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Laplace Transforms for the Nabla-Difference operator

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Abstract

A definition for the Laplace transform corresponding to the nabla difference operator is given. Several properties of this Laplace transform are established. Further, a definition for the discrete nabla Mittag-Leffler function is provided. Our results are then shown to be robust enough to lead to a practical method for solving initial value problems for discrete fractional nabla difference equations of order ν, $0 < \nu < 1$.

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1 Introduction

The continuous fractional calculus has been well developed (see the books by Miller and Ross [24], Oldham and Spanier [27], and Podlubny [28]). Recently there has been a great deal of interest in the discrete fractional calculus (see the papers by Atici and Eloe [1, 2, 3, 4, 5], Goodrich [9, 10, 11, 12, 13, 14, 15, 16, 17], Miller and Ross [25], and M. Holm [19, 20, 21]). The discrete delta fractional calculus has been studied in Atici and Eloe [1, 2, 3, 4], Goodrich [9, 10, 12, 13, 15, 16], Miller and Ross [25], and M. Holm [19, 20, 21]. The disadvantage of the discrete delta fractional calculus is the shifting of domains when one goes from the domain of the function to the domain of its delta fractional difference. This problem is not as great with the fractional nabla difference as noted by Atici and Eloe in [5]. In this paper we continue the study of the discrete fractional nabla calculus. We then define the corresponding nabla Laplace transform motivated by the very general definition of the delta Laplace transform that was first defined in a very general way by Bohner and Peterson [7]. Several properties of this nabla Laplace transform are then derived. Fractional nabla Taylor monomials are defined and formulas for their nabla Laplace transforms are presented. Then the discrete nabla version of the Mittag–Leffler function and its nabla Laplace transform is obtained. Finally, a variation of constants formula for an initial value problem for a ν^{th}, $0 < \nu < 1$, order nabla fractional difference equation is given along with some applications.

2 Preliminary Definitions

We first give some notation and state elementary results concerning the nabla calculus (for the corresponding results for the delta calculus see, for example, Kelley and Peterson [22]), which we will use in this paper. Let $a \in \mathbb{R}$, then we define $\mathbb{N}_a = \{a, a+1, a+2, \cdots\}$. For an arbitrary function $f : \mathbb{N}_a \to \mathbb{R}$ we define the nabla operator (backwards difference operator), ∇, by $(\nabla f)(t) := f(t) - f(t-1)$ for all $t \in \mathbb{N}_{a+1}$. For convenience, we adopt the convention that $\nabla f(t) := (\nabla f)(t)$. The operator ∇^n is then defined recursively by $\nabla^n f(t) := \nabla\big(\nabla^{n-1} f(t)\big)$ for $t \in \mathbb{N}_{a+n}$, $n \in \mathbb{Z}^+$. In addition, we take ∇^0 to be the identity operator. Motivated by results in time scales (see Bohner and Peterson [7]) we define the *backward jump function* by $\rho : \mathbb{N}_a \to \mathbb{N}_a$, given by $\rho(t) = \max\{a, t-1\}$.

Also we let f^ρ denote the composition function $f \circ \rho$. We say F is an anti-nabla difference of f on $\mathbb{N}_a$ provided $\nabla F(t) = f(t)$ for $t \in N_{a+1}$. If $c, d \in \mathbb{N}_a$, then we define the definite nabla integral of $f : \mathbb{N}_a \to \mathbb{R}$ by

$$\int_c^d f(t)\nabla t = \begin{cases} \sum_{t=c+1}^d f(t), & \text{if} \quad c < d \\ 0, & \text{if} \quad c = d \\ -\sum_{t=d+1}^c f(t), & \text{if} \quad d < c. \end{cases}$$

The fundamental theorem of the nabla calculus is then given as follows:

Theorem 1 (Fundamental Theorem of Nabla Calculus)**.** *Let $f : \mathbb{N}_a \to \mathbb{R}$ and let F be a nabla antidifference of f on $\mathbb{N}_a$, then for any $c, d \in \mathbb{N}_a$,*

$$\int_c^d f(t)\nabla t = F(d) - F(c).$$

Given two functions $u, v : \mathbb{N}_a \to \mathbb{R}$ and $t \in \mathbb{N}_{a+1}$, we have the (nabla) product rule

$$\nabla\big(u(t)v(t)\big) = u(t)\nabla v(t) + v(t-1)\nabla u(t).$$

This leads to the (nabla) integration (summation) by parts formula

$$\int_b^c u(t)\nabla v(t)\nabla t = u(t)v(t)\Big|_b^c - \int_b^c v(t-1)\nabla u(t)\nabla t. \tag{1}$$

3 Fractional Sums and Fractional Differences

With the relevant preliminaries established, we are now ready to develop what we mean by fractional differences and fractional sums. In the parlance of fractional difference equations, these are analogous to fractional derivatives and fractional integrals, respectively. To do this we must recall the definition of the *rising factorial function*, which is defined by

$$k^{\overline{n}} := k(k+1)\cdots(k+n-1) = \frac{(k+n-1)!}{(k-1)!},$$

when n a non-negative integer. Readers familiar with the Pochhammer function may recognize this notation in its alternative form, $(k)_n$. See Knuth [23].

We will now establish a result analogous to the Cauchy formula for repeated integration. This result will motivate our definition of fractional integrals. For completeness we prove this result.

