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Dealing with Undesirable Outputs in DEA: A Slacks-based Measure (SBM) Approach

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Abstract

In this paper, we propose a new non-parametric DEA scheme for measuring efficiency in the presence of undesirable outputs, based on a slack-based measure (SBM) developed in Tone (2001). We further extend our scheme to cope with non-separable desirable and undesirable outputs. Then we compare our approach with some other methods proposed for this purpose thus far.

Keywords: Undesirable outputs, non-separable outputs, DEA, SBM

1 Introduction

In accordance with the environmental conservation awareness in our modern society, undesirable outputs of productions and social activities, e.g., air pollutants and hazardous waste, have been widely recognized as societal evils. Thus, development of technologies with less undesirable outputs is an important subject of concern in every area of production. DEA (data envelopment analysis) usually assumes that producing more outputs relative to less input resources is a criterion of efficiency. In the presence of undesirable outputs, however, technologies with more good (desirable) outputs and less bad (undesirable) outputs relative to less input resources should be recognized as efficient. In the DEA literature, several authors have proposed

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methods for this purpose, e.g., Färe *et al.*(1989), Scheel (2001), and Seiford and Zhu (2002), among others.

This paper deals with the same problem by applying a slacks-based measure of efficiency (SBM) that the author proposed in Tone (2001). The SBM is non-radial and non-oriented, and utilizes input and output slacks directly in producing an efficiency measure. In this paper, SBM is modified so as to account for undesirable outputs.

This paper unfolds as follows. Section 2 extends the SBM to the case including undesirable outputs. The economic interpretation of the model is demonstrated in Section 3 that enhances the rationale of the method. Extensions to various returns to scale assumptions, non-separable output problems and weight restrictions are discussed in Section 4. Illustrative examples are presented in Section 5. Then, we compare our method with previously proposed ones in Section 6. Section 7 concludes this paper.

2 An SBM with undesirable outputs

Suppose that there are n DMUs (decision making units) each having three factors: inputs, good outputs and bad (undesirable) outputs, as represented by three vectors $\boldsymbol{x} \in R^m$, $\boldsymbol{y}^g \in R^{s_1}$ and $\boldsymbol{y}^b \in R^{s_2}$, respectively. We define the matrices X, Y^g and Y^b as follows.

$X = [\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n] \in R^{m\times n}$, $Y^g = [\boldsymbol{y}_1^g, \cdots, \boldsymbol{y}_n^g] \in R^{s_1\times n}$, and $Y^b = [\boldsymbol{y}_1^b, \cdots, \boldsymbol{y}_n^b] \in R^{s_2\times n}$. We assume $X > 0$, $Y^b > 0$ and $Y^b > 0$.

The production possibility set (P) is defined by

$$P = \{(\boldsymbol{x}, \boldsymbol{y}^g, \boldsymbol{y}^b) | \boldsymbol{x} \geq X\boldsymbol{\lambda},\ \boldsymbol{y}^g \leq Y^g\boldsymbol{\lambda},\ \boldsymbol{y}^b \geq Y^b\boldsymbol{\lambda},\ \boldsymbol{\lambda} \geq 0\}, \tag{1}$$

where $\boldsymbol{\lambda} \in R^n$ is the intensity vector. Notice that the above definition corresponds to the constant returns to scale technology. We discuss the other return to scale cases in Section 4.

Definition 1 (Efficient DMU) *A DMU_o $(\boldsymbol{x}_o, \boldsymbol{y}_o^g, \boldsymbol{y}_o^b)$ is efficient in the presence of undesirable outputs if there is no vector $(\boldsymbol{x}, \boldsymbol{y}^g, \boldsymbol{y}^b) \in P$ such that $\boldsymbol{x}_o \geq \boldsymbol{x}$, $\boldsymbol{y}_o^g \leq \boldsymbol{y}^g$ and $\boldsymbol{y}_o^b \geq \boldsymbol{y}^b$ with at least one strict inequality.*

In accordance with this definition, we modify the SBM in Tone (2001) as follows.

$$\text{[SBM]}\ \rho^* = \min \frac{1 - \frac{1}{m}\sum_{i=1}^{m} \frac{s_i^-}{x_{io}}}{1 + \frac{1}{s_1+s_2}\left(\sum_{r=1}^{s_1} \frac{s_r^g}{y_{ro}^g} + \sum_{r=1}^{s_2} \frac{s_r^b}{y_{ro}^b}\right)} \tag{2}$$

$$\text{subject to} \quad \boldsymbol{x}_o = X\boldsymbol{\lambda} + \boldsymbol{s}^- \tag{3}$$

$$\boldsymbol{y}_o^g = Y^g\boldsymbol{\lambda} - \boldsymbol{s}^g \tag{4}$$

$$\boldsymbol{y}_o^b = Y^b\boldsymbol{\lambda} + \boldsymbol{s}^b \tag{5}$$

$$\boldsymbol{s}^- \geq 0,\ \boldsymbol{s}^g \geq 0,\ \boldsymbol{s}^b \geq 0,\ \boldsymbol{\lambda} \geq 0.$$

The vectors $\boldsymbol{s}^- \in R^m$ and $\boldsymbol{s}^b \in R^{s_2}$ correspond to excesses in inputs and bad outputs, respectively, while $\boldsymbol{s}^g \in R^{s_1}$ expresses shortages in good outputs. The objective function (2) strictly decreases with respect to s_i^- $(\forall i)$, s_r^g $(\forall r)$ and s_r^b $(\forall r)$ and the objective value satisfies $0 < \rho^* \leq 1$. Let an optimal solution of the above program be $(\boldsymbol{\lambda}^*,\ \boldsymbol{s}^{-*},\ \boldsymbol{s}^{g*},\ \boldsymbol{s}^{b*})$. Then, we have:

