x^2 - 4, y^2 - 1 \rangle \) and conversely \( x^2 - 4, y^2 - 1 \in \langle f_1, f_2 \rangle \) proves that \( \langle f_1, f_2 \rangle = \langle x^2 - 4, y^2 - 1 \rangle \). Therefore \( V(f_1, f_2) \) consists of the four points \( (2, 1), (-2, 1), (2, -1) \) and \((-2, -1)\).

Obviously the example is rather simple and we could have solved the problem without any knowledge of the term ‘ideal.’ However, the solution approach of transforming a given system of polynomial equations into a simpler system with an identical solution set works much more generally. Under some mild conditions it is always possible to find an alternative basis for a polynomial system that can be solved easily. Such a good basis is the \textit{Gr"obner basis under lexicographic monomial order}. So why exactly do we care about Gr"obner bases?

### 2.2 The Exact Computation of All Solutions

For the remainder of this paper we restrict ourselves to square systems of polynomial equations, that is, \( k = r = n \). In a slight abuse of notation, we call an ideal regular if its complex variety has finitely many complex solutions which are locally unique in the sense that the Jacobian has full rank at all solutions. That is, if \( I = \langle f_1, \ldots, f_n \rangle \), we say that \( I \) is regular if \( f(x) = 0 \Rightarrow D_x f(x) \) has full rank \( n \). With this definition we can state the key result for our analysis in this paper.

**Lemma 1 (Shape Lemma)** \textit{Let \( I \) be a regular ideal in \( \mathbb{Q}[x_1, \ldots, x_n] \) with all \( d \) complex roots of \( I \) having distinct \( x_n \) coordinates. Then the reduced Gr"obner basis of \( I \) in the lexicographic term order has the shape}

\[
G = \{ x_1 - q_1(x_n), x_2 - q_2(x_n), \ldots, x_{n-1} - q_{n-1}(x_n), r(x_n) \}
\]

\textit{where \( r \) is a polynomial of degree \( d \) and the \( q_i \) are polynomials with a degree of at most \( d - 1 \).}
If the Shape Lemma holds finding all solutions to a polynomial system of equations reduces to finding all solutions to a single equation, a task for which there exist efficient numerical methods. The question is therefore whether the assumptions of the lemma, namely (i) the ideal being regular, and (ii) all roots having distinct $x_n$ coordinates, are easy to verify and likely to hold in economic models.

The first condition typically holds if one only considers real solutions (for example, generically, competitive and Nash equilibria are locally unique and finite in number). But it is sometimes difficult to verify that in fact there are only finitely many complex solutions. An easy way to ensure that all solutions are regular is to add the following additional polynomial equation to the original system, 

$$1 - t \det [D_x f(x)] = 0.$$ 

There cannot be a solution in $t$ and $x$ which is not locally unique.

Condition (ii) holds for a wide variety of problems. In case the condition does not hold, we can always add an additional equation,

$$y - \sum_{i=1}^{n} \alpha_i x_i = 0.$$ 

For generic $(\alpha_1, \ldots, \alpha_n)$ all solutions to $f(x) = 0$ and $y - \sum_{i=1}^{n} \alpha_i x_i = 0$ will have distinct $y$-coordinates. Hence the Shape Lemma holds for the larger system with $y$ as the last coordinate.

2.3 Bounding the Number of Zeros

The Shape Lemma implies that the number of real solutions to $f_1(x) = \ldots = f_n(x) = 0$ is equal to the number of real roots of the last polynomial $r(x_n)$. The Fundamental Theorem of Algebra, see Sturmfels (2002) states that any univariate polynomial, $\sum_{i=0}^{d} a_i z^i$, with $a_i \in \mathbb{R}$ for all $i$, has $d$ complex roots (counting multiplicities). There a better bounds available for the number of real zeros. Define the number of sign changes of $r$ to be the number of elements of \(\{a_i \neq 0, i = 0, \ldots, d - 1 : \text{sign}(a_i) = -\text{sign}(a_{i+1})\}\). The classical Descartes's Rule of Signs, see Sturmfels (2002), states that the number of real positive zeros of $r$ does not exceed the number of sign changes. This bound is remarkable because it bounds the number of real zeros. It is possible that a polynomial system is of very high degree and has many solutions but the Descartes bound on the number of zeros of the representing polynomial proves that the system has a single real positive solution.

Moreover, if all $a_i \in \mathbb{Q}$, Sturm’s theorem provides an algorithm to determine the exact number of real roots of any univariate polynomial in a particular interval (see again Sturmfels (2002)). This fact implies that it is possible to say with certainty how many equilibria there are for a given economic model.
2.4 A Parametric Shape Lemma

What makes Gröbner basis particularly useful is that in fact, one can compute a Gröbner basis for polynomials whose coefficients are parameters. The following lemma generalizes the Shape Lemma from above and allows us to represent equilibria of parameterized classes of economic models in the shape form. For the statement of this lemma we extend the definition of the polynomial ring $\mathbb{K}[x]$ with coefficients in the field $\mathbb{K}$ to allow for coefficients that are polynomials in parameters $e_1, \ldots, e_m$. We denote this ring by $\mathbb{K}[e; x]$.

**Lemma 2 (Parameterized Shape Lemma)**

Let $E \subset \mathbb{R}^n$, be an open set of parameters, $(x_1, \ldots, x_n) \in \mathbb{C}^n$ a set of variables and let $f_1, \ldots, f_n \in \mathbb{K}[e_1, \ldots, e_m; x_1, \ldots, x_n]$. Assume that for each $\bar{e} = (\bar{e}_1, \ldots, \bar{e}_m) \in E$, the ideal $I(\bar{e}) = \langle f_1(\bar{e} \cdot \cdot \cdot), \ldots, f_n(\bar{e} \cdot \cdot \cdot) \rangle$ is regular and all complex solutions have distinct $x_n$ coordinates. Then there exist $r, v_1, \ldots, v_{n-1} \in \mathbb{K}[e; y]$ and $\rho_1, \ldots, \rho_{n-1} \in \mathbb{K}[e]$, not identical equal to zero, such that for all $\bar{e} \in E$ with $\rho_l(\bar{e}) \neq 0$, for all $l$ and with $r(\bar{e}, .)$ no identically equal to zero , the following holds.

$$\{ x \in \mathbb{C}^n : f_1(\bar{e}, x) = \ldots = f_n(\bar{e}, x) = 0 \}$$

$$= \{ x \in \mathbb{C}^n : \rho_1(\bar{e}) x_1 = v_1(\bar{e}; y), \ldots, \rho_{n-1}(\bar{e}) x_{n-1} = v_{n-1}(\bar{e}; y) \text{ for } r(\bar{e}; x_n) = 0 \}.$$

We illustrate the lemma with several examples below. Basically, what it says is that one can treat the coefficients of a polynomial system as parameters and obtain a Gröbner basis where the coefficients of the polynomials in the basis are polynomials in the parameters.

There may be some parameters for which this is not the correct Gröbner basis. However, it suffices to assume that $\rho_l(e) \neq 0$, for all $l$ and that at $\bar{e}$ the polynomial $r$ is not identically equal to zero , together with the fairly strong assumption that $I$ is regular at all $e \in E$. Since this turns out to be an important but slightly complicated issue, we discuss it more formally in the Appendix.

The assumption of the lemma that $I$ is regular for all parameters in $E$ is obviously a very strong assumptions. In economic applications, one can typically show regularity for ‘almost all’ parameter values, i.e. for parameters outside of a closed set of measure zero. So the naturally the question arises what happens at some $\bar{e}$ where $I$ is not regular, but where in any open neighborhood of $\bar{e}$ there is a regular $I$. As the following example shows, one cannot find the solution set $V(I)$ from knowing $V(G)$, even if $G$ is regular. Suppose as a trivial example $I = \langle e(x + y), xy + e \rangle \subset \mathbb{Q}[e; x, y]$. A valid Göbner basis for almost all parameters $e$ is given by $G = \langle y^2 - e, x + z \rangle$. Clearly, for $e = 0$, this is not the correct Gröbner basis. This fact does not contradict the lemma since $I$ is not regular.

Instead of assuming the regularity of the ideal $I$ we could compute the set of parameter values for which the Gröbner basis may not specialize correctly. Cox et al. (1997, pp. 283 – 284) describe an algorithm for this purpose.

Having a shape representation of the polynomial system in parameters has two advantages. Often one can use Descartes’ method to derive bounds on the number of equilibria
uniform over all (or almost all) parameters. Secondly, one can search for parameters for which the system has ‘a singularity’. At these points the number of solution typically changes (more precisely, if the number of solutions changes it must be at a singularity).

2.5 Detecting Multiple Solutions

Given a Gröbner basis in parameters with univariate representation \( r(e; x_n) \), a singularity (of the Gröbner basis) occurs at \( \bar{e}, \bar{x}_n \) if and only if

\[
\begin{align*}
  r(\bar{e}, \bar{x}_n) &= 0, \\
  \frac{\partial r(\bar{e}, \bar{x}_n)}{\partial x_n} &= 0.
\end{align*}
\]

(1)

Given we start with parameters \( \hat{e} \) for which there exists a unique solution to the system of equations, (robust) multiplicity can only occur in a convex set \( E \) with \( \hat{e} \in E \), if the above system of equations has a solution in \( E \). Note that at points where a singularity occurs in the original system, Lemma 2 above no longer holds and we cannot say anything about how the Gröbner basis looks like and if the solution sets coincide. However, in order for there to be an open set of parameters for which the \( f \)-system has multiple solutions (we refer to this as robust multiplicity), there must be an open set for which the Gröbner basis has multiple solutions. This can only occur if the Gröbner system has a singularity. So how can one detect a singularity in the Gröbner basis?

For the case of a single free parameter these are two equations in two unknowns and the equations typically have finitely many solutions that can be found by computing the Gröbner basis of this new system. In the applications below, we focus on this simple case. For several parameters, the solution set is infinite (i.e. positive dimensional). Aubry et al. (2002) develop an algorithm to find representative solutions of the positive dimensional system.