Theorem 2 (Repeated Summation Rule)**.** *Let $f : \mathbb{N}_a \to \mathbb{R}$ be given, then*

$$\int_b^t \int_b^{\tau_1} \cdots \int_b^{\tau_{n-1}} f(\tau_n)\nabla\tau_n \cdots \nabla\tau_2\nabla\tau_1 = \frac{1}{(n-1)!}\int_b^t (t-\rho(\tau_1))^{\overline{n-1}} f(\tau_1)\nabla\tau_1.$$

Proof. We will prove this by induction on n for $n \geq 1$, the $n = 1$ case being trivially true. Assuming the conclusion for some $n \geq 1$, we let

$$y(t) = \int_b^t \int_b^{\tau_1} \cdots \int_b^{\tau_{n-1}} \int_b^{\tau_n} f(\tau_{n+1}) \nabla\tau_{n+1} \nabla\tau_n \cdots \nabla\tau_2 \nabla\tau_1.$$

Let $g(\tau_n) = \int_b^{\tau_n} f(\tau_{n+1}) \nabla\tau_{n+1}$, then it follows from the induction assumption that

$$y(t) = \frac{1}{(n-1)!} \int_b^t (t - \rho(\tau_1))^{\overline{n-1}} g(\tau_1) \nabla\tau_1.$$

It can be shown that $\frac{1}{(n-1)!}(t - \rho(\tau_1))^{\overline{n-1}} = -\frac{1}{n!}\nabla(t - \tau_1)^{\overline{n}}$, so if we take $u(\tau_1) = g(\tau_1)$ and $\nabla v(\tau_1) = \frac{1}{(n-1)!}(t - \rho(\tau_1))^{\overline{n-1}}$, then $\nabla u(\tau_1) = f(\tau_1)$, and $v(\tau_1) = -\frac{1}{n!}(t - \tau_1)^{\overline{n}}$. Using the integration by parts formula (1), it follows that

$$\begin{aligned} y(t) &= \int_b^t u(\tau_1) \nabla v(\tau_1) \nabla\tau_1 \\ &= \left(-\frac{1}{n!}(t - \tau_1)^{\overline{n}} \int_b^{\tau_1} f(\tau_{n+1}) \nabla\tau_{n+1} \right)\Bigg|_b^t + \frac{1}{n!} \int_b^t (t - \rho(\tau_1))^{\overline{n}} f(\tau_1) \nabla\tau_1. \end{aligned}$$

Evaluating the first term at $\tau_1 = t$, we have $0^{\overline{n}} = 0$, and at $\tau_1 = b$, the integral evaluates to zero. The desired result follows. □

Having previously defined $\nabla^n f(t)$ for n a non-negative integer, we use the above result to define $\nabla^n f(t)$ for negative integers n.

Definition 3 (Integral Sum)**.** *Let $f : \mathbb{N}_a \to \mathbb{R}$ be given and $n \in \mathbb{Z}^+$, then*

$$\nabla_a^{-n} f(t) := \frac{1}{(n-1)!} \int_a^t (t - \rho(s))^{\overline{n-1}} f(s) \nabla s.$$

With the integer order nabla differences and sums defined, we want to define the fractional order nabla differences and sums. To do this we extend the definition of the falling function $k^{\overline{n}} = \frac{(k+n-1)!}{(k-1)!}$ in the obvious way by defining

$$k^{\overline{\nu}} := \frac{\Gamma(k+\nu)}{\Gamma(k)}, \quad \text{for all} \quad k, \nu \in \mathbb{R} \tag{2}$$

such that $\Gamma(k + \nu)$ is defined. Here we adopt the common convention that $\frac{\Gamma(b)}{\Gamma(c)} := 0$ whenever $\Gamma(b)$ is defined and $\Gamma(c)$ is undefined. The motivation for this convention is the property that $\lim_{x \to c} \frac{\Gamma(b)}{\Gamma(x)} = 0$. Later we will use the generalized power rule (which is easily proved using (2))

$$\nabla(t - a)^{\overline{\mu}} = \mu(t - a)^{\overline{\mu-1}}, \quad t \in \mathbb{N}_a, \tag{3}$$

where $a, \mu \in \mathbb{R}$ such that both sides of the equation are defined. A generalization of this is the useful formula

$$\nabla^n (t - a)^{\overline{\mu}} = (\mu - n + 1)^{\overline{n}} (t - a)^{\overline{\mu-n}}. \tag{4}$$

which is easily proved by induction on n.

Now we can define fractional nabla sums and nabla differences.

Definition 4 (Nabla Fractional Sum)**.** *Let $f : \mathbb{N}_a \to \mathbb{R}$ be given and $\nu \in \mathbb{R}^+$, then*

$$\nabla_a^{-\nu} f(t) := \frac{1}{\Gamma(\nu)} \int_a^t (t-\rho(s))^{\overline{\nu-1}} f(s) \nabla s.$$

We then define the nabla fractional difference in terms of a nabla fractional sum.

Definition 5 (Nabla Fractional Difference)**.** *Let $f : \mathbb{N}_a \to \mathbb{R}$, $v \in \mathbb{R}^+$ and choose N such that $N-1 < \nu \leq N$. Then, for $t \in \mathbb{N}_{a+N}$,*

$$\nabla_a^{\nu} f(t) := \nabla^N \nabla_a^{-(N-\nu)} f(t).$$

We will return to the fractional case in the next section; however, we must digress momentarily to give a Leibniz rule for differences acting on indefinite integrals. In a later section, we will generalize this Leibniz rule to fractional differences acting on indefinite integrals. For completeness we prove the following result:

Theorem 6 (Leibniz Rule)**.** *Let $g : \mathbb{N}_a \times \mathbb{N}_a \to \mathbb{R}$, then for $t \in \mathbb{N}_{b+1}$, $b \in \mathbb{N}_a$,*

$$\begin{aligned} \nabla_t \int_b^t g(s,t) \nabla s &= g(t,t-1) + \int_b^t \nabla_t\, g(s,t) \nabla s \\ &= g(t,t) + \int_b^{t-1} \nabla_t\, g(s,t) \nabla s. \end{aligned}$$

where ∇_t denotes the nabla difference with respect to t.