Theorem 1 *The DMU$_o$ is efficient in the presence of undesirable outputs if and only if $\rho^* = 1$, i.e., $\boldsymbol{s}^{-*} = 0,\ \boldsymbol{s}^{g*} = 0$ and $\boldsymbol{s}^{b*} = 0$.*

If the DMU$_o$ is inefficient, i.e., $\rho^* < 1$, it can be improved and become efficient by deleting the excesses in inputs and bad outputs, and augmenting the shortfalls in good outputs via the following SBM-projection:

$$\boldsymbol{x}_o \leftarrow \boldsymbol{x}_o - \boldsymbol{s}^{-*} \tag{6}$$

$$\boldsymbol{y}_o^g \leftarrow \boldsymbol{y}_o^g + \boldsymbol{s}^{g*} \tag{7}$$

$$\boldsymbol{y}_o^b \leftarrow \boldsymbol{y}_o^b - \boldsymbol{s}^{b*} \tag{8}$$

Using the transformation by Charnes and Cooper (1962), we arrive at an equivalent linear program in t, Λ, $\boldsymbol{S}^-$, $\boldsymbol{S}^g$ and $\boldsymbol{S}^b$ as displayed below.

$$\text{[LP]}\ \tau^* = \min t - \frac{1}{m}\sum_{i=1}^{m} \frac{S_i^-}{x_{io}} \tag{9}$$

$$\text{subject to} \quad 1 = t + \frac{1}{s_1+s_2}\left(\sum_{r=1}^{s_1} \frac{S_r^g}{y_{ro}^g} + \sum_{r=1}^{s_2} \frac{S_r^b}{y_{ro}^b}\right) \tag{10}$$

$$\boldsymbol{x}_o t = X\Lambda + \boldsymbol{S}^- \tag{11}$$

$$\boldsymbol{y}_o^g t = Y^g\Lambda - \boldsymbol{S}^g \tag{12}$$

$$\boldsymbol{y}_o^b t = Y^b\Lambda + \boldsymbol{S}^b \tag{13}$$

$$\boldsymbol{S}^- \geq 0,\ \boldsymbol{S}^g \geq 0,\ \boldsymbol{S}^b \geq 0,\ \Lambda \geq 0,\ t > 0. \tag{14}$$

Let an optimal solution of [LP] be (t^*, Λ^*, S^{-*}, S^{g*}, S^{b*}). Then we have an optimal solution of [SBM] as defined by

$$\rho^* = \tau^*,\ \lambda^* = \Lambda^*/t^*,\ s^{-*} = S^{-*}/t^*,\ s^{g*} = S^{g*}/t^*,\ S^{b*} = s^{b*}/t^*. \quad (15)$$

(See Tone (2001) for detail). The existence of (t^*, Λ^*, S^{-*}, S^{g*}, S^{b*}) with $t^* > 0$ is guaranteed by [LP].

3 Economic interpretations

We have the dual program of [LP] as follows:

$$\begin{aligned}
& \xi^* = \max\ \xi && (16)\\
\text{subject to}\quad & \xi + \boldsymbol{v}\boldsymbol{x}_o - \boldsymbol{u}^g\boldsymbol{y}_o^g + \boldsymbol{u}^b\boldsymbol{y}_o^b = 1 && (17)\\
& -\boldsymbol{v}X + \boldsymbol{u}^g Y^g - \boldsymbol{u}^b Y^b \le \mathbf{0} &&\\
& \boldsymbol{v} \ge \frac{1}{m}[1/\boldsymbol{x}_o] &&\\
& \boldsymbol{u}^g \ge \frac{\xi}{s}[1/\boldsymbol{y}_o^g] &&\\
& \boldsymbol{u}^b \ge \frac{\xi}{s}[1/\boldsymbol{y}_o^b], &&
\end{aligned}$$

where $s = s_1 + s_2$ and the notation $[1/\boldsymbol{x}_o]$ stands for the row vector ($1/x_{1o}$, $\cdots$, $1/x_{mo}$).

The dual variable vectors $\boldsymbol{v} \in R^m$, $\boldsymbol{u}^g \in R^{s_1}$ and $\boldsymbol{u}^b \in R^{s_2}$ correspond to the constraints (11), (12) and (13), respectively. By eliminating ξ, this program is equivalent to the following:

$$\begin{aligned}
\text{[DualLP]}\quad & \max\ \boldsymbol{u}^g\boldsymbol{y}_o^g - \boldsymbol{v}\boldsymbol{x}_o - \boldsymbol{u}^b\boldsymbol{y}_o^b && (18)\\
\text{subject to}\quad & \boldsymbol{u}^g Y^g - \boldsymbol{v}X - \boldsymbol{u}^b Y^b \le \mathbf{0} && (19)\\
& \boldsymbol{v} \ge \frac{1}{m}[1/\boldsymbol{x}_o] && (20)\\
& \boldsymbol{u}^g \ge \frac{1 + \boldsymbol{u}^g\boldsymbol{y}_o^g - \boldsymbol{v}\boldsymbol{x}_o - \boldsymbol{u}^b\boldsymbol{y}_o^b}{s}[1/\boldsymbol{y}_o^g] && (21)\\
& \boldsymbol{u}^b \ge \frac{1 + \boldsymbol{u}^g\boldsymbol{y}_o^g - \boldsymbol{v}\boldsymbol{x}_o - \boldsymbol{u}^b\boldsymbol{y}_o^b}{s}[1/\boldsymbol{y}_o^b]. && (22)
\end{aligned}$$

The dual variables $\boldsymbol{v}$ and $\boldsymbol{u}^b$ can be interpreted as the virtual prices (costs) of inputs and bad outputs, respectively, while $\boldsymbol{u}^g$ denotes the price of good outputs. The dual program aims at obtaining the optimal virtual costs and prices for DMU_o so that the profit $\boldsymbol{u}^g\boldsymbol{y}^g - \boldsymbol{v}\boldsymbol{x} - \boldsymbol{u}^b\boldsymbol{y}^b$ does not exceed zero for every DMU (including DMU_o) and maximizes the profit $\boldsymbol{u}^g\boldsymbol{y}^g_o - \boldsymbol{v}\boldsymbol{x}_o - \boldsymbol{u}^b\boldsymbol{y}^b_o$ for the DMU_o concerned. Apparently, the optimal profit is at best zero and hence $\xi^* = 1$ for the SBM efficient DMUs.

