On a risk model with debit interest and dividend payments

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\textbf{A R T I C L E  I N F O}

\textbf{Article history:}
Received 3 June 2005
Received in revised form 21 January 2008
Accepted 14 February 2008
Available online 10 March 2008

\textbf{MSC:}
0167-6687

\textbf{A B S T R A C T}

We consider the compound Poisson risk model with debit interest and dividend payments. The model assumes that the company is allowed to borrow at some debit interest rate when the surplus turns negative, and that the premium incomes are paid out as dividends to shareholders when the surplus reaches a horizontal barrier of level $b$. We first derive integro-differential equations for the expected discounted value of all dividends until absolute ruin, $V_b(u)$, which is twice continuously differentiable. In the case of exponential claim amounts, we obtain explicit expressions for $V_b(u)$ and the optimal barrier $b^*$ which maximizes $V_b(u)$. We then perform a similar study for the Gerber–Shiu expected discounted penalty function. Again, when claims are exponentially distributed, we are able to find explicit expressions for the joint distribution of the surplus just prior to absolute ruin and the deficit at absolute ruin, which is a special case of the Gerber–Shiu function.

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1. Introduction

In the compound Poisson risk model, the surplus process of an insurance company is given by

$$ R(t) = u + ct - S(t) = u + ct - \sum_{k=1}^{N(t)} X_k, \quad t \geq 0, $$

(1.1)

where $u \geq 0$ is the initial surplus, $c > 0$ is the rate of premium, $N(t)$ is the Poisson claim-number process with intensity $\lambda > 0$, and \{\text{Z}_k, k = 1, 2, \ldots\}, independent of the claim-number process, are independent and identically distributed (i.i.d.) claim-size random variables with common distribution function $F$ and probability density function $f$. Note that the compound Poisson risk model is also known as the classical risk model. In this paper, we consider the surplus process (1.1) with two additional features, namely debit interest and dividend payments.

The feature of debit interest assumes that the company is allowed to borrow money at a debit interest rate $\beta > 0$ to pay claims when the surplus turns negative. As the company will pay the debts from its premium income, the negative surplus may return to a positive level. When the premium income is not enough to pay the debit interest (that is, the surplus falls below $-c/\beta$), absolute ruin is said to occur. In recent years, the issue of absolute ruin has received considerable attention in the actuarial literature. Some related results can be found in Dassios and Embrechts (1989), Embrechts and Schmidli (1994), Zhang and Wu (1999), Cai et al. (2006), Cai (2007) and Gerber and Yang (2007).

For the feature of dividend payments, it is assumed that dividends are paid to shareholders according to the so-called barrier strategy. Under the barrier strategy, the premium incomes no longer go into the surplus process but are paid out...
as dividends when the surplus reaches a horizontal barrier of level \( b \), that is, dividends are paid continuously at rate \( c \) and the surplus process remains at level \( b \) until the next claim occurs. This dividend-payment strategy was first proposed by De Finetti (1957) for a Bernoulli model. Since then, research on risk models in the presence of a constant barrier has been carried out extensively. For example, see Bühlmann (1970), Paulsen and Gjessing (1997), Lin et al. (2003), Gerber and Shiu (2004), Zhou et al. (2006), Thonhauser and Albrecher (2007), and Yuen et al. (2007, in press).

Incorporating the above-mentioned two features into \( R(t) \) of (1.1), the resulting surplus process \( R_b(t) \) can be described by

\[
dR_b(t) = \begin{cases} 
-dS(t), & R_b(t) = b, \\
(cdt - dS(t), & 0 \leq R_b(t) < b, \\
(\beta R_b(t) + c)dt - dS(t), & R_b(t) < 0,
\end{cases}
\]

where \( b \) is the constant level of dividend barrier, \( \beta \) is the debit interest rate, \( c \) is the premium rate, and \( S(t) \) is the aggregate claim-amount process given in (1.1). With the initial surplus \( R_b(0) = u \), (1.2) determines the capped surplus process \( R_b(t) \). For simplicity, \( R_b(t) \) is also referred to as the surplus process (1.2).