Proof. For $t \in \mathbb{N}_{b+1}$, $b \in \mathbb{N}_a$,

$$\begin{aligned} \nabla_t \int_b^t g(s,t) \nabla s &= \sum_{s=b+1}^{t} g(s,t) - \sum_{s=b+1}^{t-1} g(s,t-1) \\ &= \left(\sum_{s=b+1}^{t} g(s,t) \right) - \left(\sum_{s=b+1}^{t} g(s,t-1) \right) + g(t,t-1) \\ &= g(t,t-1) + \int_a^t \nabla_t\, g(s,t) \nabla s. \end{aligned}$$

The second equality follows similarly. □

4 Laplace Transform

Having established the necessary preliminaries, we are now ready to discuss the most important application of this material: the Laplace transform. The

Laplace transform, as in the standard calculus, will provide us with an elegant way to solve initial value problems for a fractional nabla difference equation. In this section, we will lay the groundwork for this method, prove the basic properties, and establish a means in which to solve various initial-value (nabla) fractional difference equations. We begin this section by defining the (nabla) exponential function $e_p(t,a)$, as the solution to the following initial value problem:

$$\nabla f(t) = pf(t), \quad t \in \mathbb{N}_{a+1}, \quad f(a) = 1,$$

where $p \in \mathcal{R} := \mathbb{C}\backslash\{1\}$. $\mathcal{R}$ is called the set of regressive (complex) constants. It is easy to show that

$$f(t) = (1-p)^{a-t}$$

is the unique solution of this IVP. Hence we can simply define the exponential function as follows:

Definition 7 (Exponential Function)**.** *For $p \in \mathcal{R}$, we define the nabla exponential function by*

$$e_p(t,a) := (1-p)^{a-t}, \quad \textit{for} \quad t \in \mathbb{N}_a.$$

Unlike the exponential function, e^{pt}, on $\mathbb{R}$, this function does not satisfy the law of exponents $e^{pt}e^{rt} = e^{(r+p)t}$. Instead, we find that $e_p(t,a)e_r(t,a) = e_{p+r-pr}(t,a)$. This leads us to define the addition $\oplus$ on $\mathcal{R}$ by

$$a \oplus b = a + b - ab.$$

It is easy to see that with this addition, $\mathcal{R}$ is an Abelian group and we get the law of exponents:

$$e_p(t,a)e_r(t,a) = e_{p\oplus r}(t,a).$$

Furthermore, the additive inverse of $p \in \mathcal{R}$ is given by $\ominus a := \frac{-a}{1-a}$. Now, motivated by the definition of the Laplace transform on a time scale given in [7] we define the (nabla) Laplace transform operator $\mathcal{L}$ as follows:

Definition 8. *For a function $f : \mathbb{N}_a \to \mathbb{R}$ and $s \in \mathcal{R}$, we define the Laplace transform of f by*

$$\mathcal{L}_a\{f\}(s) := \int_a^\infty e^\rho_{\ominus s}(t,a)f(t)\nabla t.$$

While this form will prove to be convenient at times, it is also important to consider its equivalent form

$$\mathcal{L}_a\{f\}(s) = \sum_{k=1}^{\infty}(1-s)^{k-1}f(a+k)$$

which is easily verified. Linearity of this transform follows from the above form. However, what is not immediately evident is the existence and uniqueness of this transform, which we establish in the next two theorems.

Definition 9. *A function $f : \mathbb{N}_a \to \mathbb{R}$ is said to be of exponential order α if there exist constants $M > 0$ and $T \in \mathbb{N}_a$ such that $|f(t)| \le M|e_\alpha(t,a)|$ for all $t \in \mathbb{N}_T$.*

Theorem 10 (Existence of Laplace Transform)**.** *Given a function of exponential order α, its Laplace transform exists for $\left|\frac{1-s}{1-\alpha}\right| < 1$.*

Proof. Let f be a function of exponential order α, so there are constants $M > 0$ and $T \in \mathbb{N}_a$ such that $|f(t)| \le M|e_\alpha(t,a)|$ for all $t \in \mathbb{N}_T$. Fix an integer N such that it is both greater than one and greater than $T - a$, so $a + N > T$. Then for all $k \ge N$, we have that

$$\big|f(a+k)\big| \le M|e_\alpha(a+k,a)| = \frac{M}{|1-\alpha|^k}.$$

Multiplying both sides by $|1-s|^{k-1}$ and taking the sum from $k = N$ to infinity, it follows that for $\left|\frac{1-s}{1-\alpha}\right| < 1$,

$$\begin{aligned}\sum_{k=N}^{\infty}\Big|(1-s)^{k-1}f(a+k)\Big| &\le M\frac{1}{|1-s|}\sum_{k=N}^{\infty}\left(\frac{|1-s|}{|1-\alpha|}\right)^k \\ &= M\frac{1}{|1-s|}\cdot\left(\frac{|1-s|}{|1-\alpha|}\right)^N\sum_{k=0}^{\infty}\left(\frac{|1-s|}{|1-\alpha|}\right)^k \\ &< \infty.\end{aligned}$$

It follows that $\mathcal{L}_a\{f\}(s)$ converges absolutely for $\left|\frac{1-s}{1-\alpha}\right| < 1$. □

