Characterizing Tipping in a Stochastic Reduced Stommel-Type Model in Higher-Dimensions

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Abstract During the workshop on Climate Modeling in Nonsmooth Systems, one of the major discussions involved investigating including more realistic elements, such as fluctuations and time variation, in nonsmooth models that undergo a sudden transition, with an emphasis on conceptual climate models. A number of models were discussed, including the Stommel 1961 model, the Paillard 1997 model, the Eisenman–Wettlaufer 2009 model, and the Hogg 2008 model.

1 Introduction

There has been significant recent interest in classifying the various ways in which a dynamical system may undergo a critical transition, where there is a sudden large change in the state of the system as a parameter is varied; see Kuehn [7]. Conceptual climate models may provide pertinent examples of systems which may undergo a critical transition: there is currently broad scientific and public interest in whether sudden transitions may occur in certain climate systems, including the "thermohaline" circulation in the Atlantic; see [1, 8].

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2 The Stommel 1961 Model

During the workshop one major discussion involved investigating how fluctuations influence tipping in nonsmooth models. For example, in the case of periodic fluctuations Zhu, Kuske, and Erneux studied how the frequency of additive periodic forcing affects the timing of tipping from a smooth saddle-node bifurcation in a canonical system with a slowly drifting bifurcation parameter; see [15].

Given the range of conceptual models that include non-smooth dynamics in climate dynamics [5, 6, 10, 14], one could also ask how different type of fluctuations and time dependence will influence their behaviour. Here, we give a description of the Stommel model, as an illustration of the appearance of nonsmooth dynamics in climate models; see details in [14]. The Stommel model is a conceptual model of the ocean's thermohaline circulation, which is a part of the global ocean circulation that drives global ocean currents via density gradients determined by salt and heat fluxes; see Rahmstorf [12] for a brief description of the thermohaline circulation, and Dijkstra [3] for an expository analysis of the dynamics of the Stommel model.

In Stommel's model the Northern Hemisphere is represented by two well-mixed ocean boxes connected on the surface by an overflow and at the bottom by a capillary tube [14]; see Fig. 1. The temperature and salinity in the polar region box are given by T_p and S_p, respectively. Likewise, the temperature and salinity in the equatorial box are given by T_e and S_e. The equations for the model can be expressed in dimensionless form, after rescaling Dijkstra [3], as

$$\frac{dT}{dt} = \eta_1 - T(1 + |T - S|),$$
$$\frac{dS}{dt} = \eta_2 - S(\eta_3 + |T - S|),$$

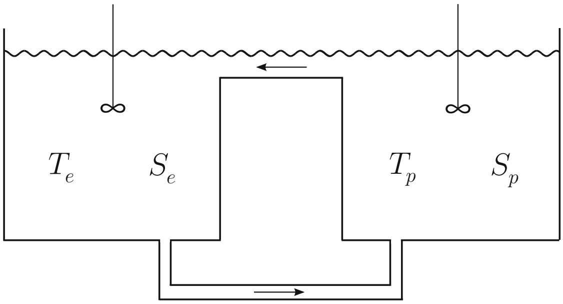

Fig. 1 Diagram of the two-box Stommel model, based on [14, Fig. 5]: (T_e, S_e) represent equatorial temperature and salinity, respectively, and (T_p, S_p) represent polar temperature and salinity. The boxes are connected on top by an overflow and on the bottom by a capillary tube. The flow rate between the two boxes is assumed to be equal and of opposite signs through the overflow and capillary tube

where $T \equiv T_e - T_p$ represents the temperature difference between the equatorial and polar boxes, $S \equiv S_e - S_p$ represents the corresponding salinity difference, and (η_1, η_2, η_3) are constants.

The Stommel model is of particular interest in the context of conceptual climate models due to the fact that the hysteresis observed in the model has consistently been observed in models with increased complexity as well, in certain parameter ranges in several intermediate-complexity models [13] and in some global circulation models [4, 9]. This consistency of results throughout the hierarchy of model complexity has lead to significant scientific consensus on the possibly that hysteresis is physically possible in the system [11].

Mathematically, the Stommel model is a nonsmooth system, with a switch wherever $T = S$. To simplify the location of the switch, one may define $V \equiv T - S$ so that the system becomes

$$\frac{dT}{dt} = \eta_1 - T(1 + |V|), \tag{1}$$

$$\frac{dV}{dt} = \eta_1 - \eta_2 + \eta_3(T - V) - T - V\,|V|\,. \tag{2}$$

There are two fold bifurcations in the (η_2, V) plane, one smooth and one nonsmooth; we are ultimately interested in determining which terms of the model are essential to this behavior and how one might embed similar bifurcation behavior in a higher-dimensional system. For the physically interesting values of the parameters, the Stommel model has either one or three fixed points. As the parameter η_2 varies, two of these experience a boundary equilibrium bifurcation (BEB) when $V = 0$ leading to a non-smooth fold bifurcation where both are annihilated/created [2]. The nature of this transition can be studied through normal form analysis, presented in [2, Chap. 5]. One of these fixed points coalesces with the third at a standard smooth saddle node bifurcation. For other values of the parameters η_1 and η_3, the BEB leads instead to a persistence of the fixed point, again in line with the normal form analysis in [2]. We are ultimately interested in determining which terms of the model are essential to this behavior and how one might embed similar bifurcation behavior in a higher-dimensional system. It is also of interest as to whether some of the more subtle dynamics associated with the nonsmooth bifurcation is realistic in a climate model. We can also explore whether similar behaviour is observed in more regular systems, for example, if the $|V|$ term is replaced by $\sqrt{\epsilon^2 + V^2}$, $\epsilon > 0$.

3 Workshop Discussion and Preliminary Steps

The discussion and ongoing study of the workshop participants centers on the idea of generating a reduced form of the Stommel model and determining possible behaviors of models with similar characteristics in higher dimensions. The goal of identifying an appropriate reduced model is to provide a well-understood basis upon which more realistic fluctuations and time dependence can be built. In this section we outline the preliminary steps we have begun to take toward embedding the dynamics of the Stommel model in higher dimensions. Certainly the nonsmooth fold, and resulting hysteresis, observed in the Stommel model is generic to BEBs in many higher dimensional nonsmooth systems [2]. However, it is not clear at present whether the hysteretic behaviour identified by Dijkstra in experimental runs of global ocean circulation models, is due to a BEB as modelled in the Stommel system, or to the more usual mechanisms seen in smooth systems, associated with the existence of multiple saddle node bifurcations.

Our first step was to determine the reduced form of the Stommel model in one dimension,

$$\frac{dx}{dt} = -(\mu + 1) + 2\,|x| - H(x)x^2,$$

where $H(x)$ is the Heaviside function. This equation retains the Stommel model skeleton and has no attractors other than the stationary points. In two dimensions, the bifurcations in the original bifurcation diagram (one smooth fold and one nonsmooth fold) are preserved if the terms of the system (1), (2) are reduced to

$$\frac{dT}{dt} = \eta_1 - T(1 + |V|),$$
$$\frac{dV}{dt} = \eta_1 - \eta_2 - T,$$

which can be shown using the trace and determinant of the Jacobian.

At the workshop we also discussed possible behaviors of maps with similar forms. Going forward, we have begun working on classifying behavior which may occur in higher dimensional versions of the reduced Stommel model without changing the projected dynamics in the (η_2, V) plane. Future planned steps include investigating the influence of fluctuations and additional time variation.

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