Constraints (20), (21) and (22) restrict the dual variables to the positive orthant. Using this framework, we can incorporate other important developments related to the virtual dual variables into the SBM model, e.g., the assurance region methods (Thompson *et al.* (1986)). These modifications will contribute to the enhancement of the potential application of the model substantially.

4 Extensions

This section first discusses the returns to scale issues, and then extends our scheme to deal with non-separable 'good' and 'bad' outputs. Lastly, the weight restriction problem, i.e., setting different weights to inputs/outputs is analyzed.

4.1 Returns to scale (RTS) issues

Although we discussed our bad outputs model under the constant returns to scale (CRS) assumption, we can incorporate other RTS by adding the following constraint to [SBM] and hence to the definition of the production possibility set P,

$$L \leq \boldsymbol{e}\boldsymbol{\lambda} \leq U, \tag{23}$$

where $\boldsymbol{e} = (1, \cdots, 1) \in R^n$ and, $L(\leq 1)$ and $U(\geq 1)$ are respectively the lower and upper bounds to the intensity $\boldsymbol{\lambda}$.

The cases $(L = 1,\ U = 1)$, $(L = 0,\ U = 1)$ and $(L = 1,\ U = \infty)$ correspond to the variable (VRS), the decreasing (DRS) and the increasing (IRS) RTS, respectively.

The definition of the efficiency status is the same as described in [Definition 1] and Theorem 1 holds in these cases, too. The addition of the RTS constraint (23) brings out modification in [DualLP] among which we observe the VRS case, i.e., $e\lambda = 1$, as representative. The [DualLP] turns out to:

$$\begin{aligned} \text{[DualLP-VRS]} \quad & \max \ u^g y_o^g - v x_o - u^b y_o^b + w & (24) \\ \text{subject to} \quad & u^g Y^g - vX - u^b Y^b + we \leq 0 & \\ & v \geq \frac{1}{m}[1/x_o] & (25) \\ & u^g \geq \frac{1 + u^g y_o^g - v x_o - u^b y_o^b + w}{s}[1/y_o^g] & (26) \\ & u^b \geq \frac{1 + u^g y_o^g - v x_o - u^b y_o^b + w}{s}[1/y_o^b], & (27) \end{aligned}$$

where $w \in R$ is the dual variable corresponding to the constraint $e\lambda = 1$. Apparently, the optimal objective function value is at best zero and attains zero if and only if the DMU$_o$ is efficient under the VRS. Let an optimal solution for [DualLP-VRS] be $(v^*, u^{g*}, u^{b*}, w^*)$, then, if DMU$_o$ is efficient, we have the following inequality.

$$u^{g*} y_o^g - v^* x_o - u^{b*} y_o^b \geq u^{g*} y_j^g - v^* x_j - u^{b*} y_j^b \quad (\forall j). \tag{28}$$

Thus, the price interpretation of the role of the dual variables remains valid in this case, too.

4.2 Non-separable 'good' and 'bad' outputs

It is often observed that a certain 'bad' outputs are not separable from the corresponding 'good' outputs. Hence, reducing bad outputs is inevitably accompanied by reduction in good outputs. In this section, we discuss this non-separable case. For this, we decompose the set of good and bad outputs (Y^g, Y^b) into (Y^{Sg}, Y^{Sb}) and (Y^{NSg}, Y^{NSb}), where $(Y^{Sg} \in R^{s_{11} \times n}, Y^{Sb} \in R^{s_{12} \times n})$ and $(Y^{NSg} \in R^{s_{21} \times n}, Y^{NSb} \in R^{s_{22} \times n})$ denote the separable and non-separable good and bad outputs, respectively. For the separable outputs (Y^{Sg}, Y^{Sb}), we have the same structure of production as (Y^g, Y^b) in P. However, the non-separable outputs (Y^{NSg}, Y^{NSb}) need handling differently. A reduction of the bad outputs y^{NSb} is designated by αy^{NSb} with $0 \leq \alpha \leq 1$, which is accompanied by a proportionate reduction in the good outputs y^{NSg} as denoted by αy^{NSg}. Although in this case we assume the same proportionate rate α in bad and good outputs, we can set other relationships between the two, e.g., αy^{NSb} and βy^{NSg} with $0 \leq \alpha, \beta \leq 1$.