Theorem 11 (Uniqueness of the Laplace Transform)**.** *$\mathcal{L}_a\{f\}(s) = 0$ if and only if $f(t) = 0$ for $t \in \mathbb{N}_{a+1}$.*

Proof. The backward direction is trivial. We now consider the forward direction. Assume that $\mathcal{L}_a\{f\}(s) = 0$. This means that $\sum_{k=1}^{\infty}(1-s)^{k-1}f(a+k) = 0$. Let $z = 1-s$, shift the index of the sum down by 1 and let $b_k = f(a+k+1)$, so we now have $\sum_{k=0}^{\infty} b_k z^k = 0$. Suppose, for a contradiction, that there is an integer α such that $b_\alpha \neq 0$, and without loss of generality, suppose α is the smallest such integer. Since $s \neq 1$, then $z \neq 0$, so we can divide both sides of our sum by z^α and let $p(z) = \sum_{k=\alpha}^{\infty} b_k z^{k-\alpha} = 0$. Let $\{z_n\}_{n=1}^{\infty}$ be a sequence whose limit is zero such that $z_n \neq 0$ for all n. By definition of p, it follows that $\lim_{n\to\infty} p(z_n) = b_\alpha$. However, this implies that $b_\alpha = 0$, which is a contradiction. We therefore have that $b_k = 0$ for all k, so $f(t) = 0$ for $t \in \mathbb{N}_{a+1}$. □

The consequence of the above result establishes the rule that two functions, f and g, whose Laplace Transformations are equal are unique on the domain $\mathbb{N}_{a+1}$. This, however, says nothing about the values $f(a)$ and $g(a)$, which must be independently verified in order to establish that f and g are equal on their domains.

5 Fractional Taylor Monomials

We now define nabla Taylor monomials and later generalize them for non-integer orders. Once established, we will determine their Laplace transforms.

Definition 12 (Taylor Monomials)**.** *For all non-negative integers n, we denote the n-th nabla Taylor monomial by $h_n(t,a)$, defined recursively by*

$$\begin{cases} h_0(t,a) := 1 \\ h_n(t,a) := \int_a^t h_{n-1}(\tau,a)\nabla\tau, \quad n \geq 1. \end{cases}$$

In the next theorem we establish a formula for the nabla Taylor monomials.

Theorem 13. *For all non-negative integers n, we have that $h_n(t,a) = \frac{(t-a)^{\overline{n}}}{n!}$.*

Proof. The proof of this result is by induction, where the $n = 0$ case is trivial. Now, assume that $h_n(t,a) = \frac{(t-a)^{\overline{n}}}{n!}$ for some $n \geq 0$. We then consider

$$h_{n+1}(t,a) = \int_a^t h_n(\tau,a)\nabla\tau = \frac{1}{n!}\int_a^t (\tau-a)^{\overline{n}}\nabla\tau.$$

By the power rule (3), it can be shown that $\frac{1}{n+1}(\tau-a)^{\overline{n+1}}$ is an antidifference of $(\tau-a)^{\overline{n}}$, so

$$h_{n+1}(t,a) = \frac{1}{n!}\left(\left.\frac{(\tau-a)^{\overline{n+1}}}{n+1}\right|_a^t\right) = \frac{(t-a)^{\overline{n+1}}}{(n+1)!}.$$

The result follows. □

Before we generalize the Taylor monomials for arbitrary values of n, we first determine the Laplace Transform of the integer-order Taylor monomials.

Theorem 14. *For all non-negative integers n, we have that*

$$\mathcal{L}_a\left\{h_n(\cdot,a)\right\}(s) = \frac{1}{s^{n+1}}, \quad \textit{for} \quad |1-s| < 1.$$

Proof. The proof is by induction on n. By definition, $h_0(t,a) = 1$, so

$$\mathcal{L}_a\left\{1\right\}(s) = \sum_{k=1}^{\infty}(1-s)^{k-1} = \frac{1}{1-(1-s)} = \frac{1}{s},$$

for $|1-s| < 1$. Suppose now that $\mathcal{L}_a\left\{h_n(\cdot,a)\right\}(s) = \frac{1}{s^{n+1}}$ for some $n \geq 0$ and $|1-s| < 1$. Then, consider $\mathcal{L}_a\left\{h_{n+1}(\cdot,a)\right\}(s) = \int_a^\infty e^{\rho}_{\ominus s}(t,a)h_{n+1}(t,a)\nabla t$. We will apply the integration by parts formula (1) with $u(t) = h_{n+1}(t,a)$ and

$\nabla v(t) = e^{\rho}_{\ominus s}(t,a)$. It then follows that $\nabla u(t) = h_n(t,a)$ and it can be shown that $v(t) = -\frac{1}{s}e_{\ominus s}(t,a)$ is a (nabla) antidifference of $e^{\rho}_{\ominus s}(t,a)$. This means that

$$\begin{aligned}\mathcal{L}_a\left\{h_{n+1}(\cdot,a)\right\}(s) &= -\frac{1}{s}e_{\ominus s}(t,a)h_{n+1}(t,a)\Big|_a^{\infty} + \frac{1}{s}\int_a^{\infty} e^{\rho}_{\ominus s}(t,a)h_n(t,a)\nabla t \\ &= -\frac{1}{s}(1-s)^{t-a}h_{n+1}(t,a)\Big|_a^{\infty} + \frac{1}{s}\mathcal{L}_a\left\{h_n(\cdot,a)\right\}(s).\end{aligned}$$

Evaluating the first term as $t \to \infty$, given the assumption that $|1-s| < 1$, means that the term goes to zero. Likewise, it is easy to show that $h_n(a,a) = 0$ for all $n \geq 1$, thus we have that $\mathcal{L}_a\left\{h_{n+1}(\cdot,a)\right\}(s) = \frac{1}{s^{n+2}}$, completing the proof. □

Next we define the fractional order nabla Taylor monomials.