Define \( T^b_u = \inf\{t : R_b(t) < -c/\beta\} \) as the time of absolute ruin. Let \( D(t) \) be the total dividends paid to time \( t \), and \( D^b_u \) be the present value of all dividends payable to shareholders till time \( T^b_u \) calculated at constant force of interest \( \delta > 0 \). Then,

\[
D^b_u = \int_0^{T^b_u} e^{-\delta t} dD(t).
\]

Apparently, the present value \( c/\delta \) of a continuous perpetuity with payment rate \( c \) is an upper bound for \( D^b_u \). Denote the expected discounted value of all dividends until absolute ruin by \( V_b(u) = E[D^b_u] \). Here, the optimal dividend barrier strategy is defined as the barrier level \( b^* \) which maximizes \( V_b(u) \).

The rest of this paper proceeds as follows. In Section 2, integro-differential equations for \( V_b(u) \) are derived. For an exponential claim-amount distribution, explicit expressions for \( V_b(u) \) and the optimal dividend barrier \( b^* \) are given in Section 3. Finally, the Gerber–Shiu expected discounted penalty function under the capped surplus process (1.2) is discussed in Section 4.

2. Integro-differential equations

In this section, we derive integro-differential equations for \( V_b(u) \), and show that \( V_b(u) \) is twice differentiable. The results are summarized in the following theorem.

**Theorem 2.1.** The expected discounted value of all dividends until absolute ruin \( V_b(u) \) is twice differentiable on \((-c/\beta, b)\), and satisfies the following integro-differential equations:

\[
(\beta u + c)V_b(u) = (\lambda + \delta)V_b(u) - \lambda \int_0^{u+\frac{c}{\beta}} V_b(u-x)f(x)dx, \quad -c/\beta < u < 0,
\]

\[
cV_b(u) = (\lambda + \delta)V_b(u) - \lambda \int_0^{u+\frac{c}{\beta}} V_b(u-x)f(x)dx, \quad 0 < u < b,
\]

with the following conditions:

- (A1) \( V_b(u) \big|_{u=b} = 1 \) (boundary condition);
- (A2) \( V_b(-\frac{c}{\beta}) = 0 \) (boundary condition);
- (A3) \( V_b(0-) = V_b(0+) \) (continuity condition);
- (A4) \( V_b(0-) = V_b(0+) \) (smoothness condition).

**Proof.** Consider \( R_b(t) \) in a small time interval \((0, dt] \) and conditioning on the time and the amount of the first claim, we have

\[
V_b(u) = (1 - \lambda dt)e^{-\delta dt}V_b \left( \frac{c}{\beta} + u \right) e^{\delta dt} - \frac{c}{\beta}
+ \lambda dt e^{-\delta dt} \int_0^{\frac{c}{\beta} + u} e^{\delta x} \left( \frac{c}{\beta} + u \right) e^{\delta dt} - \frac{c}{\beta} f(x)dx + o(dt),
\]

for \(-c/\beta < u < 0\), and

\[
V_b(u) = (1 - \lambda dt)e^{-\delta dt}V_b(u + cdt) + \lambda dt e^{-\delta dt} \int_0^{u + cdt} \left( V_b(u + cdt - x) f(x)dx + o(dt),
\]

for \(0 < u < b\). Let \( u_{dt} = (u + (c/\beta))e^{-\delta dt} - c/\beta \). It is clear that \( u_{dt} \uparrow u \) as \( dt \downarrow 0 \). Replacing \( u \) by \( u_{dt} \) in (2.3) and by \( u - cdt \) in (2.4) yields

\[
V_b(u_{dt}) = (1 - \lambda dt)e^{-\delta dt}V_b(u) + \lambda dt e^{-\delta dt} \int_0^{u + c/\beta} V_b(u - x)f(x)dx + o(dt),
\]
for $-c/\beta < u \leq 0$, and

$$V_b(u-cd) = (1 - \lambda dr)e^{-\beta dr}V_b(u) + \lambda dr e^{-\beta dr} \int_0^{b+u/c} V_b(u-x)f(x)dx + o(dr).$$

for $0 < u < b$. Hence, the right continuity follows from (2.3) and (2.4), and the left continuity follows from (2.5) and (2.6). Furthermore, dividing $dr$ on both sides of (2.3)–(2.6) and letting $dr \to 0$, it can be shown that $V_b(u)$ is differentiable on $(-c/\beta, b)$, and that (2.1) and (2.2) hold. The twice differentiability of $V_b(u)$ is simply due to (2.1) and (2.2).