Now, the new production possibility set P_{NS} under the VRS is defined by:

$$\begin{aligned} P_{NS} &= \Big\{(\boldsymbol{x}, \boldsymbol{y}^{Sg}, \boldsymbol{y}^{Sb}, \boldsymbol{y}^{NSg}, \boldsymbol{y}^{NSb})| \\ &\quad \boldsymbol{x} \geq X\boldsymbol{\lambda},\ \boldsymbol{y}^{Sg} \leq Y^{Sg}\boldsymbol{\lambda},\ \boldsymbol{y}^{Sb} \geq Y^{Sb}\boldsymbol{\lambda},\ \boldsymbol{y}^{NSg} \leq Y^{NSg}\boldsymbol{\lambda}, \\ &\quad \boldsymbol{y}^{NSb} \geq Y^{NSb}\boldsymbol{\lambda},\ \boldsymbol{e\lambda} = 1,\ \boldsymbol{\lambda} \geq \boldsymbol{0}\Big\}. \end{aligned} \tag{29}$$

Basically this definition is a natural extension of P in (1). We alter the definition of the efficiency status in the non-separable case as follows:

Definition 2 (NS-efficient) *A DMU_o $(\boldsymbol{x}_o, \boldsymbol{y}_o^{Sg}, \boldsymbol{y}_o^{Sb}, \boldsymbol{y}_o^{NSg}, \boldsymbol{y}_o^{NSb})$ is called NS-efficient if and only if (1) for any α $(0 \leq \alpha < 1)$, we have*

$$(\boldsymbol{x}_o, \boldsymbol{y}_o^{Sg}, \boldsymbol{y}_o^{Sb}, \alpha\boldsymbol{y}_o^{NSg}, \alpha\boldsymbol{y}_o^{NSb}) \notin P_{NS},$$

and (2) there is no $(\boldsymbol{x}, \boldsymbol{y}^{Sg}, \boldsymbol{y}^{Sb}, \boldsymbol{y}^{NSg}, \boldsymbol{y}^{NSb}) \in P_{NS}$ such that

$$\boldsymbol{x}_o \geq \boldsymbol{x},\ \boldsymbol{y}_o^{Sg} \leq \boldsymbol{y}^{Sg},\ \boldsymbol{y}_o^{Sb} \geq \boldsymbol{y}^{Sb},\ \boldsymbol{y}_o^{NSg} = \boldsymbol{y}^{NSg},\ \boldsymbol{y}_o^{NSb} = \boldsymbol{y}^{NSb}$$

with at least one strict inequality.

An SBM with non-separable outputs can be implemented by the program in $(\boldsymbol{\lambda},\ \boldsymbol{s}^-,\ \boldsymbol{s}^{Sg},\ \boldsymbol{s}^{Sb},\ \alpha)$ as below:

[SBM-NS]

$$\rho^* = \min \frac{1 - \frac{1}{m}\sum_{i=1}^{m} \frac{s_i^-}{x_{io}}}{1 + \frac{1}{s}\Big(\sum_{r=1}^{s_{11}} \frac{s_r^{Sg}}{y_{ro}^{Sg}} + \sum_{r=1}^{s_{12}} \frac{s_r^{Sb}}{y_{ro}^{Sb}} + (s_{21} + s_{22})(1-\alpha)\Big)} \tag{30}$$

subject to

$$\boldsymbol{x}_o = X\boldsymbol{\lambda} + \boldsymbol{s}^-$$
$$\boldsymbol{y}_o^{Sg} = Y^{Sg}\boldsymbol{\lambda} - \boldsymbol{s}^{Sg}$$
$$\boldsymbol{y}_o^{Sb} = Y^{Sb}\boldsymbol{\lambda} + \boldsymbol{s}^{Sb}$$
$$\alpha\boldsymbol{y}_o^{NSg} \leq Y^{NSg}\boldsymbol{\lambda} \tag{31}$$
$$\alpha\boldsymbol{y}_o^{NSb} \geq Y^{NSb}\boldsymbol{\lambda} \tag{32}$$
$$\boldsymbol{e\lambda} = 1$$
$$\boldsymbol{s}^- \geq \boldsymbol{0},\ \boldsymbol{s}^{Sg} \geq \boldsymbol{0},\ \boldsymbol{s}^{Sb} \geq \boldsymbol{0},\ \boldsymbol{\lambda} \geq \boldsymbol{0},\ 0 \leq \alpha \leq 1,$$

where $s = s_{11} + s_{12} + s_{21} + s_{22}$.

The objective function is strictly monotone decreasing with respect to s_i^- $(\forall i)$, s_r^{Sg} $(\forall r)$, s_r^{Sb} $(\forall r)$ and α. Let an optimal solution for [SBM-NS] be $(\rho^*,\ \boldsymbol{\lambda}^*,\ \boldsymbol{s}^{-*},\ \boldsymbol{s}^{Sg*},\ \boldsymbol{s}^{Sb*},\ \alpha^*)$, then we have $0 < \rho^* \leq 1$ and the following theorem holds:

Theorem 2 *The DMU$_o$ is NS-efficient if and only if $\rho^* = 1$, i.e., $\boldsymbol{s}^{-*} = \boldsymbol{0}$, $\boldsymbol{s}^{Sg*} = \boldsymbol{0}$, $\boldsymbol{s}^{Sb*} = \boldsymbol{0}$ and $\alpha^* = 1$.*

If the DMU$_o$ is NS-inefficient, i.e., $\rho^* < 1$, it can be improved and become NS-efficient by the following NS-projection:

$$\boldsymbol{x}_o \leftarrow \boldsymbol{x}_o - \boldsymbol{s}^{-*} \tag{33}$$
$$\boldsymbol{y}_o^{Sg} \leftarrow \boldsymbol{y}_o^{Sg} + \boldsymbol{s}^{Sg*} \tag{34}$$
$$\boldsymbol{y}_o^{Sb} \leftarrow \boldsymbol{y}_o^{Sb} - \boldsymbol{s}^{Sb*} \tag{35}$$
$$\boldsymbol{y}_o^{NSg} \leftarrow \alpha^* \boldsymbol{y}_o^{NSg} \tag{36}$$
$$\boldsymbol{y}_o^{NSb} \leftarrow \alpha^* \boldsymbol{y}_o^{NSb}. \tag{37}$$

It should be noted that, from (31) and (32), it holds that

$$\boldsymbol{s}^{NSg*} \equiv -\alpha^* \boldsymbol{y}_o^{NSg} + Y^{NSg}\boldsymbol{\lambda}^* \geq \boldsymbol{0} \tag{38}$$
$$\boldsymbol{s}^{NSb*} \equiv \alpha^* \boldsymbol{y}_o^{NSb} - Y^{NSb}\boldsymbol{\lambda}^* \geq \boldsymbol{0}. \tag{39}$$

This means that the some of slacks in non-separable good and bad outputs may remain positive even after the projection, and that these slacks are not accounted for in the NS-efficiency score. This is because we assume proportionate reduction (α^*) in these outputs. Thus, we apply the SBM to the separable outputs, whereas we employ the radial approach to the non-separable outputs.