3 Gröbner Basis Computation with SINGULAR

As of the writing of this paper the computer algebra system SINGULAR is considered to be among the leading if not the best freely available software package for Gröbner basis computations. Decker and Lossen (2006), Greuel and Pfüster (2007), and Greuel et al. (2005) provide detailed descriptions of this software. SINGULAR has many capabilities that we do not need for our objective of finding all economic equilibria. And so here we provide only information on the software that is needed for the computation of Gröbner bases and all solutions to square systems of polynomial equations.

Consider the following system of three equations in the three unknowns \( x, y, z \),

\[
\begin{align*}
  x - yz^3 - 2z^3 + 1 &= -x + yz - 3z + 4 = x + yz^9 = 0.
\end{align*}
\]

The polynomials on the left-hand side of the three equations define a polynomial ideal. As a first step we compute a Gröbner basis for this ideal. We enter the following commands in
In Singular, we first have to declare a base-ring which we call $R$ in this example. A zero in the ring declaration indicates that we consider polynomials over the rational numbers. We highly recommend this declaration since only then the computation is exact. We denote the unknowns by $x, y, z$ and use 'lp' to instruct Singular that we use the lexicographic monomial order for the variables, see the Appendix for background on monomial orderings. This first command line should only be altered to change the number (or names) of the variables or to introduce parameters. Next an ideal, here called $I$, is defined via a list of the polynomials that form a basis of the ideal. Note that Singular requires the signs * and ** to indicate the multiplication and power operation, respectively. The command groebner($I$) computes a Gröbner basis for $I$, by first choosing an 'optimal algorithm' under several available in Singular and then applying it.

To see the output, we type 'G;' at the Singular command line and obtain the following output.

```plaintext
g1=2z11+3z9-5z8+5z3-4z2-1
g2=2y+18z10+25z8-45z7-5z6+5z5-5z4+5z3+40z2-31z-6
g3=2x-2z9-5z7+5z6-5z5+5z4-5z3+5z2+1
```

Contrary to the input format the output of Singular does not use the multiplication and power signs. For example, $2z11$ is to be read as $2 \cdot z^{11}$. In this example the Shape Lemma holds. There are 11 solutions since $g1$ is a polynomial of degree 11 in the last variable $z$. Note that by Descartes’ bound at most 3 of them can be real and positive. To solve numerically for all complex solutions, we first need to load a library in Singular.

```plaintext>
> LIB "solve.lib";
```

The following command gives all complex solutions.

```plaintext>
> solve(G);
```
Only one of the 11 solutions is real, the other 10 are complex. SINGULAR prints them all. The unique real solution is \((\frac{1}{2}, -\frac{1}{2}, 1)\).

3.1 Parameters

The following variation of the previous example illustrates how to introduce parameters. Let the coefficient of the monomial \(x\) in the last equation be a free parameter, i.e. the last equation becomes 

\[ ex + yz^9 = 0. \]

In SINGULAR we now need to declare the parameter as part of the ring in the initial command line. The declaration ‘\(\texttt{R=(0,e)}\)’ states that all polynomial coefficients contain only elements of \(\mathbb{Q}\) and the parameter \(e\).

\[
\text{ring } \texttt{R=(0,e),(x,y,z),lp}; \\
\text{ideal } \texttt{I=(}} \\
x-y*z**3-2*z**3+1, \\
-x+y*z-3*z+4, \\
e*x+y*z**9); \\
\text{ideal } \texttt{G=groebner(I)}; \\
\]

\[
\text{G;} \\
\text{G[1]}=2z11+3z9-5z8+(5e)*z3+(-4e)*z2+(-e) \\
\text{G[2]}=(-e2-e)*y+(-8e-10)*z10+(-10e-15)*z8+(20e+25)*z7+(5e)*z6+(-5e)*z5 \\
+((5e)*z4+(5e)*z3+(-20e2-20e)*z2+(16e2+15e)*z+(3e2+3e) \\
\text{G[3]}=(-e-1)*x+2z9+5z7-5z6+5z5-5z4+5z3-5z2-1
\]

SINGULAR produces a Gröbner basis for the ideal of parameterized polynomials. Observe that the univariate representation \(G[1]\) is a polynomial of degree 11 for any value of \(e\). Figure 1 shows the real roots of the univariate representation for \(e \in [-5, 1]\). For positive values of \(e\) \(G[1]\) has the unique solution \(z = 1\). For non-positive values of \(e\) there are multiple solutions. However, we cannot conclude that as a result the entire system of equations always has the same number of real solutions. Recall that the parameterized Shape Lemma holds only for a generic set of parameter values. For fixed values of the parameter the parameterized Gröbner basis may not specialize to the correct basis. Here this difficulty becomes obvious. Observe that the leading term of \(G[2]\) is \(e(-e - 1)y\) and so for \(e \in \{-1, 0\}\) the variable \(y\) no longer appears. The same is true for the variable \(x\) in \(G[3]\) for \(e = -1\). Figures 2 and 3 show the real solutions for \(G[2]\) and \(G[3]\), respectively, for \(e \in [-5, 1]\).

As \(e \to -1\) the values of \(y\) and \(x\) grow unbounded in two of the three solutions. Only in one solution their values remain bounded. For \(e = -1\) both variables no longer appear in
the Gröbner basis. As $e \searrow 0$ the values of $y$ and $x$ remain bounded in all three solutions.

Instead of using the parameterized basis we need to resolve the original system for $e = 0$ and $e = -1$. For $e = 0$ the resulting Gröbner basis is as follows.

\begin{align*}
G[1] &= 2z^3 + 3z - 5 \\
G[2] &= y \\
G[3] &= x + 3z - 4
\end{align*}

There is a unique real solution, $(1, 0, 1)$. This indicates that as $e \searrow$ two of the three solutions do not converge to a solution even though all three solutions remain finite. Only the solution with $z = 1$ converges to a solution of the original system at $e = 0$.

For $e = -1$ the Gröbner basis is as follows.

\begin{align*}
G[1] &= 2z^9 + 5z^7 - 5z^6 + 5z^5 - 5z^4 + 5z^3 - 5z^2 - 1 \\
G[2] &= 33y + 320z^8 + 10z^7 + 790z^6 - 765z^5 + 740z^4 - 715z^3 + 690z^2 - 665z - 94 \\
G[3] &= 33x + 10z^8 - 10z^7 + 35z^6 - 60z^5 + 85z^4 - 110z^3 + 135z^2 + 5z + 28
\end{align*}

There is a unique real solution, $(-3.37023, -4.63605, 0.965189)$. 

Figure 1: Real solutions for $z$ depending on the parameter $e$
3.2 Critical Points

If along a path of parameters the number of real solutions changes then there must be a critical point, that is, system (1) with the parameters and the last variable $x_n$ as its unknowns must have a solution. For a single parameter this system consists of two equations in two unknowns. For our example the appropriate SINGULAR code is then as follows.

```
ring R=0,(e,z),lp;
ideal I=(
2*z**11+3*z**9-5*z**8+5*e*z**3-4*e*z**2-e,
11*2*z**10+9*3*z**8-8*5*z**7+3*5*e*z**2-2*4*e*z);
ideal G=groebner(I);
solve (G);
```

Observe that we now must declare the parameter $e$ as one of the two variables. This system has two real solutions, $(e,z) = (0,0)$ and $(e,z) = (-\frac{9}{7},1)$. For $e = 1$ we have seen that there is a unique solution and so there must be a unique solution for all $e > 0$. In order for there to be multiple solutions for any $\bar{e} > 0$, there must be a singularity between that $\bar{e}$ and $e = 1$, which is not the case. For all positive parameters the system has a unique solution. At $e = -\frac{2}{7} \approx -1.28571$ there is a critical point but the number of solutions is three both for smaller and larger values of $e$. At the point there is a multiple solution.
3.3 Failure of Shape Lemma

The Shape Lemma rests on the two assumptions that the ideal $I$ is regular and that all solutions have distinct values for the last variable. Here we illustrate with two simple examples what can happen when the assumptions are not satisfied.

Consider the system of equations

$$x^2 - y = y - 4 = 0.$$ 

This system has the two solutions $(-2, 4)$ and $(2, 4)$. If $y$ is the last coordinate then both solutions have the same value for the last coordinate. No polynomial that is linear in $x$ of the form $x - q(y)$ can yield two different solutions for $x$ for the same value of $y$.

It is easy to see (without any computation) that the Gröbner basis is as follows.

$G[1] = y - 4$
$G[2] = x^2 - 4$

The Shape Lemma fails, $G[2]$ is not linear in $x$. But observe that if we reorder the variables then the Shape Lemma holds since now $x$ is the last coordinate and both solutions have different values for $x$.

Consider the system of equations

$$x^2(x - y) = y^2(x + y - 1) = 0.$$ 

Figure 3: Solutions for $G[3]=0$ for real values of $z$
Clearly for $x = y = 0$ the solutions are not locally unique in our sense. The Shape Lemma fails and the Gröbner basis is as follows.

$G[1] = 2y^5 - 5y^4 + 4y^3 - y^2$
$G[2] = xy^2 + y^3 - y^2$
$G[3] = x^3 - x^2$y

However, it is easy to see that the system has finitely many solutions. To solve for them we need to compute what is called the ‘radical’ of the system, see the Appendix. After loading the library ‘solve.lib’ we can also use the command radical and obtain a Gröbner basis with multiple zeros eliminated:

```
ideal J=groebner(radical(I));
J[1] = 2y^3 - 3y^2 + y
J[2] = x + 2y^2 - 2y
```

The Shape Lemma holds once we compute the radical of the ideal. This approach works if and only if the all solutions have distinct last coordinate and there are finitely many. Our notion of regular ideal requires two elements, namely finitely many solutions and full rank of the Jacobian at all solutions. The latter condition automatically holds if we first compute the radical of the ideal.