To complete the proof, we need to show that the boundary conditions (A1) and (A2) hold. Conditioning on the first claim in $(0, dr)$, we have

$$V_b(b) = (1 - \lambda dr)(cdr + e^{-\beta dr}V_b(b)) + \lambda dr e^{-\beta dr} \int_0^{b+u/c} V_b(b-x)f(x)dx + o(dr).$$

Dividing $dr$ and letting $dr \to 0$, we get

$$(\lambda + \delta)V_b(b) - \lambda \int_0^{b+u/c} V_b(u-x)f(x)dx = c.$$ Comparing it to (2.2) with $u \to b$, we obtain $V_b(u)|_{u=b} = 1$. Finally, it follows from (2.1) that $V_b(-c/\beta) = 0$ as $u \to -c/\beta$. □

As (2.1) and (2.2) are homogeneous integro-differential equations of order 1, $V_b(u) = \gamma_1 h_1(u), -c/\beta < u < 0$, and $V_b(u) = \gamma_2 h_2(u), 0 \leq u < b$, where $\gamma_1$ and $\gamma_2$ are arbitrary constants, and $h_1(u)$ and $h_2(u)$ are unique solutions to the integro-differential equations

$$(\beta u + c)h_1(u) = (\lambda + \delta)h_1(u) - \lambda \int_0^{u+\lambda/c} h_1(u-x)f(x)dx, \quad u > -\frac{c}{\beta},$$

$$\beta h_2(u) = (\lambda + \delta)h_2(u) - \lambda \int_0^{u+\lambda/c} h_2(u-x)f(x)dx, \quad u > -\frac{c}{\beta},$$ respectively. Let $x = u + c/\beta$ and $l_i(x) = h_i(x - c/\beta), i = 1, 2$. Then, one needs to solve

$$\beta i_1(x) = (\lambda + \delta) i_1(x) - \lambda \int_0^{x} i_1(x-y)f(y)dy, \quad x > 0,$$

$$\beta i_2(x) = (\lambda + \delta) i_2(x) - \lambda \int_0^{x} i_2(x-y)f(y)dy, \quad x > 0.$$ Following the method of Lin et al. (2003) (see also Bühlmann (1970, Section 6.4.9)), one can show that the general solution to (2.8) is proportional to the product of an exponential function and the distribution function of a compound geometric distribution. However, it is very difficult to find the general solution to (2.7).

3. Analysis of $V_b(u)$ with exponential claims

In this section, we assume that the claim amounts are exponentially distributed with $f(x) = e^{-\alpha x}, x > 0$. We first derive explicit expressions for $V_b(u)$, and then study the problem of optimal dividends.

Since

$$\frac{d}{dx} \left( \int_0^{x} l_i(x-y)ae^{-\alpha y}dy \right) = \alpha \left( l_i(x) - \int_0^{x} l_i(x-y)ae^{-\alpha y}dy \right),$$

differentiating both sides of (2.7) and (2.8) yields

$$\beta i_1'(x) = (\lambda + \delta - \beta - \beta \alpha) i_1'(x) - \delta \alpha i_1(x) = 0, \quad x > 0,$$

$$\beta i_2'(x) = (\lambda + \delta - \beta - \beta \alpha) i_2'(x) - \delta \alpha i_2(x) = 0, \quad x > 0.$$ Let $z = -\alpha x$ and $g(z) = i_1(-z/\alpha)$. Then, (3.1) can be rewritten as

$$g''(z) + \left(1 - \frac{\lambda + \delta}{\beta} - z \right) g'(z) - \left( \frac{\delta}{\beta} \right) g(z) = 0.$$ By Salter (1960), the solution of this equation has the form

$$g(z) = C_1 G\left( -\frac{\delta}{\beta}, 1 - \frac{\lambda + \delta}{\beta}; z \right) + C_2 (-\alpha x)^{\frac{\lambda + \delta}{\alpha \beta}} G\left( \frac{\lambda}{\beta}, 1 + \frac{\lambda + \delta}{\beta}; z \right),$$

where

$$G(a, b; x) = \frac{\tau(b)}{\tau(b-a)\tau(a)} \int_0^1 e^{\beta x}e^{(1-t)(b-a)}dt, \quad b > a > 0,$$
is the standard confluent hypergeometric function and its infinite-series form is
\[ G(a, b; x) = \sum_{n=0}^{\infty} \frac{a(a+1) \cdots (a+n-1)}{n!b(b+1) \cdots (b+n-1)} x^n. \]