In the case that the proportionate reduction rate (α) has a lower bound, we replace the constraint $0 \leq \alpha \leq 1$ by $\alpha_{min} \leq \alpha \leq 1$, where α_{min} is the lower bound for the reduction rate of the non-separable good (bad) outputs.

We further observe the dual of [SBM-NS] after transforming the fractional program into the corresponding equivalent linear program in the similar manner as in the case [LP].

$$\begin{aligned}
&[\text{DualLP-NS}]\\
&\max\ \xi && (40)\\
&\text{subject to}\\
&\xi = 1 + \boldsymbol{u}^{Sg}\boldsymbol{y}_o^{Sg} + \boldsymbol{u}^{NSg}\boldsymbol{y}_o^{NSg} - \boldsymbol{v}\boldsymbol{x}_o - \boldsymbol{u}^{Sb}\boldsymbol{y}_o^{Sb} - \boldsymbol{u}^{NSb}\boldsymbol{y}_o^{NSb} + w\\
&\boldsymbol{u}^{Sg}Y^{Sg} + \boldsymbol{u}^{NSg}Y^{NSg} - \boldsymbol{v}X - \boldsymbol{u}^{Sb}Y^{Sb} - \boldsymbol{u}^{NSb}Y^{NSb} + w\boldsymbol{e} \leq \boldsymbol{0}\\
&\boldsymbol{v} \geq \frac{1}{m}[1/\boldsymbol{x}_o],\ \boldsymbol{u}^{Sg} \geq \frac{\xi}{s}[1/\boldsymbol{y}_o^{Sg}],\ \boldsymbol{u}^{Sb} \geq \frac{\xi}{s}[1/\boldsymbol{y}_o^{Sb}] && (41)\\
&\boldsymbol{u}^{NSg} \geq \boldsymbol{0},\ \boldsymbol{u}^{NSb} \geq \boldsymbol{0}\\
&\text{with additional constraints corresponding to } 0 \leq \alpha \leq 1. && (42)
\end{aligned}$$

Thus, we have the same economic interpretations as in the case [DualLP]. We are looking for the optimal virtual prices $\boldsymbol{v}$ (for inputs), $\boldsymbol{u}^{Sg}$ (for separable good outputs), $\boldsymbol{u}^{NSg}$ (for non-separable good outputs), $\boldsymbol{u}^{Sb}$ (for separable bad outputs) and $\boldsymbol{u}^{NSb}$ (for non-separable bad outputs) under the VRS that maximize the virtual profit $\boldsymbol{u}^{Sg}\boldsymbol{y}_o^{Sg} + \boldsymbol{u}^{NSg}\boldsymbol{y}_o^{NSg} - \boldsymbol{v}\boldsymbol{x}_o - \boldsymbol{u}^{Sb}\boldsymbol{y}_o^{Sb} - \boldsymbol{u}^{NSb}\boldsymbol{y}_o^{NSb} + w$, while keeping the virtual profit of every DMU (including DMU$_o$) non-positive.

4.3 Imposing weights to inputs and/or outputs

If putting preference (or importance) on input/output items is requested, we can impose weights to the objective function in (2) as follows:

$$[\text{SBM}]\ \rho^* = \min \frac{1 - \frac{1}{m}\sum_{i=1}^{m}\frac{w_i^- s_i^-}{x_{io}}}{1 + \frac{1}{s_1+s_2}\left(\sum_{r=1}^{s_1}\frac{w_r^g s_r^g}{y_{ro}^g} + \sum_{r=1}^{s_2}\frac{w_r^b s_r^b}{y_{ro}^b}\right)}, \tag{43}$$

where w_i, w_r^g and w_r^b are the weights to the input i, the desirable output r, and the undesirable output r, respectively, and $\sum_{i=1}^{m} w_i^- = m$, $w_i^- \geq 0\ (\forall i), \sum_{r=1}^{s_1} w_r^g + \sum_{r=1}^{s_2} w_r^b = s_1 + s_2$, $w_r^g \geq 0\ (\forall r),$, $w_r^b \geq 0\ (\forall r)$.

5 Illustrative examples

We present numerical examples that illustrate the aforementioned schemes. All sample problems were solved under the variable returns to scale (VRS) assumption, i.e., $\boldsymbol{e\lambda} = 1$.

5.1 Separable bad outputs models

Table 1 exhibits a simple data set composed of a single input (with value=1) and two outputs: one desirable and one undesirable. We solved this problem

Table 1: Separable bad outputs case: Data set

	Input	Output	
		Desirable	Undesirable
DMU	x	y^{Good}	y^{Bad}
A	1	1	1
B	1	2	1
C	1	6	2
D	1	8	4
E	1	9	7
F	1	5	2
G	1	4	3
H	1	6	4
I	1	4	6

by employing the weight selection described in Section 4.3. We placed three ratios of weights on 'good' and 'bad' outputs: (1:0.3), (1:1) and (1:3). Table 2 reports the results: the efficiency (ρ^*), and the projected good and bad outputs. DMUs B, C, D and E are efficient in this separable outputs model. It is observed that, as the weight moves from good to bad, the emphasis of projection changes from enlargement of the good output to reduction of the bad output. Figures 1, 2 and 3 illustrate these changes.