4 Multiple Perfect Bayesian Equilibria

As a first application of the Gröbner bases methods we compute equilibria in a game-theoretic model with cheap talk. Specifically, we compute Bayesian Nash equilibria for the arms race game of Baliga and Sjöström (2004), hereafter BS. For this purpose we first summarize the computationally relevant aspects of the game. For many additional details and particularly the interpretation of the game we refer the reader to the original paper.

4.1 An Arms Race Game with Cheap Talk

Two players simultaneously and independently choose between building a new weapons system ($B$) and not building new weapons ($N$). If both players choose $N$ then the payoff to each of them is 0. A player who chooses $N$ while its opponent chooses $B$ suffers a loss $d > 0$. A player who chooses $B$ while its opponent chooses $N$ receives a gain of $\mu > 0$. Player $i$’s cost of acquiring a new weapons system is $c_i \geq 0$, $i = 1, 2$. Player $i$’s payoffs are summarized in the following payoff matrix, where player $i$ chooses a row and its opponent
The cost $c_i$ are player $i$'s private information. We refer to $c_i$ as player $i$'s type. Each player knows its own type $c_i$ but not the other player’s type $c_j$. The types $c_1$ and $c_2$ are i.i.d. with a continuous cumulative distribution function $F$. The function $F$ has compact support $[0, \bar{c}]$ with $F(0) = 0$, $F'(c) > 0$ for $0 < c < \bar{c}$, and $F(\bar{c}) = 1$. Also, $\bar{c} < d$. All parameters and functions are common knowledge with the exception of the types $c_1$ and $c_2$.

For their analysis of the arms race game BS introduce a key assumption, the multiplier condition for the distribution function $F$. This condition requires that $F(c)d \geq c$ for all $c \in [0, \bar{c}]$. BS show that if the multiplier condition is satisfied, then for any $\mu > 0$ there is a unique Bayesian-Nash equilibrium. In this equilibrium all players choose to build a new weapons system (action $B$), regardless of their type. In the language of BS the only equilibrium outcome in the game is an “arms race.” This outcome is inefficient because all types prefer $(N, N)$ to the equilibrium $(B, B)$.

After the analysis of the described equilibrium outcome BS introduce cheap talk to their game. The cheap-talk extension of the arms race game consists of three stages. In stage zero, nature chooses the types $c_1$ and $c_2$. In stage one, the players simultaneously and publicly announce messages. Two messages are sent in equilibrium, a conciliatory message or an aggressive message. These messages do not affect the possible payoffs but they may convey information about what the players intend to do in the future. Finally, in stage two, the players simultaneously choose either $B$ or $N$ and receive the corresponding payoff from their respective payoff matrices. The main theorem of BS states that in this cheap-talk extension of the game there exists a perfect Bayesian equilibrium in which arms races can be avoided with high probability if $\mu > 0$ is sufficiently small.

**Baliga and Sjöström (2004, Theorem 2)** Suppose the multiplier condition is satisfied. For any $\delta > 0$ there is a $\bar{\mu} > 0$ such that if $0 < \mu < \bar{\mu}$ then there is a perfect Bayesian equilibrium of the cheap-talk extension of the arms race game where $N$ is played with at least probability $1 - \delta$.

The theorem and the construction of the equilibrium rely crucially on the following lemma.

**Baliga and Sjöström (2004, Lemma 1)** Suppose the multiplier condition is satisfied.
For sufficiently small $\mu > 0$, there exists a triple $(c_L, c^*, c^H)$ such that

$$\mu < c_L < c^* < c^H < \bar{c}$$

(2)

$$[F(c^H) - F(c^L)] c^L = (1 - F(c^H)) \mu$$

(3)

$$[1 - 2(F(c^H) - F(c^L))] c^H = F(c^L)\bar{d}$$

(4)

$$(1 - F(c^H)) (\mu - c^*) + F(c^L)(-c^*) = F(c^L)(-d)$$

(5)

If $\mu \to 0$ then $c^H \to 0$.

This lemma builds the foundation for the equilibrium announcement made in stage one and the actions played on the equilibrium path in stage two. A complete description of the equilibrium is not needed for our analysis. Informally, the following features of the equilibrium are important. The values $c_L$ and $c^H$ represent cut-offs in the type space of a player at which it changes its message in stage one. (At the cut-off $c^*$ a player is indifferent between $B$ and $N$ in stage 1 if both players sent a conciliatory message.) Equations (3), (4), and (5) determine the values $c_L$, $c^H$, and $c^*$. BS show that in equilibrium, if both players have a type exceeding $c^H$ then they both send a conciliatory message in stage one and play $N$ in stage two. In the interpretation of BS, contrary to the original game without cheap-talk, there is now an equilibrium in which an arms race is avoided. And if $c^H \to 0$ then the probability of player $i$ having a type $c_i > c^H$ tends to one, $1 - F(c^H) \to 1$. This is the key result of the paper.

BS point out that there may be other perfect Bayesian equilibria of the cheap-talk extension of the arms race game. They mention a second equilibrium with cut-offs $(\hat{c}_L, \hat{c}^*, \hat{c}^H)$ which also satisfy equations (3) – (5). This second equilibrium has the property that $(\hat{c}_L, \hat{c}^H) \to (0, c^\text{med})$ as $\mu \to 0$, where $c^\text{med}$ is the median cost given by $F(c^\text{med}) = \frac{1}{2}$. Thus, in this second equilibrium an arms race can be avoided for small $\mu$ only roughly a quarter of the time. BS make no statements whether there are other equilibria than the two described ones.

### 4.2 Polynomial Specification of the Game

Any solution to equations (3) – (5) that also satisfies the inequalities (2) yields an equilibrium of the structure described by BS. Clearly we cannot hope to solve these three equations in three unknowns for general functions $F$. But if $F$ is a polynomial function on its support $[0, \bar{c}]$ then the equations (3) – (5) are polynomial equations and we can apply the Gröbner bases methods.

Suppose the distribution of types is uniform on the interval $[0, 1]$ and so $F(c) = c$ for
\( c \in [0,1]. \) Then equations (3) – (5) have the following form.

\[
\begin{align*}
(c^H - c^L) c^L &= (1 - c^H) \mu \\
[1 - 2 (c^H - c^L)] c^H &= c^L d \\
(1 - c^H) (\mu - c^*) + c^L (-c^*) &= c^L (-d)
\end{align*}
\]  

(6) (7) (8)

Using the simplified notation \( m = \mu \) and \((c(3), c(2), c(1)) = (c^H, c^*, c^L)\) we write the system in SINGULAR as follows.

```plaintext
int n = 3;
ring R= (0,m,d),(c(3),c(2),c(1)),lp;
option(redSB);
ideal I =
\( (c(3)-c(1))*c(1)-(1-c(3))*m, \)
\( (1-2*(c(3)-c(1)))*c(3)-c(1)*d, \)
\( (1-c(3))*(m-c(2))-c(1)*c(2)+c(1)*d, \)
ideal G=groebner(I);
```

SINGULAR produces the following output.

\[
\begin{align*}
G[1] &= (-2*m+d-1)*c(1)**3+(2*m*d+m)*c(1)**2+(m2*d-2*m2-m)*c(1)+(m2) \\
G[2] &= (m+1)*c(2)+(m-d)*c(1)+(-m*d-m) \\
G[3] &= (-2*m2-2*m)*c(3)+(2*m-d+1)*c(1)**2+(-m*d)*c(1)+(2*m2+m)
\end{align*}
\]

With \( \bar{c} = 1 \) as the upper bound for the support of the distribution \( F \) the assumption \( \bar{c} < d \) implies \( d > 1 \). For sufficiently small \( m = \mu \) the univariate polynomial \( G[1] \) then has two sign changes and so there can be at most two positive real solutions. For fixed values of \( d \) and \( m \) we can easily find all zeros of the Gröbner basis. Figure 4 displays the positive real solutions for \( d = 1.5 \) and small values of \( \mu \). (The third solution has negative real values for all three variables for \( \mu < 0.25 \).) There are three curves, one for each cut-off value \( c^L < c^* < c^H \). The respective upper branches of the curves for \( c^L \) and \( c^* \) together with the lower branch of the curve for \( c^H \) represent the equilibrium solution that is emphasized and deemed desirable by BS. (The three branches are marked with a point at \( \mu = 0.024 \).) Note that as in their lemma \( c^H \to 0 \) as \( \mu \to 0 \). The respective other branches of the three curves represent the second equilibrium which has the property that \( c^H \to \frac{1}{2} \) as \( \mu \to 0 \). At \( \mu = \frac{16}{143+19\sqrt{57}} \approx 0.0558568 \) the two respective branches meet for each cut-off value. The univariate polynomial \( G[1] \) has a double solution \( c^L = 0.101588 \). As \( \mu \) increases further the solutions become complex. For example, for \( \mu = 0.0559 \) the two solutions for \( c^L \) become \( 0.101611 \pm 0.0023494 i \). The system (6) – (8) no longer has a positive real solution. Note that this observation does not violate the lemma of BS since its claim holds only for sufficiently small \( \mu > 0 \).
We next show that the condition \( \bar{c} < d \) is important. When \( d = \bar{c} = 1 \) the Gröbner basis is sufficiently simple to allow for closed-form expressions in \( \mu \) for all three solutions of the system (6) – (8). The solution \((c^L, c^*, c^H) = (1, 1, 1)\) violates the inequalities (2). Figure 5 displays the remaining two solutions, which are

\[
\left( \frac{1}{4} \left( 1 - \sqrt{1-8\mu} \right), \frac{1 + 7\mu - (1 - \mu)\sqrt{1-8\mu}}{4(1 + \mu)}, \frac{1}{2} \right)
\]

and

\[
\left( \frac{1}{4} \left( 1 + \sqrt{1-8\mu} \right), \frac{1 + 7\mu + (1 - \mu)\sqrt{1-8\mu}}{4(1 + \mu)}, \frac{1}{2} \right).
\]

For \( \mu > 0.125 \) both solutions for \( c^L \) and \( c^* \) are complex. Note that \( c^H = \frac{1}{2} \) and so as \( \mu \to 0 \) it holds that \( c^H \not\to 0 \). Thus the theorem of BS no longer holds for \( d = \bar{c} = 1 \).