Definition 15 (Fractional Order Taylor Monomials)**.** *Let $\nu \in \mathbb{R}\backslash\{-1,-2,\cdots\}$, then the ν-th Taylor monomial is given by $h_\nu(t,a) := \frac{(t-a)^{\overline{\nu}}}{\Gamma(\nu+1)}$.*

Before we continue, we will establish a lemma which will prove useful in finding the Laplace transform of the fractional order nabla Taylor monomials.

Lemma 16. *For $\nu \in \mathbb{R}\backslash\mathbb{Z}$ and $n \geq 0$, we have that*

$$(1+\nu)^{\overline{n}} = \frac{(-1)^n\Gamma(-\nu)}{\Gamma(-\nu-n)}.$$

Proof. The proof is by induction, where the $n = 0$ case is easily established. Suppose the assertion is true for some $n \geq 0$. Then,

$$\begin{aligned}(1+\nu)^{\overline{n+1}} &= (1+\nu)^{\overline{n}}(\nu+n+1) \\ &= \frac{(-1)^n\Gamma(-\nu)(\nu+n+1)}{\Gamma(-\nu-n)} \\ &= \frac{(-1)^{n+1}\Gamma(-\nu)}{\Gamma(-\nu-(n+1))}.\end{aligned}$$

The result follows. □

We now determine the Laplace transform of the fractional order nabla Taylor monomial.

Theorem 17. *For ν a non-integer real number, we have that*

$$\mathcal{L}_a\left\{h_\nu(\cdot,a)\right\}(s) = \frac{1}{s^{\nu+1}}, \quad \textit{for} \quad |1-s| < 1.$$

Proof.

$$\begin{aligned}
\mathcal{L}_a\left\{h_\nu(\cdot,a)\right\}(s) &= \sum_{k=1}^{\infty}(1-s)^{k-1}h_\nu(a+k,a)\\
&= \sum_{k=1}^{\infty}(1-s)^{k-1}\frac{\Gamma(k+\nu)}{\Gamma(k)\Gamma(\nu+1)}\\
&= \sum_{k=0}^{\infty}(1-s)^{k}\frac{(1+\nu)^{\overline{k}}}{\Gamma(k+1)}\\
&= \sum_{k=0}^{\infty}(-(1-s))^{k}\frac{\Gamma(-\nu)}{\Gamma(k+1)\Gamma(-\nu-k)} \quad \text{(by Lemma 16)}\\
&= \sum_{k=0}^{\infty}\binom{-(\nu+1)}{k}(-(1-s))^{k}\\
&= \frac{1}{s^{\nu+1}} \quad \text{(by the Generalized Binomial Theorem)}
\end{aligned}$$

□

From the results of Theorems 14 and 17, we state the following corollary:

Corollary 18. *For $\nu \in \mathbb{R}\backslash\{-1,-2,-3,\dots\}$, we have that*

$$\mathcal{L}_a\left\{h_\nu(\cdot,a)\right\}(s) = \frac{1}{s^{\nu+1}}, \quad \textit{for} \quad |1-s|<1.$$

6 Convolution

We are now ready to investigate one of the most important properties in solving initial-value fractional nabla difference equations: convolution. This definition is motivated by the desire to express the fractional nabla sums and fractional nabla differences as convolutions of arbitrary functions and Taylor monomials. As a consequence, the resulting properties that stem from this definition are in fact consistent with the standard convolution.

Definition 19. *For $f,g:\mathbb{N}_a\to\mathbb{R}$ and all $t\in\mathbb{N}_{a+1}$, we define the convolution of f and g by*

$$(f*g)(t) := \int_a^t f(t-\rho(s)+a)g(s)\nabla s.$$

Theorem 20. *Let $\nu\in\mathbb{R}\backslash\{0,-1,-2,\cdots\}$ and $f:\mathbb{N}_a\to\mathbb{R}$, then*

$$\nabla_a^{-\nu}f(t) = (h_{\nu-1}(\cdot,a)*f(\cdot))(t).$$

Proof.

$$\begin{aligned}(h_{\nu-1}(\cdot,a)*f(\cdot))(t) &= \int_a^t h_{\nu-1}(t-\rho(s)+a,a)f(s)\nabla s\\ &= \frac{1}{\Gamma(\nu)}\int_a^t (t-\rho(s))^{\overline{\nu-1}}f(s)\nabla s\\ &= \nabla_a^{-\nu}f(t).\end{aligned}$$

□

Theorem 21 (Convolution Theorem)**.** *For $f,g:\mathbb{N}_a\to\mathbb{R}$, we have that*

$$\mathcal{L}_a\{f*g\}(s)=\mathcal{L}_a\{f\}(s)\mathcal{L}_a\{g\}(s).$$

Proof.

$$\begin{aligned}\mathcal{L}_a\{f*g\}(s) &= \sum_{k=1}^{\infty}(1-s)^{k-1}(f*g)(a+k)\\ &= \sum_{k=1}^{\infty}(1-s)^{k-1}\sum_{r=a+1}^{a+k}f(a+k-\rho(r)+a)g(r)\\ &= \sum_{k=1}^{\infty}\sum_{r=1}^{k}(1-s)^{k-1}f(k-\rho(r)+a)g(r+a)\\ &= \sum_{r=1}^{\infty}\sum_{k=r}^{\infty}(1-s)^{k-1}f(k-\rho(r)+a)g(r+a)\\ &= \sum_{r=1}^{\infty}(1-s)^{r-1}g(r+a)\sum_{k=1}^{\infty}(1-s)^{k-1}f(k+a)\\ &= \mathcal{L}_a\{g\}(s)\mathcal{L}_a\{f\}(s).\end{aligned}$$

□

With the above result and the uniqueness of the Laplace transform, it follows that the convolution product is commutative and associative.