Note that \( V_b(u) = l_1(u + c/b) = g(-\alpha(u + c/b)) \) for \(-c/b < u < 0\). Due to the boundary condition (A2), we see that \( c_1 = 0 \). Thus,
\[ V_b(u) = c_2 \left( -\alpha \left( u + \frac{c}{b} \right)^{\frac{2+\delta}{\beta}} G \left( \frac{\lambda}{\beta}, 1 + \frac{\lambda + \delta}{\beta}; -\alpha \left( u + \frac{c}{b} \right) \right) \right), \quad -\frac{c}{b} < u < 0. \]

On the other hand, it follows from (3.2) that \( l_2(x) \) takes the form
\[ l_2(x) = c_3 e^{x^2} + c_4 e^{2x}, \quad x > 0, \]
where \( r_1 > 0 \) and \( r_2 < 0 \) are the two roots of the equation
\[ cr^2 - (\lambda + \delta - c\alpha)r - \delta\alpha = 0. \]

Therefore,
\[ V_b(u) = c_3 e^{r_1(u + \frac{c}{b})} + c_4 e^{r_2(u + \frac{c}{b})}, \quad 0 \leq u \leq b. \]

Using (A1)–(A4), one can determine the constants \( c_i, i = 2, 3, 4 \). As a result, \( V_b(u) \) has the form
\[ V_b(u) = \begin{cases} \dfrac{(r_1 - r_2) \left( u + \frac{c}{b} \right)^{\frac{2+\delta}{\beta}} G \left( \frac{\lambda}{\beta}, 1 + \frac{\lambda + \delta}{\beta}; -\alpha \left( u + \frac{c}{b} \right) \right)}{h(b)}, & -\frac{c}{b} < u < 0, \\ \dfrac{(r_1 + \alpha)G_1 - \frac{\lambda\alpha}{\beta + \lambda + \delta}G_2}{h(b)} e^{r_1u} - \dfrac{(r_2 + \alpha)G_1 - \frac{\lambda\alpha}{\beta + \lambda + \delta}G_2}{h(b)} e^{r_2u}, & 0 \leq u \leq b, \end{cases} \]

where
\[ h(b) = r_1 \left( (r_1 + \alpha)G_1 - \frac{\lambda\alpha}{\beta + \lambda + \delta}G_2 \right) e^{r_1b} - r_2 \left( (r_2 + \alpha)G_1 - \frac{\lambda\alpha}{\beta + \lambda + \delta}G_2 \right) e^{r_2b}, \]
\[ G_1 = G \left( \frac{\lambda}{\beta}, 1 + \frac{\lambda + \delta}{\beta}; -\alpha \frac{c}{b} \right), \]
\[ G_2 = G \left( 1 + \frac{\lambda}{\beta}, 2 + \frac{\lambda + \delta}{\beta}; -\alpha \frac{c}{b} \right). \]

We now turn to the optimal dividend barrier \( b^* \). For a given \( u > -\frac{c}{\beta} \), we want to find \( b^* \) that maximizes \( V_b(u) \), that is, \( V_{b^*}(u) = \sup_b V_b(u) \). It follows from (3.4) that
\[ b^* = \frac{1}{r_1 - r_2} \ln \left( \frac{r_1^2 \left( (r_2 + \alpha)G_1 - \frac{\lambda\alpha}{\beta + \lambda + \delta}G_2 \right)}{r_2^2 \left( (r_1 + \alpha)G_1 - \frac{\lambda\alpha}{\beta + \lambda + \delta}G_2 \right)} \right), \]
which is independent of \( u \). Substituting \( b^* \) of (3.5) into the second equality of (3.4), we obtain
\[ V_{b^*}(b^*) = \frac{r_1 + r_2}{r_1 r_2} = \frac{c}{\delta} + \frac{\lambda + \delta}{\delta\alpha}. \]