This completes our analysis of the cheap-talk game. Note that in the analyzed range for \( \mu > 0 \) no coefficient of a leading term in the Gröbner basis ever has the value 0. The parameterized Gröbner basis specializes correctly for all analyzed parameter values. But for \( \mu = 0 \) the Gröbner basis does not specialize since the linear term in \( c[3] \) has the coefficient 0 in \( G[3] \). A separate analysis shows that there is a regular solution at \((0, 0, \frac{1}{2})\) and a double solution at \((0, 0, 0)\).
5 Multiple Nash Equilibria in Strategic Market Games

Following Shapley and Shubik (1977) strategic market games have become an important tool to understand the foundations of price formation in markets. Such games provide strategic underpinnings for the general equilibrium paradigm. A key result (see e.g. Postlewaite and Schmeidler (1978)) in this framework states that as the number of agents becomes large, prices and allocations in all non-trivial Nash equilibria in the strategic market game approximate Walrasian equilibria in the corresponding general equilibrium model of the economy. This approximation result naturally leads to further research questions such as the following. How many agents are needed in the game for the Nash equilibrium to be a reasonable approximation of the Walrasian equilibrium? What happens in the presence of multiple Nash and Walrasian equilibria? Do some Walrasian equilibria get better approximated by the market game equilibrium than others? Of course, little if anything can be said in general about these issues but we may hope for answers to these questions for specific economic models. For such an analysis we need a method to solve for all Nash equilibria in market games and all Walrasian equilibria in the corresponding general equilibrium model. We now describe the application of Gröbner basis methods to address these issues in the context of a simple model. We begin with the computation of equilibria in a general equilibrium model and subsequently compute equilibria for the corresponding market game.
5.1 Multiple Arrow-Debreu Equilibria

Suppose there are two types of agents and two commodities. Utility functions are

\[ u^1(c_1, c_2) = -\frac{64}{2} c_1^{-2} - \frac{1}{2} c_2^{-2}, \quad u^2(c_1, c_2) = -\frac{1}{2} c_1^{-2} - \frac{64}{2} c_2^{-2}. \]  

Parameterized individual endowments are

\[ e^1 = (1 - e, e), \quad e^2 = (e, 1 - e) \]

for the parameter \( e \in [0, 1] \). We denote the endowment and consumption of agent \( h \) in good \( l \) by \( e_{hl} \) and \( c_{hl} \), respectively. The price of good \( l \) is \( p_l \). Using the necessary and sufficient first-order conditions for the agents’ utility maximization problems in addition to the market-clearing equations yields the following equilibrium system, where \( \lambda_h \) denotes the Lagrange multiplier for the budget constraint of agent \( h \),

\[
\begin{align*}
64c_{11}^{-3} - \lambda_1 p_1 &= 0 \\
c_{12}^{-3} - \lambda_1 p_2 &= 0 \\
p_1(c_{11} - e_{11}) + p_2(c_{12} - e_{12}) &= 0 \\
c_{21}^{-3} - \lambda_2 p_1 &= 0 \\
64c_{22}^{-3} - \lambda_2 p_2 &= 0 \\
p_1(c_{21} - e_{21}) + p_2(c_{22} - e_{22}) &= 0 \\
c_{11} + c_{21} - e_{11} - e_{21} &= 0 \\
c_{12} + c_{22} - e_{12} - e_{22} &= 0.
\end{align*}
\]

We transform this equilibrium system into a much simpler system of polynomial equations. By Walras’ law we can normalize prices by setting \( p_1 = 1 \) and eliminate the budget equation of agent 2. This normalization allows us to eliminate the Lagrange multipliers and the first optimality condition of each agent. Next, the market-clearing equation allows us to eliminate the consumption variables of the second agent. And finally we substitute \( p_2 = q^3 \) and write the remaining equations in terms of the excess demand \( x_l = c_{1l} - e_{1l} \) of agent 1 to obtain the following polynomial system.

\[
\begin{align*}
(1 - e + x_1) - 4(e + x_2)q &= 0 \\
4(e - x_1) - (1 - e - x_2)q &= 0 \\
x_1 + x_2q^3 &= 0.
\end{align*}
\]

The corresponding SINGULAR code is as follows.

```plaintext
int n = 3;
ring R= (0,e),(x(1),x(2),q),lp;
option(redSB);
```
ideal I = (
  -4*(e+x(2))*q+(1-e+x(1)),
  -(1-e-x(2))*q+4*(e-x(1)),
  x(1)+x(2)*q**3);
ideal G=groebner(I);

The resulting Gröbner basis is as follows.

\begin{align*}
G[1] &= (-15e-1)q^3 + 4q^2 - 4q + (15e+1) \\
G[2] &= (-225e-15)x(2) + (60e+4)q^2 - 16q + (-225e^2-30e+15) \\
G[3] &= 15x(1) + 4q + (-15e-1)
\end{align*}

The univariate representation has three roots,

\begin{align*}
1, \quad & \frac{3 - 15e - \sqrt{5}\sqrt{1 - 42e - 135e^2}}{2(1 + 15e)}, \quad \frac{3 - 15e + \sqrt{5}\sqrt{1 - 42e - 135e^2}}{2(1 + 15e)}
\end{align*}

Figure 6 displays the real roots of $G[1]$ for small values of $e$. For $e \leq \frac{1}{45}$ all three roots are real and positive. The corresponding values for the excess demand variables, $x_1, x_2$, lead to positive consumption values for both agents. Thus there are three Arrow-Debreu equilibria. For $e = \frac{1}{45}$ the polynomial $G[1]$ has $q = 1$ as a triple root. For $e > \frac{1}{45}$, two of three roots are complex and so $q = 1$ remains as the unique Arrow-Debreu equilibrium. Note that for
positive values of $e$ all leading terms are never zero and so the parameterized Gröbner basis specializes correctly.

For our now following examination of strategic market games we focus on one particular parameter value for which there are three equilibria. Table 1 reports the equilibrium values for $e = \frac{1}{51} \approx 0.0196078$.

<table>
<thead>
<tr>
<th>$q$</th>
<th>$p_2$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$c_{11}$</th>
<th>$c_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.740536</td>
<td>0.406105</td>
<td>-0.111202</td>
<td>0.273825</td>
<td>0.869190</td>
<td>0.293433</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-0.180392</td>
<td>0.180392</td>
<td>0.8</td>
<td>0.2</td>
</tr>
<tr>
<td>1.35037</td>
<td>2.46241</td>
<td>-0.273825</td>
<td>0.111202</td>
<td>0.706567</td>
<td>0.13081</td>
</tr>
</tbody>
</table>

Table 1: Arrow-Debreu equilibria for $e = \frac{1}{51}$

5.2 A Strategic Market Game

We consider a simple sell-all strategic market game in the spirit of Postlewaite and Schmeidler (1978). Here we summarize the computationally relevant aspects of the game. For many additional details we refer the reader to the original paper.

Consider two types of agents with utility functions and endowments as specified in our discussion of Arrow-Debreu equilibria. The so-called $N$th replica of the economy consists of $N$ identical agents within each type. An individual is now characterized by his type $h = 1, 2$ which determines his utility function and endowment as well as by his number in the replica, $m$. Each individual $i = (h, m) \in \mathcal{I}$ has a strategy consisting of bids $b_{i1}$ and $b_{i2}$ for commodities 1 and 2, respectively. (We consider a sell-all game and so sales are not a choice variable. This fact simplifies the notation.) Prices are determined by supply and demand as follows,

$$p_l((b_i)_{i \in \mathcal{I}}) = \frac{\sum_{i \in \mathcal{I}} b_{il}}{\sum_{i \in \mathcal{I}} e_{il}}.$$ 

The budget constraints are

$$c_{il} = \frac{b_{il}}{p_l((b_j)_{j \in \mathcal{I}})}, \quad b_{i1} + b_{i2} = p_1((b_j)_{j \in \mathcal{I}})e_{i1} + p_2((b_j)_{j \in \mathcal{I}})e_{i2}.$$ 

We are interested in computing all symmetric Nash equilibria. (The problem of asymmetric equilibria is another question that could also be addressed using our methods.) For the specific example, with aggregate endowments in each good being $N$, we can rewrite the budget constraint for agent $i$,

$$N(b_{i1} + b_{i2}) = e_{i1} \sum_{j \in \mathcal{I}} b_{j1} + e_{i2} \sum_{j \in \mathcal{I}} b_{j2},$$

$$c_{il} = N \frac{b_{il}}{\sum_{j \in \mathcal{I}} b_{jl}}.$$
In a symmetric Nash equilibrium we identify an agent simply by his type and, in a slight abuse of notation, write $b_{hl}$, $h = 1, 2$, for the bid of an agent of type $h$. With $\lambda$ denoting the Lagrange multiplier, the first order conditions for any agent of type 1 with respect to bid $l$ are then

$$k_{1l} \frac{(N b_{2,l} + (N - 1) b_{1,l})/(N b_{1,l} + N b_{2,l})^2}{(b_{1,l}/(N b_{1l} + N b_{2l}))^3} - \lambda (N - e_1^l) = 0$$

where $k_{11} = 64$ and $k_{12} = 1$ are the respective weights in the utility functions of type 1 agents for consumption in goods 1 and 2, see (9). Eliminating the multiplier $\lambda$, we obtain the following polynomial equation that characterizes optimal choices for agent 1,

$$64 b_{12}^3 (N - e_1^2) (N b_{21} + (N - 1) b_{11}) (b_{11} + b_{21})$$

$$= b_{11}^3 (N - e_1^1) (N b_{22} + (N - 1) b_{12}) (b_{12} + b_{22}) \quad (12)$$