Table 2: Separable bad outputs case: Results

	Weights to Good and Bad Outputs								
	1:0.3			1:1			1:3		
		Projected			Projected			Projected	
	ρ^*	y^{Good}	y^{Bad}	ρ^*	y^{Good}	y^{Bad}	ρ^*	y^{Good}	y^{Bad}
A	0.565	2	1	0.667	2	1	0.8	2	1
B	1	-	-	1	-	-	1	-	-
C	1	-	-	1	-	-	1	-	-
D	1	-	-	1	-	-	1	-	-
E	1	-	-	1	-	-	1	-	-
F	0.867	6	2	0.909	6	2	0.914	5	1.75
G	0.634	7	3	0.706	6	2	0.727	6	2
H	0.796	8	4	0.80	6	2	0.727	6	2
I	0.527	8.67	6	0.60	8	4	0.615	6	2

Figure 1: yGood:yBad = 1:0.3

Figure 2: yGood:yBad = 1:1

Figure 3: yGood:yBad = 1:3

5.2 Non-separable outputs model

Suppose that the outputs in Table 1 are non-separable. In this case, we cannot reduce the bad output without worsening the good output. We applied [SBM-NS] to this data set. Table 3 exhibits the results of two cases: one with the lower bound of reduction rate $\alpha_{min} = 0$ and the other with $\alpha_{min} = 0.8$.

In this model, the NS-efficient DMUs are A, B and C while D and E are no longer efficient. Figure 4 shows the projection of inefficient DMUs in the

Table 3: Non-separable outputs case: Results

	$\alpha_{min} = 0$				$\alpha_{min} = 0.8$			
			Projected				Projected	
	ρ^*	α^*	y^{Good}	y^{Bad}	ρ^*	α^*	y^{Good}	y^{Bad}
A	1	1	-	-	1	1	-	-
B	1	1	-	-	1	1	-	-
C	1	1	-	-	1	1	-	-
D	0.571	0.25	2	1	0.833	0.8	6.4	3.2
E	0.538	0.14	1.29	1	0.833	0.8	7.2	5.6
F	0.75	0.67	3.33	1.33	0.833	0.8	4	1.6
G	0.6	0.33	1.33	1	0.833	0.8	3.2	2.4
H	0.571	0.25	1.5	1	0.833	0.8	4.8	3.2
I	0.545	0.17	0.667	1	0.833	0.8	3.2	4.8

case in which $\alpha_{min} = 0$. DMU D is projected to D_1 (=B). DMUs E, H, G and I are projected onto the efficient portion of the line segment passing through A and B, while DMU F is directed to F_1 on the line segment connecting B and C. They are all radially projected to the frontiers. In the case where $\alpha_{min} = 0.8$, all inefficient DMUs are eventually proportionally reduced in the good and bad outputs by the rate 0.8.

Figure 4: Non-separable outputs

5.3 An example with both separable and non-separable outputs

Table 4 exhibits a data set with both non-separable outputs (y^{Good} and y^{Bad}) and separable outputs (z^{Good} and z^{Bad}) that are produced from two inputs x_1 and x_2. Although the separable outputs can be increased or decreased independently, the non-separable outputs can be changed only proportionally. (In this case we assume a proportional reduction.)

We applied the [SBM-NS] model in (30) to this data set with $\alpha_{min} = 0.7$ and obtained the results as displayed in Table 5. DMU B, C and D are NS-efficient. DMU A has DMU B as its reference and reduces inputs to $x_1 \to 2$ and $x_2 \to 3$ by deleting input surpluses, and enlarges the separable z^{Good} to 6, while non-separable outputs remain unchanged. Remaining inefficient

Table 4: Non-separable and separable outputs case: Data set

	Input		Output			
			Non-separable		Separable	
DMU	x_1	x_2	y^{Good}	y^{Bad}	z^{Good}	z^{Bad}
A	3	4	1	1	2	2
B	2	3	2	1	6	1
C	5	1	6	2	8	1
D	3	2	8	4	10	5
E	5	5	9	7	6	4
F	6	5	5	2	7	5
G	7	3	4	3	2	1
H	3	7	6	4	3	1
I	5	8	4	6	4	2

DMUs E, F, G, H and I reduced non-separable outputs proportionally by the rate α^*, as cited in Table 5. The reduction rate is at the lower bound ($\alpha^* = \alpha_{min} = 0.7$) for DMUs E, G, H and I, while $\alpha^* = 0.75$ for DMU F. Inefficient DMUs projected other inputs and outputs non-radially.

6 Comparisons with other methods

Several authors have proposed efficiency measures in the presence of undesirable outputs. A conventional and traditional way to handle this problem is to shift undesirable outputs to inputs and to apply traditional DEA models to the data set. In the VRS environment, Seiford and Zhu (2002) proposed a method that first multiplies each undesirable output by -1 and then finds a proper translation vector to let all negative undesirable outputs be positive. Interestingly, Scheel (2001) pointed out that these two transformations (position change and translation) give the same efficient frontiers, although the Seiford and Zhu method is only valid under the VRS condition. However, resulting efficiency scores for inefficient DMUs are different according to the model employed. Another conventional way is to invert the undesirable output value and treat it as a desirable one. This operation may cause deformation of the efficient frontiers due to the non-linear transformation and hence gives a different identification of the efficiency status and efficiency score.