Let \( \hat{V}_b(u) \) be the expected discounted value of all dividends until ruin for the compound Poisson risk model with constant dividend barrier \( \hat{b} \) (but without debit interest), and the corresponding optimal barrier be \( \hat{b}^* \). Then, \( \hat{V}_{b^*}(u) = \sup_b \hat{V}_b(u) \). In the literature, it is well known that
\[ \hat{b}^* = \frac{1}{r_1 - r_2} \ln \left( \frac{r_1^2 (r_2 + \alpha)}{r_2^2 (r_1 + \alpha)} \right). \]

Furthermore, it is easy to check that \( \hat{V}_{b^*}(\hat{b}^*) = V_{b^*}(b^*) \). For optimal dividend results for the compound Poisson risk model with a constant dividend barrier (but without debit interest), see Gerber et al. (2008), and references therein. We end the section by comparing the two optimal dividend barriers, \( b^* \) and \( \hat{b}^* \). Note that \( G_1 > 0, G_2 > 0, \) and
\[ (r_1 + \alpha)G_1 - \frac{\lambda\alpha}{\beta + \lambda + \delta}G_2 \geq (r_1 + \alpha)G_1 - \alpha G_1 \geq 0. \]

Since \( r_1 + \alpha > r_2 + \alpha, \) we have
\[ \frac{(r_2 + \alpha)G_1 - \frac{\lambda\alpha}{\beta + \lambda + \delta}G_2}{(r_1 + \alpha)G_1} < \frac{(r_2 + \alpha)G_1}{(r_1 + \alpha)G_1} < r_2 + \alpha \]
\[ \frac{(r_1 + \alpha)G_1 - \frac{\lambda\alpha}{\beta + \lambda + \delta}G_2}{(r_1 + \alpha)G_1} < r_1 + \alpha, \]
which implies that \( b^* < \hat{b}^* \).
4. Gerber–Shiu expected discounted penalty function

In this section, we are interested in studying the Gerber and Shiu (1998) expected discounted penalty function for the surplus process (1.2) given by

$$W_\eta(u) = E[e^{-\eta T_b^\rho U} w(R_b(T_b^\rho U), |R_b(T_b^\rho U)|) I(T_b^\rho < \infty) | R_b(0) = u].$$  \hspace{1cm} (4.1)$$

where $T_b^\rho$ is the time of absolute ruin defined in Section 2, the penalty function $w$ is a nonnegative measurable function, the parameter $\eta > 0$ may be interpreted as the force of interest, and $I(\lambda)$ is the indicator function of event $A$. As the surplus prior to absolute ruin $R_b(T_b^\rho U)$ could be negative and the deficit at absolute ruin $|R_b(T_b^\rho U)|$ is at least $c/\beta$, the domain of the penalty function $w(x_1, x_2)$ is defined by $x_1 \geq -c/\beta$ and $x_2 > c/\beta$.

It should be pointed out that the Gerber–Shiu expected discounted penalty function embraces many important actuarial functions including the probability of absolute ruin ($\eta = 0$ and $w(R_b(T_b^\rho U), |R_b(T_b^\rho U)|) \equiv 1$), the Laplace transform of $T_b^\rho$ ($w(R_b(T_b^\rho U), |R_b(T_b^\rho U)|) \equiv 1$), the distribution of $R_b(T_b^\rho U)$ ($\eta = 0$ and $w(R_b(T_b^\rho U), |R_b(T_b^\rho U)|) = I(R_b(T_b^\rho U) \leq x)$), the distribution of $|R_b(T_b^\rho U)|$ ($\eta = 0$ and $w(R_b(T_b^\rho U), |R_b(T_b^\rho U)|) = I(|R_b(T_b^\rho U)| \leq y)$), and the joint distribution of $R_b(T_b^\rho U)$ and $|R_b(T_b^\rho U)|$ ($\eta = 0$ and $w(R_b(T_b^\rho U), |R_b(T_b^\rho U)|) = I(R_b(T_b^\rho U) \leq x, |R_b(T_b^\rho U)| \leq y)$). Note that the ruin probability of absolute ruin for (1.2) equals one, that is, $Pr[T_b^\rho < \infty] = 1$ for $b < \infty$.