Budget equations (11) for $h = 1$, the optimality condition (12) for agents of type 1 and the corresponding condition for agents of type 2 characterize symmetric Nash equilibria. These are three polynomial equations in the four unknowns $b_{hl}$ for $h = 1, 2$ and $l = 1, 2$. To obtain a square system of equations we need to normalize the bids. We add the equation

$$b_{11} + b_{12} + b_{21} + b_{22} = 10. \quad (13)$$

The resulting system of equations still allows for a continuum of solutions, for example there are continua with $b_{11} = b_{21} = 0$ or $b_{11} = b_{22} = 0$. So we need to add the condition that all bids are different from zero. For this purpose we introduce the variable $t$ and require as the final equation that

$$1 - t b_{11} b_{21} b_{12} b_{22} = 0. \quad (14)$$

In SINGULAR we denote by $b(1)$ and $b(2)$ bids by agent 1 for goods 1 and 2 and by $b(3)$ and $b(4)$ the bids of agent 2 for these two goods. We also include equations for the consumption allocations of the two agents.

```plaintext
int m = 4;
ring R= (0,n),(t,c(1..m),b(1..m)),lp;
ideal I =(
64*(n-1/51)*(b(1)+b(3))*((n-1)*b(1)+n*b(3))\*b(2)**3-(n-50/51)*
((n-1)*b(2)+n*b(4))*((n-1)*b(4)+n*b(2))*b(1)**3,
1/64*(n-50/51)*(n*b(1)+(n-1)*b(3))*((n-1)*b(4)+n*b(2))*b(3)**3-(n-1/51)*b(2)+b(4))\*
((n-1)*b(4)+n*b(2))*b(3)**3,
1/51*b(1)+50/51*b(2)-50/51*b(3)-1/51*b(4),
10-b(1)-b(2)-b(3)-b(4),
1-t*b(1)*b(2)*b(3)*b(4),
c(1)*b(1)+b(3)-b(1),
```

24
The resulting expression for the univariate representation is rather messy and so we do not display it here. We can find all roots of $G[1]$ as a function of the parameter $n$. For $n = 1$ there is a unique positive solution corresponding to a unique Nash equilibrium. For $n > 1$ there are 3 solutions corresponding to three symmetric Nash equilibria. Table 2 reports agent 1’s equilibrium consumption for a few selected values of $n$. We observe that the three Nash equilibria converge to the three Walrasian equilibria of the corresponding Arrow-Debreu economy.

### Table 2: Equilibria for various values of $N$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$c_1(1)$</th>
<th>$c_2(1)$</th>
<th>$c_1(2)$</th>
<th>$c_2(2)$</th>
<th>$c_1(3)$</th>
<th>$c_2(3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.882646</td>
<td>0.117354</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>2</td>
<td>0.812542</td>
<td>0.187458</td>
<td>0.868703</td>
<td>0.259155</td>
<td>0.740845</td>
<td>0.131297</td>
</tr>
<tr>
<td>10</td>
<td>0.801999</td>
<td>0.198001</td>
<td>0.875683</td>
<td>0.299961</td>
<td>0.700039</td>
<td>0.124317</td>
</tr>
<tr>
<td>100</td>
<td>0.800193</td>
<td>0.199807</td>
<td>0.869945</td>
<td>0.294277</td>
<td>0.705723</td>
<td>0.130055</td>
</tr>
<tr>
<td>1000</td>
<td>0.800019</td>
<td>0.199981</td>
<td>0.869267</td>
<td>0.293519</td>
<td>0.706481</td>
<td>0.130733</td>
</tr>
<tr>
<td>WE</td>
<td>0.8</td>
<td>0.2</td>
<td>0.869190</td>
<td>0.293433</td>
<td>0.706567</td>
<td>0.130810</td>
</tr>
</tbody>
</table>

6 Multiple Steady States in OLG Models

Overlapping generations ("OLG") models are a workhorse for both theoretical and applied analysis in economics, particularly in public finance, monetary theory and macroeconomics. Robust examples of OLG economies with a continuum of competitive equilibria are well known in the economics literature. For example, Kehoe and Levine (1990) construct robust examples of realistically calibrated OLG models with agents living for three periods in which indeterminacy of equilibria occurs. They point out that the possibility of a continuum of competitive equilibria poses a serious challenge to applied equilibrium modeling in the spirit of the influential work by Auerbach and Kotlikoff (1987). As one possible escape from the indeterminacy of equilibria, Kehoe and Levine (1990) suggest to focus on stationary equilibria (steady states). Kehoe and Levine (1984) show that steady states are generally determinate. Clearly the presence of multiple (albeit finitely many) steady states also poses problems to the application of OLG models in policy analysis. Unfortunately conditions that ensure uniqueness of steady states are extremely restrictive, see Kehoe et al. (1991). Beyond these conditions little if anything is known about multiplicity of steady state equilibria in
OLG models. In the following we show how Gröbner bases methods can address this important problem.

6.1 A Stationary OLG Model

We consider the so-called “double-ended infinity model” (Kehoe et al. (1991), Geanakoplos (2008)) in which discrete time runs from minus infinity to plus infinity, \( t \in \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \). At each time \( t \) a representative agent is born and lives for \( N \geq 2 \) periods. Each period those agents who are alive receive an endowment \( e_a \) that depends solely on their age, \( a = 1, \ldots, N \). An agent’s utility is time separable with the utility of an agent born at time \( t \) given by

\[
U_t(c) = \sum_{a=1}^{N} u(c_a(t + a - 1)),
\]

where \( c_a(t + a - 1) \) denotes the consumption of an agent born at time \( t \) in period \( t + a - 1 \) (when he is in period \( a \) of his life). For simplicity and without loss of generality we assume that agents do not discount and have time-invariant utility. The computational methods also apply to more general models.

A competitive equilibrium is defined as usual by market clearing and agent optimality, that is, it is given by a sequence of prices and consumption allocations,

\[
(p(t), (\bar{c}_a(t))_{a=1}^{N})_{t \in \mathbb{Z}}
\]

such that for each \( t \),

\[
\sum_{a=1}^{N} (\bar{c}_a(t) - e_a) = 0
\]

and

\[
(\bar{c}_1(t), \ldots, \bar{c}_N(t + N - 1)) \in \arg \max_{c(t), \ldots, c(t + N - 1)} U_t(c(t), \ldots, c(t + N - 1))
\]

subject to

\[
\sum_{a=1}^{N} p(t + a - 1) (c(t + a - 1) - e_a) = 0.
\]

The computation of a general competitive equilibrium requires us to calculate infinitely many prices and consumption values. Clearly our methods cannot compute arbitrary non-stationary equilibria. But they allow us to find all stationary equilibria.

A steady state, or stationary equilibrium, is a collection of consumption allocations for all agents and all ages as well as prices such that market clearing and agent optimality holds and for all \( t \in \mathbb{Z} \),

\[
\frac{p_{t+1}}{p_t} = q > 0 \quad \text{and} \quad \bar{c}_a(t) = c_a.
\]

There are two types of such stationary equilibria. Kehoe and Levine (1984) prove that (generically) \( q = 1 \) corresponds to the unique “monetary” steady state and that there is an odd number of “real” (non-monetary) steady states with \( q \neq 1 \).
This completes our concise description of steady states in our stationary OLG pure exchange economy. For much more detailed treatments of OLG economies we refer the interested reader to Ljungqvist and Sargent (2000) and Geanakoplos (2008). These treatments relate our model to the classic OLG model of Samuelson (1958) and Gale (1973) with fiat money.

6.2 Polynomial Equilibrium System

In applied research modelers often assume that agents’ per period utility function is given by $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ for $0 < \sigma \neq 1$ (and log utility for $\sigma = 1$). For $\sigma \in \mathbb{N}$ this utility function leads to polynomial equilibrium equations. Using the necessary and sufficient first-order conditions for agents’ utility maximization problems in addition to the market-clearing equations yields the following equilibrium system,

$$c_{a+1}\sigma - c_a\sigma = 0, \quad a = 1, \ldots, N-1,$$

$$\sum_{a=1}^{N} q^{a-1}(c_a - e_a) = 0,$$

$$\sum_{a=1}^{N} (c_a - e_a) = 0.$$

Evidently the system always has a solution with $q = 1$ and $c_a = \frac{1}{N} \sum_{a=1}^{N} e_a$. This solution is the unique monetary steady state. It is called the “golden rule” steady state and is Pareto efficient.

We now investigate the number of real steady states in this economy. For the purpose of finding all equilibria with SINGULAR we define $w = q^{1/\sigma}$ and rewrite the system of equilibrium equations as follows,

$$c_{a+1}w - c_a = 0, \quad a = 1, \ldots, N-1,$$

$$\sum_{a=1}^{N} w^{\sigma(a-1)}(c_a - e_a) = 0,$$

$$\sum_{a=1}^{N} c_a - e_a = 0.$$

For integer-valued $\sigma$ this system is polynomial. We do not examine the case of log utility since it is well-known that there is a unique real steady state, see Kehoe et al. (1991). Instead we begin our analysis with $\sigma = 2$ and subsequently examine larger levels of risk aversion. To give the reader some idea about the needed SINGULAR code and the resulting output we show the code and output for $N = 3$ and $\sigma = 2$. As in Section 5.1 we use excess demand variables $x(a)$ for $c_a - e_a$. We denote the endowment parameters $e_1, e_2, e_3$ by $e, f, g$, respectively.
int n = 4;
ring R = (0,e,f,g,b),x(1..n),lp;
option(redSB);
ideal I =
-(f+x(2))*x(4)+(e+x(1)),
-(g+x(3))*x(4)+(f+x(2)),
x(1)+x(2)*x(4)**2+x(3)**4,
x(1)+x(2)+x(3));
ideal G=groebner(I);

The resulting Gröbner basis is as follows.