Table 5: Non-separable and separable outputs case: Results ($\alpha_{min} = 0.7$)

			Projection					
			Input		Output			
	Score				Non-separable		Separable	
DMU	ρ^*	α^*	x_1	x_2	y^{Good}	y^{Bad}	z^{Good}	z^{Bad}
A	0.35	1	2	3	1	1	6	1
B	1	1	2	3	2	1	6	1
C	1	1	5	1	6	2	8	1
D	1	1	3	2	8	4	10	5
E	0.41	0.7	2.72	2.28	6.3	4.9	8.87	3.87
F	0.37	0.75	3.5	2	3.75	1.5	7	1
G	0.35	0.7	5	1	2.8	2.1	8	1
H	0.62	0.7	3	3.85	4.2	2.8	5.48	1
I	0.29	0.7	2.13	2.87	2.8	4.2	6.53	1.53

Färe *et al.* (1989) was the first paper to treat this subject systematically. They treat desirable and undesirable outputs asymmetrically, resulting in an enhanced hyperbolic output efficiency measure. This approach needs to solve a non-linear programming problem. As for the non-separable models, Scheel (2001) proposed a radial and output-oriented method, whereas Färe *et al.* (2003) developed a directional vector approach in output-orientation. The non-separable outputs models have different efficient frontiers than the separable outputs case. In discussing efficiency, it is important to define the production possibility set and its efficient frontier, and then to identify the method of measuring the efficiency of inefficient DMUs.

We would like to observe this problem from another standpoint. Most (but not all) DEA models can be categorized into four classes: (1) radial and oriented, (2) radial and non-oriented, (3) non-radial and oriented, and (4) non-radial and non-oriented. Here, 'radial' means that a proportional reduction or enlargement of inputs/outputs is the main concern in measuring efficiency, while 'oriented' indicates input-oriented or output-oriented. Consequently, radial models neglect slacks and hence, when dealing with undesirable outputs, slacks in undesirable outputs are not accounted in the efficiency measure. This is a crucial shortcoming of radial models. On the other hand, the major concern of input (output)-oriented models focuses on the input (output)-side efficiency, and output (input)-side is a minor subject in measuring efficiency. Thus, only the non-radial and non-oriented models can capture all aspects of efficiency. From this viewpoint, it should be noted that our [SBM-NS] model can deal with inputs and outputs efficiency in the presence of both non-separable and separable outputs in a unified manner in any RTS environment.

7 Conclusion

In accordance with the relatively recent increase in environmental protection awareness, it is crucial to measure the relative efficiency of enterprises within a framework in which the amounts of both desirable and undesirable outputs are taken into account. In this paper we have proposed a new SBM scheme for this purpose. The new scheme provides the full efficiency score 1 to a DMU if and only if the DMU is efficient, and assigns a score less than 1 to inefficient DMUs depending on the relative magnitudes of slacks in inputs/outputs. The score is strictly monotone decreasing with respect to an increase in any slacks. Furthermore, the proposed method processes the given data set as it is, i.e., neither translation nor inversion is operated, and no position change (from output to input) is required.

This scheme can be extended to cope with co-existence of non-separable desirable and undesirable outputs as well as separable ones. It belongs to a non-radial and non-oriented index and has a reasonable LP dual interpretation as a profit maximization engine.

A major concern of our society is how to better take into considerations both economic and ecological issues. For this, as Korhonen and Luptacik (2003) remind us, *we need new indicators to measure the economic performance of the firm and the national economy, which take into account environmental aspects as well.* We hope the new methodology developed in this paper can contribute to this purpose.

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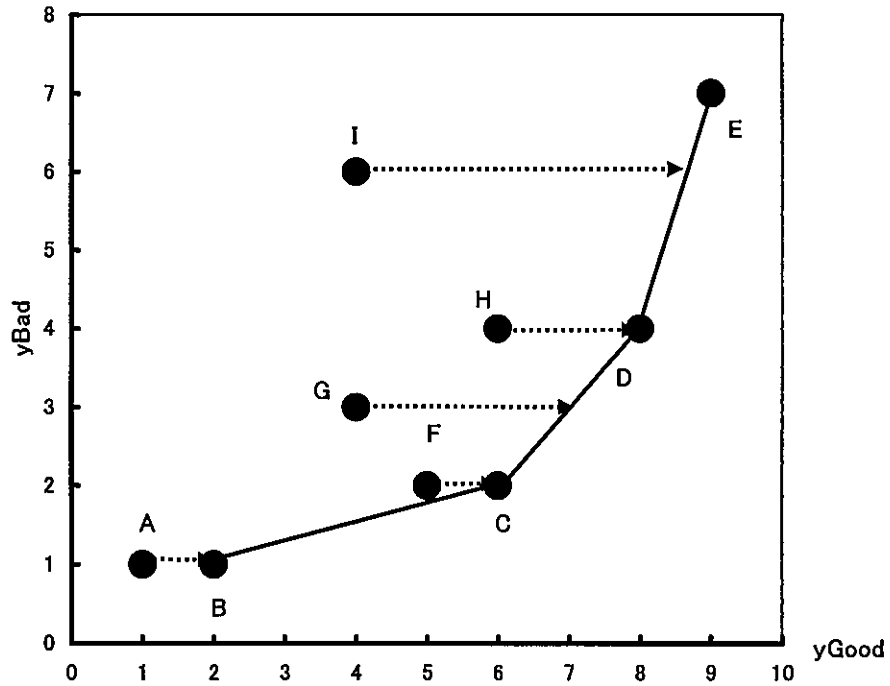