Using arguments similar to those in the proof of Theorem 2.1, we have

**Theorem 4.1.** The Gerber–Shiu expected discounted penalty function $W_\eta(u)$ is continuous on $(-c/\beta, b)$ and satisfies the following integro-differential equations:

$$(\beta u + c)W_\eta'(u) = (\lambda + \eta)W_\eta(u) - \lambda \int_0^{u+\frac{c}{\beta}} W_\eta(u-x) dF(x) - \lambda \int_{u+\frac{c}{\beta}}^{\infty} w(u, x-u) dF(x),$$

for $-c/\beta < u < 0$, and

$$cW_\eta'(u) = (\lambda + \eta)W_\eta(u) - \lambda \int_0^{u+\frac{c}{\beta}} W_\eta(u-x) dF(x) - \lambda \int_{u+\frac{c}{\beta}}^{\infty} w(u, x-u) dF(x),$$

for $0 \leq u < b$, with the following boundary conditions:

$$W_\eta(x)|_{x=b} = 0,$$

$$W_\eta\left(-\frac{c}{\beta}\right) = \frac{\lambda}{\lambda + \eta} \int_0^{\infty} w\left(-\frac{c}{\beta}, x + \frac{c}{\beta}\right) dF(x).$$

As an example, we only consider a special case of (4.1) with $\eta = 0$ and $w(R_b(T_b^\rho U), |R_b(T_b^\rho U)|) = I(R_b(T_b^\rho U) > x, |R_b(T_b^\rho U)| > y + c/\beta)$, that is, the joint survival distribution of the surplus just prior to absolute ruin and the deficit at absolute ruin. Denote

$$P_{x,y}(u) = Pr[R_b(T_b^\rho U) > x, |R_b(T_b^\rho U)| > y + c/\beta].$$

**Theorem 4.2.** Suppose that the claim-amount distribution is exponential with $F(x) = 1 - e^{-ax}$, $x > 0$. Let $\bar{F}(x) = 1 - F(x)$ and $\rho = c/\lambda - 1$. Then, the joint survival function of the surplus just prior to absolute ruin and the deficit at absolute ruin has the following form:

1. for $x \geq b$,

$$P_{x,y}(u) = 0;$$

2. for $0 \leq x < b$,

$$P_{x,y}(u) = \begin{cases} B_1(x, y) - \frac{1}{\rho} \bar{F}\left(x + y + \frac{c}{\beta}\right), & x < u \leq b, \\ B_1(x, y) - \frac{1}{\rho} \bar{F}\left(x + y + \frac{c}{\beta}\right) e^{\frac{1}{\rho} F(x-y)}, & 0 \leq u \leq x, \\ B_2(x, y) \int_{\frac{c}{\beta}}^{\infty} (\beta t + c)^{\frac{1}{\beta}} e^{-\alpha t} dt, & -\frac{c}{\beta} < u < 0, \end{cases}$$

where

$$B_2(x, y) = \frac{\lambda}{\rho} \bar{F}\left(x + y + \frac{c}{\beta}\right) c^{\frac{1}{\beta}} e^{\frac{1}{\beta} y};$$

$$B_1(x, y) = B_2(x, y) \int_{\frac{c}{\beta}}^{\infty} (\beta t + c)^{\frac{1}{\beta}} e^{-\alpha t} dt + \frac{1}{\rho} \bar{F}\left(x + y + \frac{c}{\beta}\right) e^{\frac{1}{\beta} y}.$$
(3) for \(-c/\beta < x < 0\),
\[
P_{x,y}(u) = \begin{cases} 
\frac{\lambda F}{\beta} \left( x + y + \frac{c}{\beta} \right) (\beta x + c)^{-\frac{1}{\beta}} e^{\frac{1}{\beta} x} \int_{-\frac{1}{\beta}}^{u} (\beta t + c)^{-\frac{1}{\beta}-1} e^{-\beta t} dt, & x \leq u \leq b, \\
\frac{\lambda F}{\beta} \left( x + y + \frac{c}{\beta} \right) (\beta x + c)^{-\frac{1}{\beta}} e^{\frac{1}{\beta} x} \int_{-\frac{1}{\beta}}^{u} (\beta t + c)^{-\frac{1}{\beta}-1} e^{-\beta t} dt, & -\frac{c}{\beta} < u < x;
\end{cases}
\]

(4) for \(x \leq -c/\beta\),
\[
P_{x,y}(u) = \bar{F}(y).
\]