G[1]=(-g)*x(4)**4+(e+g)*x(4)**2+(-e)
G[2]=(3*e**2+3*e*g+3*g**2)*x(3)+(-e**2*g-e*f*g-3*e*g**2-2*f*g**2-2*g**3)*
x(4)**3+(2*e**2*g+2*e*f*g+3*e*g**2+2*f*g**2+2*g**3)*x(4)**2+(e**3+e**2*f+3*e**2*g+2*e*f*g+5*e*g**2+3*f*g**2+3*g**3)*x(4)+(-2*e**3-2*e**2*f-3*e**2*g-4*e*f*g-4*e*g**2-3*f*g**2)
G[3]=(-3*e**2-3*e*g-3*g**2)*x(2)+(-2*e**2*g-2*e*f*g-3*e*g**2-2*f*g**2-g**3)*
x(4)**3+(2*e**2*g+2*e*f*g+3*e*g**2+2*f*g**2+2*g**3)*x(4)**2+(e**3+2*e**2*f+6*e**2*g+4*e*f*g+7*e*g**2+3*f*g**2+3*g**3)*x(4)+(-e**3-4*e**2*f-3*e**2*g-5*e*f*g-2*e*g**2-3*f*g**2)
G[4]=(-3*e**2-3*e*g-3*g**2)*x(1)+(e**2*g+e*f*g-f*g**2-g**3)*x(4)**3+(e**2*g+
e*f*g+3*e*g**2+2*f*g**2+2*g**3)*x(4)**2+(-e**3-e**2*f-3*e**2*g-2*e*f*g-2*e*g**2)*x(4)+(-e**3+2*e**2*f+e*f*g-2*e*g**2)

The Shape Lemma holds and the univariate representation $G[1]$ provides us with an equation for the possible equilibrium values of $w$. Note that for positive values of the parameters all leading terms are never zero and so the parameterized Gröbner basis specializes correctly.

### 6.3 Uniqueness for $\sigma = 2$

SINGULAR computes the fully parameterized Gröbner basis in a few seconds for $N = 3, 4, 5, 6$, in a few minutes for $N = 7, 8$, and in about an hour for $N = 9, 10$. When $N$ is even, the univariate representation is given by

$$r(w) = \sum_{i=1}^{N} e_i w^{2(i-1)} - \left( \sum_{i=1}^{N} e_i \right) w^{N-1}.$$
Table 3: Steady States for $e_2 = 12$

<table>
<thead>
<tr>
<th>Eq</th>
<th>$w$</th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$c_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.479331</td>
<td>1.81485</td>
<td>3.78621</td>
<td>7.89894</td>
</tr>
<tr>
<td>2</td>
<td>0.775522</td>
<td>3.41586</td>
<td>4.40460</td>
<td>5.67953</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>4.5</td>
<td>4.5</td>
<td>4.5</td>
</tr>
<tr>
<td>4</td>
<td>3.98751</td>
<td>10.2765</td>
<td>2.57718</td>
<td>0.646312</td>
</tr>
</tbody>
</table>

For odd $N$ the expression is

$$r(w) = \sum_{i=1}^{(N-1)/2} e_i w^{2(i-1)} - \left( \sum_{i=1}^{(N-1)/2} e_i + \sum_{i=(N+3)/2}^N e_i \right) w^{N-1} + \sum_{i=(N+3)/2}^N e_i w^{2(i-1)}.$$

Observe that in both cases there are exactly two sign changes among the coefficients. Thus, Descartes’ Rule bounds the number of positive real solutions by two. But we know that there must be at least two solutions. The golden rule monetary steady state is given by the solution $w = 1$. The remaining solution thus must be the only real steady state. To the best of our knowledge, this result was not previously known. In sum, direct Gröbner basis computation proves the following result.

**Proposition 1** The double-ended infinity exchange economy with $\sigma = 2$ has a unique real steady state for all $N$.

Applied researchers may consider a value of $\sigma = 2$ to be somewhat realistic for the calibrations of models. The proposition implies for the double-ended infinity model that researchers do not need to worry about multiplicity of real steady states for $\sigma = 2$. We next show that for integer-valued $\sigma \geq 3$ multiplicity of real steady states arises.

### 6.4 Larger Coefficients of Risk Aversion

For $\sigma = 3$ and $N = 3$ the univariate representation is given by

$$r(w) = e_3 w^6 - (e_1 + e_2 + e_3) w^4 + (e_1 + 2e_2 + e_3) w^3 - (e_1 + e_2 + e_3) w^2 + e_1.$$

There are four sign changes and so Descartes’ rule no longer guarantees the existence of a unique non-monetary steady state. In fact, it is a simple exercise to construct economies with four equilibria, that is one monetary and three real (non-monetary) steady states. Figure 7 shows the positive real roots of the univariate representation $r$ as a function of $e_2$ for $e_1 = 1$ and $e_3 = \frac{1}{2}$. For $e_2 < 10.575747$ there is a unique real steady state in addition to the monetary steady state ($w = 1$). For $e_2 > 10.575747$ the polynomial $r$ has four positive real roots. All four solutions for $w$ lead to positive consumption values for the $N = 3$ agents alive at any given time. Table 3 lists all four steady states for $e_2 = 12$. The
first two steady states are efficient equilibria with positive interest rates, the third steady
state is the monetary steady state with a zero interest rate, and the fourth steady state
is inefficient with a negative interest rate. Similar to the examples of indeterminacy of
equilibria in Kehoe and Levine (1990) the existence of multiple real steady states requires
substantial hump-shape in life-cycle income.

For $\sigma = 3$, unlike for the case of $\sigma = 2$, the number of sign changes in the univariate
representation is no longer independent of $N$. For example, for $N = 5$ (and $\sigma = 3$) we ob-
tain a univariate representation with 6 sign changes. Denoting the endowment parameters
$e_1, e_2, \ldots, e_5$ by $e, f, g, h, i$, respectively, the polynomial $G[1]$ is as follows.

$G[1] = (-i) x(6)^{12} + (-h) x(6)^9 + (e+f+g+h+i) x(6)^8 + (-e-f-g-h-i) x(6)^7 +$
$+ (e+f+h+i) x(6)^6 + (-e-f-g-h-i) x(6)^5 + (e+f+g+h+i) x(6)^4 +$
$+ (-f) x(6)^3 + (-e)$

There are six sign changes and so there could be up to six positive real solutions and
thus up to five real steady states in this economy.

Our last example of OLG economies outlines how we can apply the described methods
to models used in applied work. Krueger and Kubler (2005) calibrate a nine-period OLG
model to match observed US data. While they consider a stochastic model with capital
accumulation and shocks to production, we can still use the labor income from their calibra-
Table 4: Labor Endowments

<table>
<thead>
<tr>
<th>age a</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l^a$</td>
<td>1</td>
<td>1.35</td>
<td>1.54</td>
<td>1.65</td>
<td>1.67</td>
<td>1.66</td>
<td>1.61</td>
<td>1/3</td>
<td>1/3</td>
</tr>
</tbody>
</table>

Figure 4 lists these endowments. For these endowments and $\sigma = 3$ the polynomial system of our double-ended infinity model has four real solutions. Two of them are not equilibria since prices are negative. Equilibrium consumption in the unique real steady state is

$$(2.10122, 1.80705, 1.55407, 1.33651, 1.1494, 0.988487, 0.850102, 0.73109, 0.62874)$$

and the interest rate is $\frac{1}{1.16279} - 1$. The interest rate is negative and thus the equilibrium is inefficient. (We also computed equilibria for $\sigma = 4$ and $\sigma = 5$ and the real steady state remains unique.)

Clearly the analysis of a realistically calibrated applied model is greatly aided by the existence of a unique steady state. Although Kehoe and Levine (1990) demonstrate that it is straightforward to construct fairly realistic examples with multiple steady states, many applied modelers hope (or even claim) that their computed (real) steady states are unique. Of course, it remains an open question for many policy models whether computed real steady states in fact are unique. The Gröbner basis methods offer one possible approach to examine this issue in much detail.

7 Conclusion

Multiplicity of equilibria is a prevalent problem in many economic models. Often equilibria are characterized as solutions to a system of polynomial equations. Therefore, methods that allow the computation of all solutions to such systems are of great interest to economists.

In this paper we have provided an introduction to the application of Gröbner basis methods for finding all solutions to polynomial systems of equations. We have described a beautiful result from algebraic geometry, the Shape Lemma, which states under mild assumptions that a given polynomial system is equivalent to a much simpler system with identical solutions. The new system enables a fast and robust calculation of all solutions. Essentially the computation of all solutions reduces to finding all roots of a single polynomial in a single unknown. We have described the software package SINGULAR which computes the equivalent simple system. If all coefficients in the original equilibrium equations are rational numbers or parameters then the computations of SINGULAR are exact. This fact implies that the described methods cannot only be used for a numerical approximation of equilibria but in fact may allow us to prove theoretical results for the underlying economic model. Three economic applications have illustrated that without much prior knowledge
in algebraic geometry the described methods can be used to gain interesting insights into modern economic models.

In a companion paper (Kubler and Schmedders, 2007) we prove that for large classes of general equilibrium models the assumption of semi-algebraic marginal utility leads to polynomial equilibrium equations. We show that the Shape Lemma can be applied to a wide variety of GE models and thereby build the theoretical foundation for a systematic analysis of multiplicity in applied general equilibrium.

Appendix

A Mathematical Background

Section 2 of this paper contains a brief summary of definitions and concepts from the mathematical field of algebraic geometry that are important for a basic understanding of the presented Gröbner basis methods. In this Appendix we provide additional mathematical details. And once again we refer the interested reader to the textbooks by Cox et al. (1997, 1998) for an excellent introduction to computational algebraic geometry.

A.1 Ideals and Varieties

A subset $I$ of the polynomial ring $\mathbb{K}[x]$ is called an ideal if $0 \in I$, $I$ is closed under sums, $f + g \in I$ for all $f, g \in I$, and it satisfies the property that $h \cdot f \in I$ for all $f \in I$ and $h \in \mathbb{K}[x]$. It is easy to check that for given polynomials $f_1, \ldots, f_k$, the set

$$I = \{ \sum_{i=1}^{k} h_i f_i : h_i \in \mathbb{K}[x] \} = \langle f_1, \ldots, f_k \rangle$$

is in fact an ideal (as stated in Section 2). It is called the ideal generated by $f_1, \ldots, f_k$. The polynomials $f_1, \ldots, f_k$ are called a basis of $I$. This ideal $\langle f_1, \ldots, f_k \rangle$ is the set of all linear combinations of the polynomials $f_1, \ldots, f_k$, where the ‘coefficients’ in each linear combination are themselves polynomials in the polynomial ring $\mathbb{K}[x]$. The famous Hilbert Basis Theorem states that for any ideal $I \subset \mathbb{K}[x]$ there exist finitely many polynomials that generate $I$. Note that $\mathbb{K}[x]$ is itself an ideal, a basis being the constant polynomials $f = 1$.