Figure 1: yGood:yBad = 1:0.3

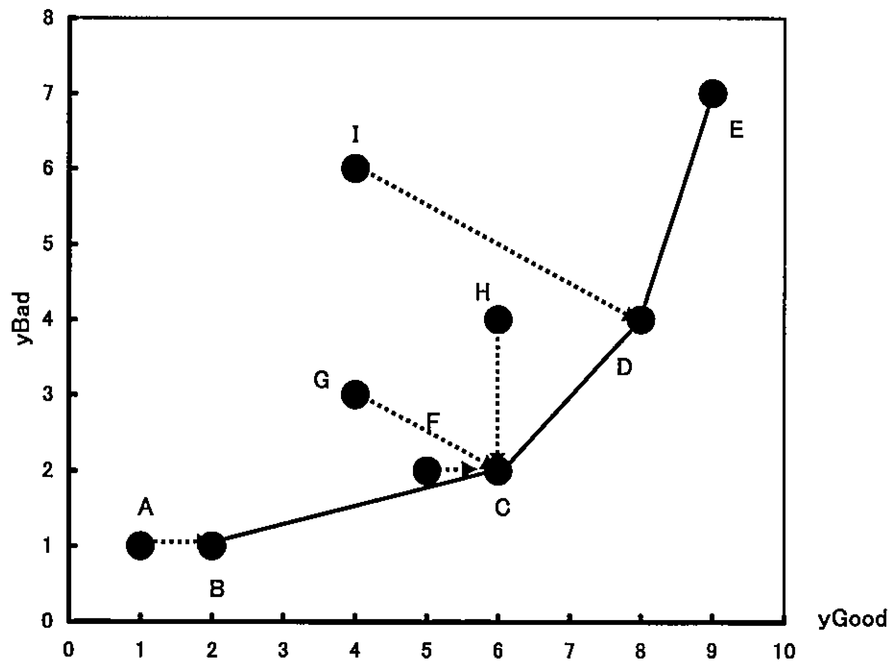

Figure 2: yGood:yBad = 1:1

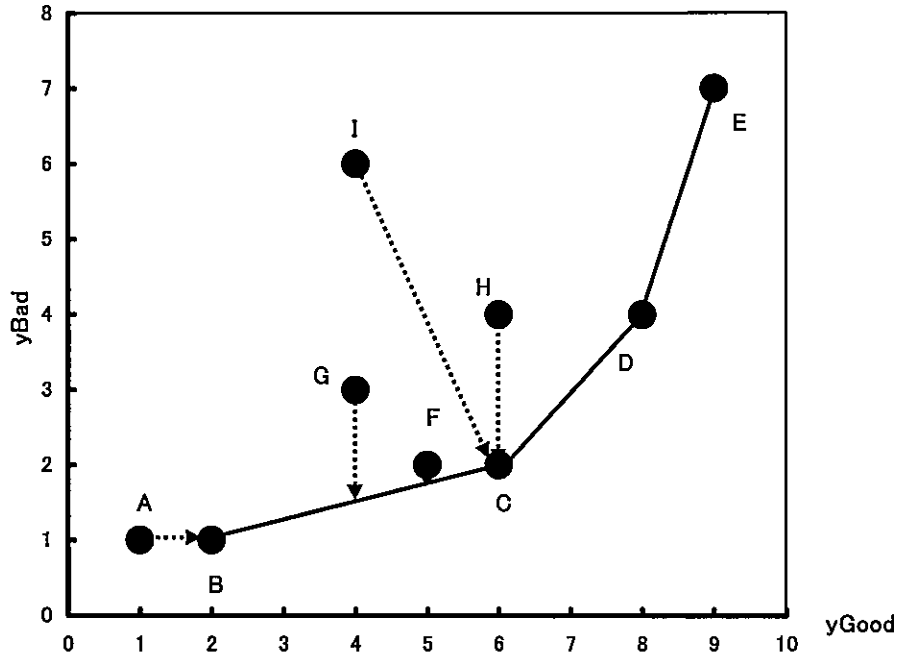

Figure 3: yGood:yBad = 1:3

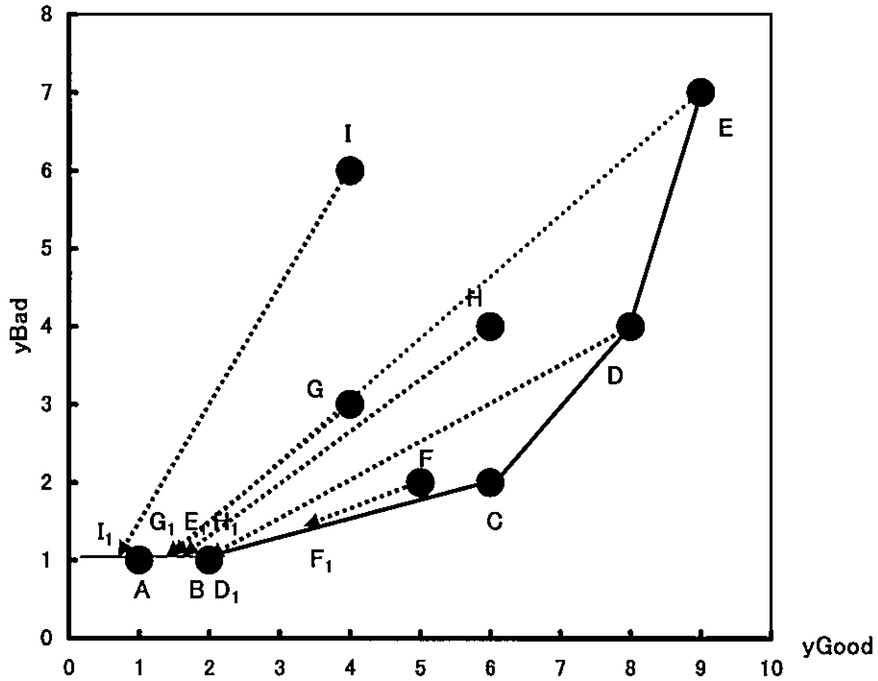

Figure 4: Non-separable outputs