7 Further Properties of the Laplace Transform

The focus of this section will be to establish many of the properties of the Laplace transform that will be useful in solving initial-value problems for fractional nabla difference equations.

Theorem 22 (Transformation of Fractional Sums)**.** *For $f:\mathbb{N}_a\to\mathbb{R}$ and $\nu\in\mathbb{R}^+$, we have that $\mathcal{L}_a\{\nabla_a^{-\nu}f\}(s)=\frac{1}{s^\nu}\mathcal{L}_a\{f\}(s)$.*

Proof.

$$\begin{aligned}
\mathcal{L}_a\left\{\nabla_a^{-\nu} f\right\}(s) &= \mathcal{L}_a\left\{h_{\nu-1}(\cdot,a) * f\right\}(s)\\
&= \mathcal{L}_a\left\{h_{\nu-1}(\cdot,a)\right\}(s)\mathcal{L}_a\left\{f\right\}(s)\\
&= \frac{1}{s^\nu}\mathcal{L}_a\left\{f\right\}(s).
\end{aligned}$$

□

Note that in the special case when ν is a positive integer, this result is consistent with the standard Laplace transform of multiple integration. We want to establish similar properties for fractional differences; however, we must first establish integer-order difference properties. While this will be easily generalized for an arbitrary integer, this will not be the case for arbitrary values of ν. In this way, we will then only consider fractional differences of order ν, where $\nu \in (0,1)$. First we find the Laplace transform of integer order nabla differences.

Theorem 23 (Transform of Nabla Difference)**.** *For $f : \mathbb{N}_a \to \mathbb{R}$, we have that*

$$\mathcal{L}_{a+1}\left\{\nabla f\right\}(s) = s\mathcal{L}_{a+1}\left\{f\right\}(s) - f(a+1).$$

Proof.

$$\begin{aligned}
\mathcal{L}_{a+1}\left\{\nabla f\right\}(s) &= \sum_{k=1}^{\infty}(1-s)^{k-1}\nabla f(a+1+k)\\
&= \sum_{k=1}^{\infty}(1-s)^{k-1}\big(f(a+1+k) - f(a+k)\big)\\
&= \mathcal{L}_{a+1}\left\{f\right\}(s) - \sum_{k=0}^{\infty}(1-s)^k f(a+1+k)\\
&= \mathcal{L}_{a+1}\left\{f\right\}(s) - f(a+1) - (1-s)\sum_{k=1}^{\infty}(1-s)^{k-1}f(a+1+k)\\
&= \mathcal{L}_{a+1}\left\{f\right\}(s)\big(1-(1-s)\big) - f(a+1)\\
&= s\mathcal{L}_{a+1}\left\{f\right\}(s) - f(a+1).
\end{aligned}$$

□

We can then generalize this result for an arbitrary number of nabla differences.

Theorem 24 (Transform of n^{th}-Order Nabla Difference)**.** *For $f : \mathbb{N}_a \to \mathbb{R}$, we have that*

$$\mathcal{L}_{a+n}\left\{\nabla^n f\right\}(s) = s^n\mathcal{L}_{a+n}\left\{f\right\}(s) - \sum_{k=1}^{n} s^{n-k}\nabla^{k-1}f(a+n).$$

Proof. The result follows from induction on n with the previous theorem as a base case. The inductive step is omitted. □

We next want to find the Laplace transform of a ν^{th} order difference where $0 < \nu < 1$. First, however, a useful lemma will be necessary.

Lemma 25 (Shifting Lemma)**.** *Given $f : \mathbb{N}_a \to \mathbb{R}$, we have*

$$\mathcal{L}_{a+1}\left\{f\right\}(s) = \frac{1}{1-s}\mathcal{L}_a\left\{f\right\}(s) - \frac{1}{1-s}f(a+1).$$

Proof.

$$\begin{aligned}\frac{1}{1-s}\mathcal{L}_a\left\{f\right\}(s) &= \frac{1}{1-s}\sum_{k=1}^{\infty}(1-s)^{k-1}f(a+k)\\ &= \frac{1}{1-s}f(a+1) + \sum_{k=1}^{\infty}(1-s)^{k-1}f(a+k+1)\\ &= \frac{1}{1-s}f(a+1) + \mathcal{L}_{a+1}\left\{f\right\}(s).\end{aligned}$$

The desired result follows from this form. □

With this, we are ready to provide the general form of the Laplace transform of a ν^{th} order, $0 < \nu < 1$, fractional order difference.