The intersection of two ideals $I$ and $J$ is the set of polynomials that belong both to $I$ and $J$. The intersection is an ideal itself, as is easy to check. The product of two ideals $I$ and $J$, $IJ$, can be defined as the set of polynomials $h = fg$ with $f \in I$ and $g \in J$.

For an ideal $I$ we denote by $V(I)$ the affine variety of $I$, the set of points where all the elements of $I$ vanish. If $I = \langle f_1, \ldots, f_k \rangle$ then we can simply write $V(I) = \{ y \in \mathbb{K}^n : f_1(y) = \ldots = f_k(y) = 0 \}$. Independently of the field $\mathbb{K}$, we refer to the complex variety of an ideal as

$$V_\mathbb{C}(I) = \{ y \in \mathbb{C}^n : f_1(y) = \ldots = f_k(y) = 0 \}.$$
If $I$ and $J$ are ideals in $\mathbb{K}[x]$, then $V(IJ) = V(I) \cup V(J)$.

The Hilbert Basis Theorem implies that any affine variety in $\mathbb{K}^n$ is the set of all solutions of a system of finitely many polynomial equations. We write $V(f_1, \ldots, f_n)$ to denote the set of solutions to $f_1(x) = \ldots = f_n(x) = 0$. If $f_1, \ldots, f_s$ and $g_1, \ldots, g_t$ are bases of the same ideal in $\mathbb{K}[x]$ then $V(f_1, \ldots, f_s) = V(g_1, \ldots, g_t)$. This relatively obvious observation is the starting point of our method to solve polynomial equations: For a given polynomial system, examine the ideal that is generated by the system and ask if there is another basis for this ideal which is simple in the sense that we can easily find all elements of the complex variety.

Before stating the main result, we need to introduce a few more concepts. An ideal $I$ is called zero-dimensional if its complex variety contains finitely many points. For an ideal $I$ the radical of $I$ is defined as $\sqrt{I} = \{ f \in \mathbb{K}[x] : \exists m \geq 1 \text{ such that } f^m \in I \}$. The radical $\sqrt{I}$ is itself an ideal and contains $I$, $I \subset \sqrt{I}$. We call an ideal $I$ radical if $I = \sqrt{I}$.

The ‘strong Nullstellensatz’ is the main reason why it is useful for us to focus on zero-dimensional radical ideals. It states that if the field $\mathbb{K}$ is algebraically closed (that is, in our framework, if $\mathbb{K} = \mathbb{C}$), then $I(V(I)) = \sqrt{I} = I$. So, radical ideals are in one-to-one correspondence with complex varieties. This fact is not true for the computationally convenient case of $\mathbb{K} = \mathbb{Q}$, but we see below that this issue is of no consequence for our analysis.

How can we check if a given ideal is radical and zero-dimensional? Given a polynomial function $f : \mathbb{C}^n \to \mathbb{C}$ partial derivatives with respect to complex numbers are defined in the usual way. Write $f = c_0(x_{-j}) + c_1(x_{-j})x_j + \ldots + c_d(x_{-j})x_j^d$, where the $c_i$ are polynomials in the variables $x_{-j} = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$. Then, $\partial f / \partial x_j := c_1(x_{-j}) + \ldots + dc_d(x_{-j})x_j^{d-1}$.

Given a system of polynomial equations $f : \mathbb{C}^n \to \mathbb{C}^n$, the Jacobian $\partial f(x)$ is defined as usual as the matrix of partial derivatives. A sufficient condition for an ideal $\langle f_1, \ldots, f_n \rangle$ to be radical and zero-dimensional is that $\det(\partial f_1(x_1), \ldots, f_n(x)) \neq 0$ whenever $f_1(x) = \ldots = f_n(x) = 0$. See Cox et al. (1998, Chapter 4.2).

### A.2 Gröbner Basis

A crucial element for the analysis of families of polynomials is the presence of an order among the monomials appearing within a polynomial. Throughout this paper we order monomials according to the lexicographic ordering, that is,

$$x^\alpha > x^\beta \iff \alpha > \beta \iff \text{The left-most non-zero entry of } \alpha - \beta \text{ is positive.}$$

For this particular monomial order we can define for any polynomial $f \in \mathbb{K}[x]$ the multidegree of $f = \sum_\alpha a_\alpha x^\alpha$, $\text{md}(f) = \max\{ \alpha \in \mathbb{Z}_+^n : a_\alpha \neq 0 \}$. That is, the multidegree of $f$ is
the largest vector of exponents among the monomials in \( f \) according to the monomial (here lexicographic) ordering. The monomial with the multidegree as its vector of exponents gives rise to the leading term of \( f \), \( \text{LT}(f) = a_{\text{md}(f)}x^{\text{md}f} \). We will see below that the leading term plays a prominent role in Buchberger’s algorithm for finding Gröbner Bases.

We denote by \( \langle \text{LT}(I) \rangle \) the set of leading terms of elements of \( I \), that is, \( \langle \text{LT}(I) \rangle = \left\{ cx^{\alpha} : \exists f \in I \text{ with } \text{LT}(f) = cx^{\alpha} \right\} \) and by \( \langle \text{LT}(I) \rangle \) the ideal generated by all the elements of \( \text{LT}(I) \). Note that if \( I = \langle f_1, \ldots, f_k \rangle \) then it is true that \( \langle \text{LT}(f_1), \ldots, \text{LT}(f_k) \rangle \subset \langle \text{LT}(I) \rangle \) but the converse often does not hold. To see this, consider the example \( f_1 = x^3 - 2xy, f_2 = x^2y - 2y^2 + x \). A quick check shows that \( x^2 \in I = \langle f_1, f_2 \rangle \). So \( x^2 \) must be in \( \text{LT}(I) \). But clearly \( x^2 \) is not an element of \( \langle \text{LT}(f_1), \text{LT}(f_2) \rangle \).

The question is if there are some polynomials \( g_1, \ldots, g_k \) which generate \( I \) and for which in fact \( \langle \text{LT}(g_1), \ldots, \text{LT}(g_k) \rangle = \langle \text{LT}(I) \rangle \). In fact, such polynomials exist and they are called a Gröbner basis for \( I \). For now it is not so clear why this should be interesting. That should become clear in the next section. The formal definition of a Gröbner bases allows us to understand the algorithm to compute one.

**Definition 1** A finite subset \( g_1, \ldots, g_s \) of an ideal \( I \) is called a Gröbner basis of \( I \) if

\[
\langle \text{LT}(g_1), \ldots, \text{LT}(g_k) \rangle = \langle \text{LT}(I) \rangle.
\]

While the definition does not require that \( g_1, \ldots, g_k \) form a basis for \( I \) this fact can be shown fairly easily.

A Gröbner basis, \( G \), is called ‘reduced’ if for all distinct \( p, q \in G \) no monomial appearing in \( p \) is a multiple of \( \text{LT}(q) \). Each ideal in \( \mathbb{K}[x_1, \ldots, x_n] \) has a unique reduced Gröbner basis in which the coefficient of the leading term of every polynomial is one.

(lexicographic) Gröbner bases are interesting because, under some fairly mild technical conditions, they reduce the problem of finding all solutions of a square polynomial system of equations to finding all zeros of a single univariate polynomial. Before discussing this result we first describe an algorithm to compute Gröbner bases.

**A.3 Buchberger’s Algorithm**

There are now a variety of methods to compute Gröbner bases. The original algorithm by Buchberger implies a constructive existence proof for Gröbner basis and allows us to derive some important properties. Therefore we briefly outline the algorithm. In order to do so, we first need to generalize the division of two polynomials in one variable to multivariate polynomials.

Given any polynomials \( f_1, \ldots, f_s \in \mathbb{K}[x] \), with \( \text{md}(f_i) \geq \text{md}(f_{i+1}) \), we can write every \( f \in \mathbb{K} \) in the form

\[
f = a_1f_1 + \ldots + a_sf_s + r,
\]

where \( a_i, r \in \mathbb{K}[x] \) and either \( r = 0 \) or \( r \) is a linear combinations of monomials, none of which is divisible by any of the leading terms \( \text{LT}(f_1), \ldots, \text{LT}(f_s) \). The polynomial \( r \) is
called the remainder of \( f \) on division by \( f_1, \ldots, f_s \). Furthermore if \( a_i f_i \neq 0 \) we must have that \( \text{md}(f) \geq \text{md}(a_i f_i) \).

A formal algorithm which computes \( a_1, \ldots, a_s \) and \( r \) given any polynomials \( f, f_1, \ldots, f_s \) is described in Cox et al. (1997, Chapter 2.3). The algorithm can be implemented exactly (that is, performing computations without numerical errors) if \( K = \mathbb{Q} \). While the algebra behind the division algorithm is very simple, the algorithm plays a crucial role in the computation of \( \text{Gröbner bases} \).

To outline Buchberger’s algorithm for the computation of a \( \text{Gröbner basis} \), we need to define an S-polynomial. Let \( f, g \in K[x_1, \ldots, x_n] \) with \( \text{md}(f) = \alpha \) and \( \text{md}(g) = \beta \). Define \( \gamma = (\gamma_1, \ldots, \gamma_n) \) by \( \gamma_i = \max\{\alpha_i, \beta_i\}, i = 1, \ldots, n \), and define

\[
S(f, g) = \frac{x^{\gamma}}{\text{LT}(f)} f - \frac{x^{\gamma}}{\text{LT}(g)} g.
\]

The S-polynomial is interesting because of the following result.

**Theorem 1** \( G \) is a \text{Gröbner basis} if and only if for each \( g_i, g_j \in G \), the remainder of \( S(g_i, g_j) \) on division by \( G \) is zero.