**Proof.** It is obvious that (1) of Theorem 4.2 holds. For \(0 \leq x < b\), \(P_{x,y}(u)\) satisfies the integro-differential equations
\[
(\beta u + c)P_{x,y}^{'}(u) = \lambda P_{x,y}(u) - \lambda \int_{0}^{u+\frac{c}{\beta}} P_{x,y}(u-s) dF(s), -\frac{c}{\beta} < u < 0,
\]
\[
cP_{x,y}^{'}(u) = \lambda P_{x,y}(u) - \lambda \int_{0}^{u+\frac{c}{\beta}} P_{x,y}(u-s) dF(s), 0 \leq u \leq x,
\]
\[
cP_{x,y}^{'}(u) = \lambda P_{x,y}(u) - \lambda \int_{0}^{u+\frac{c}{\beta}} P_{x,y}(u-s) dF(s) - \lambda \frac{\lambda}{\beta} \left( u + y + \frac{c}{\beta} \right), x < u \leq b.
\]
Differentiating both sides of system (4.2) gives
\[
(\beta u + c)P_{x,y}^{''}(u) - (\lambda - \beta - (\beta u + c)\alpha)P_{x,y}^{'}(u) = 0, -\frac{c}{\beta} < u < 0,
\]
\[
cP_{x,y}^{''}(u) - (\lambda - \alpha \alpha)P_{x,y}^{'}(u) = 0, 0 \leq u \leq x,
\]
\[
cP_{x,y}^{''}(u) - (\lambda - \alpha \alpha)P_{x,y}^{'}(u) = 0, x < u \leq b.
\]
So, \(P_{x,y}(u)\) can be expressed as
\[
P_{x,y}(u) = \begin{cases} 
A_{1} \int_{-\frac{1}{\beta}}^{u} (\beta t + c)^{-\frac{1}{\beta}-1} e^{-\beta t} dt + A_{2}, & -\frac{c}{\beta} < u < 0, \\
A_{3} e^{-\alpha (u-\frac{1}{\beta})} + A_{4}, & 0 \leq u \leq x, \\
A_{5} e^{-\alpha (u-\frac{1}{\beta})} + A_{6}, & x < u \leq b.
\end{cases}
\]

From Theorem 4.1 and (4.2), we have the following boundary conditions:
\[
P_{x,y}^{'}(b) = 0, P_{x,y} \left( -\frac{c}{\beta} \right) = 0.
\]
\[
P_{x,y}(x+) = P_{x,y}(x-), P_{x,y}^{'}(x+) + \frac{\lambda}{\beta} \left( x + y + \frac{c}{\beta} \right) = P_{x,y}^{'}(x-),
\]
\[
P_{x,y}(0+) = P_{x,y}(0-), P_{x,y}^{'}(0+) = P_{x,y}^{'}(0-).
\]

With the boundary conditions (4.3), we can determine the six constants \(A_{i}, i = 1, \ldots, 6\), and hence obtain (2) of Theorem 4.2. Along the same lines, one can show that results (3) and (4) hold. \(\square\)

As a consequence of Theorem 4.2, the probability that the surplus prior to absolute ruin equals \(b\) is given in the following corollary.

**Corollary 4.3.** The probability of the surplus just prior to absolute ruin equal to \(b\) has the form
\[
P(R_{b}(T_{u}^{b} u)) = b) = \begin{cases} 
\bar{F} \left( b + \frac{c}{\beta} \right) e^{\frac{1}{\beta} b} \left( \lambda \frac{c}{\beta} \int_{-\frac{1}{\beta}}^{0} (\beta t + c)^{-\frac{1}{\beta}-1} e^{-\beta t} dt + \frac{1}{\rho} \left( 1 - e^{-\frac{1}{\beta} b} \right) \right), & 0 < u \leq b, \\
\lambda \bar{F} \left( b + \frac{c}{\beta} \right) e^{\frac{1}{\beta} b} c^{-\frac{1}{\beta}} \int_{-\frac{1}{\beta}}^{u} (\beta t + c)^{-\frac{1}{beta}-1} e^{-\beta t} dt, & -\frac{c}{\beta} < u \leq 0.
\end{cases}
\]

**Acknowledgements**

This research was supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. HKU 7475/05H) and a grant of the Natural Science Foundation of China (10701082).
References