Theorem 26. *Given $f : \mathbb{N}_a \to \mathbb{R}$ and $0 < \nu < 1$, then for $t \in \mathbb{N}_{a+1}$, we have*

$$\mathcal{L}_{a+1}\left\{\nabla_a^{\nu} f\right\}(s) = s^{\nu}\mathcal{L}_{a+1}\left\{f\right\}(s) - \frac{1-s^{\nu}}{1-s}f(a+1).$$

Proof. Consider the following:

$$\begin{aligned}\mathcal{L}_{a+1}\left\{\nabla_a^{\nu} f\right\}(s) &= \mathcal{L}_{a+1}\left\{\nabla\nabla_a^{-(1-\nu)}f\right\}(s)\\ &= s\mathcal{L}_{a+1}\left\{\nabla_a^{-(1-\nu)}\right\}(s) - \nabla_a^{-(1-\nu)}f(a+1).\end{aligned}$$

The last line follows from Theorem 23. Furthermore, we observe that

$$\nabla_a^{-(1-\nu)}f(a+1) = f(a+1),$$

and by an application of Lemma 25, we have that

$$\begin{aligned}\mathcal{L}_{a+1}\left\{\nabla_a^{\nu} f\right\}(s) &= s\left(\frac{1}{1-s}\mathcal{L}_a\left\{\nabla_a^{-(1-\nu)}f\right\}(s) - \frac{1}{1-s}f(a+1)\right) - f(a+1)\\ &= \frac{s^{\nu}}{1-s}\mathcal{L}_a\left\{f\right\}(s) - \frac{1}{1-s}f(a+1) \quad \text{(by Theorem 22).}\end{aligned}$$

Applying Lemma 25 again and simplifying, we end up with the desired result. □

8 Generalized Power Rule

Equation (3) gives us a way to evaluate integer-order nabla differences. It fails, however, to give us a means in which to evaluate fractional ordered nabla sums and nabla differences of $(t-a)^{\overline{\mu}}$ terms. The previous section now gives us the means in which to generalize the power rule quickly and efficiently. A direct proof of this result is very tedious.

Theorem 27 (Generalized Power Rule)**.** *Let $\nu \in \mathbb{R}^+$ and $\mu \in \mathbb{R}$ such that μ and $\nu+\mu$ are not negative integers, then for $t \in \mathbb{N}_a$ we have*

(i) $\nabla_a^{-\nu}(t-a)^{\overline{\mu}} = \left(\frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}\right)(t-a)^{\overline{\mu+\nu}}$,

(ii) $\nabla_a^{\nu}(t-a)^{\overline{\mu}} = \left(\frac{\Gamma(\mu+1)}{\Gamma(\mu-\nu+1)}\right)(t-a)^{\overline{\mu-\nu}}$.

Proof. We will first establish the result for (i), after which (ii) will follow. Consider the following:

$$\begin{aligned}\mathcal{L}_a\left\{\nabla_a^{-\nu}h_\mu(\cdot,a)\right\}(s) &= \frac{1}{s^\nu}\mathcal{L}_a\left\{h_\mu(\cdot,a)\right\}(s)\\ &= \frac{1}{s^{\nu+\mu+1}}\\ &= \mathcal{L}_a\left\{h_{\nu+\mu}(\cdot,a)\right\}(s).\end{aligned}$$

By Definition 15 and the linearity of the Laplace transform, (i) holds for $t \in \mathbb{N}_{a+1}$. Observing that for $t=a$ the stated equality holds, hence (i) follows for all $t \in \mathbb{N}_a$.

Now for (ii), choose N such that $N-1<\nu\leq N$ and consider the following:

$$\begin{aligned}\nabla_a^{\nu}(t-a)^{\overline{\mu}} &= \nabla^N\nabla_a^{-(N-\nu)}(t-a)^{\overline{\mu}}\\ &= \frac{\Gamma(\mu+1)}{\Gamma(\mu+N-\nu+1)}\nabla^N(t-a)^{\overline{\mu+N-\nu}}.\end{aligned}$$

The last line follows from (i). Applying the generalized power rule for integer-order differences and the definition of the rising factorial, (ii) follows. □

9 Mittag-Leffler Function

We begin this section by proving that the exponential function as described by Defintion 7 can be written as an infinite sum of Taylor monomials (this is just the Taylor series of $e_p(t,a)$). From this result, we will define the Mittag-Leffler function, thereby generalizing the exponential function, and then determine its Laplace transform.

Theorem 28. *For $|p|<1$, we have that $e_p(t,a)=\sum_{k=0}^{\infty}p^k h_k(t,a)$ for $t\in\mathbb{N}_a$.*

Proof. We will show that $e_p(t,a)$ and $\sum_{k=0}^{\infty} p^k h_k(t,a)$ have the same Laplace transform. In order to ensure convergence, we restrict the transform domain such that $|s| < |p|$, $|1-s| < 1$, and $|1-s| < |1-p|$. First, we determine the Laplace transform of the exponential function as follows:

$$\begin{aligned}\mathcal{L}_a\{e_p(\cdot,a)\}(s) &= \sum_{k=1}^{\infty}(1-s)^{k-1}(1-p)^{-k} \\ &= \frac{1}{1-p}\sum_{k=0}^{\infty}\left(\frac{1-s}{1-p}\right)^k = \frac{1}{s-p}.\end{aligned}$$

Next, we have

$$\begin{aligned}\mathcal{L}_a\left\{\sum_{k=0}^{\infty} p^k h_k(\cdot,a)\right\}(s) &= \sum_{k=0}^{\infty} p^k \mathcal{L}_a\{h_k(\cdot,a)\}(s) \\ &= \frac{1}{s}\sum_{k=0}^{\infty}\left(\frac{p}{s}\right)^k = \frac{1}{s-p}.\end{aligned}$$

Finally, $e_p(a,a) = 1$ by definition, and $\sum_{k=0}^{\infty} p^k h_k(a,a) = 1$ since $p^0 h_0(a,a) = 1$ and $p^k h_k(a,a) = 0$ for $k \geq 1$. Therefore, we obtain the desired result on $\mathbb{N}_a$. $\square$

Next we define the Mittag-Leffler function, which is a generalization of the exponential function $e_p(t,a)$.