For a proof of this theorem see Cox et al (1997, Chapter 2.6).

Given this result, it is possible to prove that the following algorithm always produces a \( \text{Gröbner basis} \) in finitely many steps. Let \( F = \{f_1, \ldots, f_k\} \) be a basis for the ideal \( I \). We construct a set \( G \) which is a \( \text{Gröbner basis} \).

1. Set \( G := F \).
2. \( G' := G \).
3. For each pair \( p, q \in G', p \neq q \), let \( S \) denote the remainder of \( S(f, g) \) on division by \( G' \). If \( S \neq 0 \) then \( G := G \cup \{S\} \).
4. If \( G \neq G' \) goto step 2.

To prove that the algorithm works, first we show that in each iteration \( \langle G \rangle \), i.e. the ideal generated by all polynomials in the finite set \( G \) is a subset of \( I \). If the algorithm terminates, the resulting \( G \) must be a \( \text{Gröbner basis} \) by the above theorem. It is a bit more involved to show that the algorithm actually does terminate: in each iteration, we must have \( \langle \text{LT}(G') \rangle \subset \langle \text{LT}(G) \rangle \) since \( G' \subset G \). If \( G' \neq G \), the inclusion is strict. The following lemma then implies that eventually the inclusion cannot be strict, \( G' = G \) and the algorithm must stop at a finite number of iterations.

**Lemma 3 (Ascending Chains Lemma)** Let \( I_1 \subset I_2 \subset \ldots \) be an ascending chain of ideals in \( K[x_1, \ldots, x_n] \). Then there exists a \( N \geq 1 \) such that \( I_N = I_{N+1} = I_{N+2} = \ldots \).
To prove the lemma, consider the set \( \mathcal{I} = \bigcup_{i=1}^{\infty} \mathcal{I}_i \). Note that \( \mathcal{I} \) is an ideal. By the Hilbert basis theorem, \( \mathcal{I} \) must be finitely generated, i.e., there must exist \( f_1, \ldots, f_s \) such that \( \mathcal{I} = \langle f_1, \ldots, f_s \rangle \), but each of the generators must be contained in some of the \( \mathcal{I}_j \), take \( n \) to be the maximum of these \( j \)'s.

Note that while this algorithm is well defined independently of the field \( \mathbb{K} \), it can be performed exactly over \( \mathbb{Q} \). Furthermore, if the coefficients in the polynomials \( f_1, \ldots, f_k \) are parameters, the algorithm can be applied to obtain a set of polynomials \( g_1, \ldots, g_m \) whose coefficients themselves are polynomial functions of the parameters. If the coefficients of \( f_1, \ldots, f_k \) are real parameters, the coefficients of \( g_1, \ldots, g_m \) will be polynomial functions in these parameters, with rational coefficients. The result of Buchberger’s algorithm forms a Gröbner basis for \( \langle f_1, \ldots, f_k \rangle \) for all values of the parameters, except for a set that is a finite union of sets defined by polynomial equations. The division set is generic in that for specific values of the parameters (satisfying some polynomial equation) it implies division by zero and is therefore not valid. However, if we take the parameters to lie in \( \mathbb{R}^k \), the polynomials resulting from Buchberger’s algorithm for a Gröbner basis for a Zariski-open subset of \( \mathbb{R}^k \). Unless some of the polynomial functions are identical equal to zero (and the subset of valid parameters is the empty set), the set of parameters for which the resulting functions do not form a Gröbner basis has \( k \)-dimensional Lebesgue measure zero. This does not change if one considers a reduced Gröbner basis. In this case, one simply eliminates some of the generating polynomials.

So called ‘comprehensive’ Gröbner bases keep track of all sub-cases, including the non-generic ones, but we argue below that they are not needed in the multiplicity analysis.

### A.4 Elimination Ideals and the Shape Lemma

For given \( \mathcal{I} = \langle f_1, \ldots, f_s \rangle \subset \mathbb{K}[x_1, \ldots, x_n] \), define the \( l \)'th elimination ideal \( \mathcal{I}_l \) as the ideal in \( \mathbb{K}[x_{l+1}, \ldots, x_n] \) defined by \( \mathcal{I}_l = \mathcal{I} \cap \mathbb{K}[x_{l+1}, \ldots, x_n] \). In other words \( \mathcal{I}_l \) consists of all ‘consequences’ of \( f_1 = \ldots = f_s = 0 \) which eliminate the variables \( x_1, \ldots, x_l \). Each \( \mathcal{I}_l \) is an ideal, i.e. there exist polynomials \( f_1, \ldots, f_r \in \mathbb{K}[x_{l+1}, \ldots, x_n] \) that generate \( \mathcal{I}_l \). If \( \mathcal{I} \) is radical and zero-dimensional then the \((n-1)\)th elimination ideal must describe the \( x_n \)-coordinate of all possible solutions to the original system solutions. Since there are finitely many there must be a univariate polynomial that generates this (at least, if we take \( \mathbb{K} = \mathbb{C} \), this is just the product of all \((x_n - a_i)\) terms for all zeros \( a_i \)). By the strong Nullstellensatz, this polynomial must itself belong to the ideal (since adding it to the ideal does not change the solution set).

Given a set of polynomials \( G = \{ f_1, \ldots, f_r \} \) we can obviously define \( G \cap \mathbb{K}[x_{l+1}, \ldots, x_n] \) as those polynomials in \( G \) which only involve \( x_{l+1}, \ldots, x_n \). For general polynomials and \( l > 0 \) this set will generally be empty. However, not for Gröbner bases, as the following theorem shows.

**Theorem 2 (Elimination Theorem)** Let \( \mathcal{I} \subset \mathbb{K}[x_1, \ldots, x_n] \) be an ideal and let \( G \) be a lex
Gröbner basis of $I$. Then for every $0 \leq l \leq n$, $G_l = G \cap \mathbb{K}[x_{l+1}, \ldots, x_n]$ is a Gröbner basis of the $l$'th elimination ideal.

**Proof.** For $l$ between 0 and $n-1$, since $G_l \subseteq I_l$ by construction, it suffices to show that $\langle LT(I_l) \rangle = \langle LT(G_l) \rangle$. It suffices to show that in fact $\langle LT(I_l) \rangle \subset \langle LT(G_l) \rangle$. In other words, for every $f \in I_l$, $LT(f)$ is divisible by $LT(g)$ for some $g \in G_l$. Note that $f$ must also lie in $I$ which means that $LT(f)$ must be divisible by some $LT(g)$, $g \in G$. But since $f \in I_l$ this means that $LT(G)$ can only involve variables $x_{l+1}, \ldots, x_n$. But since we work under lexicographical order, $LT(g) \in \mathbb{K}[x_{l+1}, \ldots, x_n]$ must imply that $g \in \mathbb{K}[x_{l+1}, \ldots, x_s]$. □

This result now leads us to the Shape Lemma. Given a zero-dimensional radical ideal $I$, with $V(I) = \{a^1, \ldots, a^d\}$, the above theorem implies that the reduced lexicographic Gröbner basis of $I$ must contain the univariate polynomial $\Pi_{i=1}^d(x_n - a^i_n)$, which must be a polynomial over $\mathbb{K}$. If across all $i = 1, \ldots, d$, the $a^i_n$ are distinct, for each $l = 1, \ldots, n-1$, $a^i_l$ must be the unique solution to a polynomial involving only $x_l, \ldots, x_n$ with $x_{l+1} = a^i_{l+1}, \ldots, x_n = a^i_n$. But for $\mathbb{K} = \mathbb{C}$ this implies that this polynomial must be linear in $x_l$ (otherwise it has more than one solution, the ideal being radical rules out multiple zeros). The Shape Lemma form than simply follows by substituting recursively for each $x_l, l = 2, \ldots, n-1$.

**A.5 The Parametric Shape Lemma**

For $e \in E$, let $f_e = (f_1(e, \cdot), \ldots, f_n(e, \cdot))$ be a system of polynomial equations in parameters, $e$, and in the unknowns $(x_1, \ldots, x_n)$. Let $g_e = (g_1(e, \cdot), \ldots, g_n(e, \cdot))$ denote the output of Buchberger’s algorithm. The parametric shape lemma requires that for all $e \in E$, the original system $f$ is regular. If this is the case, $g_e$ is the correct reduced Gröbner basis and has the Shape-Lemma form for ‘generic’ $\bar{e}$, that is for all $\bar{e} \in E$ outside of a closed set of Lebesgue measure zero (see e.g. Cox et al. (1997) for this result). We obtain a slightly stronger result in that we characterize the set of parameters for which the output of Buchberger’s algorithm is not the correct Gröbner basis explicitly.

By the implicit function theorem, regularity of $f$ guarantees that for all $\bar{e}$, for any $\bar{x}$ that is among the finitely many solutions to $f(x) = 0$, there is an open neighborhood around $\bar{e}$ such that for all sequences of parameters $(e^i)$ in this neighborhood that converge to $\bar{e}$, there exist $(x^i)$ such that for all $i$, $f(e^i, x^i) = 0$ and $x^i \to \bar{x}$. Therefore if $f(\bar{e}, \bar{x}) = 0$, we must also have $g(\bar{e}, \bar{x}) = 0$. This is true because we can find sequences of parameters which converge to $\bar{e}$ and for which $g$ is the correct Gröbner basis at all points along the sequence.

However, in principle it can be the case that $g(\bar{e}, x) = 0$ has solutions that are not solutions to the original system. One simple way to rule this out is to require that $g$ itself is regular at $\bar{e}$. Rather than assuming this, one can actually verify this. A singularity can only occur if either a derivative with respect to $x_1, \ldots, x_{n-1}$ is equal to zero, which means that some $p_l(\bar{e}) = 0, l = 1, \ldots, n-1$ or that $r(\bar{e}, x) = 0$ and $\partial r / \partial x = 0$. If $r(\bar{e}, \cdot)$ is not identically equal to zero, the latter implies that there is a multiple solution which is ruled
out by regularity of $f_e$.

References