Definition 29 (Mittag-Leffler Function)**.** *For $|p| < 1$, $\alpha > 0$, $\beta \in \mathbb{R}$, and $t \in \mathbb{N}_a$, we define the Mittag-Leffler function to be*

$$E_{p,\alpha,\beta}(t,a) := \sum_{k=0}^{\infty} p^k h_{\alpha k+\beta}(t,a).$$

As desired, by taking $\alpha = 1$ and $\beta = 0$, we obtain $e_p(t,a)$, so this function does indeed generalize our exponential function. We will now determine the Laplace transform of the Mittag-Leffler function.

Theorem 30. *For $|p| < 1$, $\alpha > 0$, $\beta \in \mathbb{R}$, $|1-s| < 1$, and $|s^\alpha| > |p|$, we have*

$$\mathcal{L}_a\{E_{p,\alpha,\beta}(\cdot,a)\}(s) = \frac{s^{\alpha-\beta-1}}{s^\alpha - p}.$$

Proof.

$$\begin{aligned}\mathcal{L}_a\{E_{p,\alpha,\beta}(\cdot,a)\}(s) &= \sum_{k=0}^{\infty} p^k \mathcal{L}_a\{h_{\alpha k+\beta}(\cdot,a)\}(s) \\ &= \frac{1}{s^{\beta+1}}\sum_{k=0}^{\infty}\left(\frac{p}{s^\alpha}\right)^k \\ &= \frac{s^{\alpha-\beta-1}}{s^\alpha - p}.\end{aligned}$$

$\square$

10 General Solutions to Initial-Value Problems

We will now consider a general ν^{th} order fractional nabla initial-value problem and give a formula for its solution, $0 < \nu < 1$.

Theorem 31 (Variation of Constants)**.** *Let $f, g : \mathbb{N}_a \to \mathbb{R}$ and $0 < \nu < 1$. Then, for $t \in \mathbb{N}_{a+1}$, the fractional initial-value problem*

$$\begin{cases} \nabla_a^\nu f(t) + cf(t) = g(t), \quad t \in \mathbb{N}_{a+1}, \quad |c| < 1 \\ f(a+1) = A, \quad A \in \mathbb{R} \end{cases}$$

has the solution

$$f(t) = \big(E_{-c,\nu,\nu-1}(\cdot, a) * g(\cdot)\big)(t) + \big(A(c+1) - g(a+1)\big)E_{-c,\nu,\nu-1}(t,a).$$

Proof. We begin by taking the Laplace transform of both sides, starting at $a+1$. Applying Theorem 26, we end up with

$$(s^\nu + c)\mathcal{L}_{a+1}\{f\}(s) - A\left(\frac{1-s^\nu}{1-s}\right) = \mathcal{L}_{a+1}\{g\}(s).$$

Using the Lemma 25, distributing the $s^\nu + c$ term, canceling the $(1-s)^{-1}$ term on both sides of the equation, and solving for $\mathcal{L}_a\{f\}(s)$, we have

$$\begin{aligned} \mathcal{L}_a\{f\}(s) &= \frac{1}{s^\nu + c}\Big(\mathcal{L}_a\{g\}(s) - g(a+1) + A(c+1)\Big) \\ &= \mathcal{L}_a\{E_{-c,\nu,\nu-1}(\cdot,a)\}(s)\Big(\mathcal{L}_a\{g\}(s) - g(a+1) + A(c+1)\Big) \\ &= \mathcal{L}_a\Big\{\big(E_{-c,\nu,\nu-1} * g\big) + \big(A(c+1) - g(a+1)\big)E_{-c,\nu,\nu-1}\Big\}(s). \end{aligned}$$

This implies that

$$f(t) = \big(E_{-c,\nu,\nu-1}(\cdot, a) * g(\cdot)\big)(t) + \big(A(c+1) - g(a+1)\big)E_{-c,\nu,\nu-1}(t,a)$$

for $t \in \mathbb{N}_{a+1}$. □

Supposing that $c = 0$ in the above fractional initial-value problem, we have the following corollary.

Corollary 32. *Let $f, g : \mathbb{N}_a \to \mathbb{R}$ and $0 < \nu < 1$. Then, for $t \in \mathbb{N}_{a+1}$, the fractional initial-value problem*

$$\begin{cases} \nabla_a^\nu f(t) = g(t), \quad t \in \mathbb{N}_{a+1} \\ f(a+1) = A, \quad A \in \mathbb{R} \end{cases}$$

has the solution

$$f(t) = \nabla_a^{-\nu} g(t) + \big(A - g(a+1)\big)h_{\nu-1}(t,a).$$

Proof. First, we observe that

$$E_{0,\nu,\nu-1}(t,a) = \sum_{k=0}^{\infty} 0^k h_{\nu k+\nu-1}(t,a).$$

All but the $k = 0$ term are zero, which means that $E_{0,\nu,\nu-1}(t,a) = h_{\nu-1}(t,a)$. Finally, we have $\big(E_{0,\nu,\nu-1}(\cdot,a) * g(\cdot)\big)(t) = \big(h_{\nu-1}(\cdot,a) * g(\cdot)\big)(t) = \nabla_a^{-\nu} g(t)$ by Theorem 20. From this, the stated solution to the initial-value problem follows. □

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References

[1] F. M. Atici and P. W. Eloe, Two-point boundary value problems for finite fractional difference equations, *Journal of Difference Equations and Appl.* doi: 10.1080/10236190903029241.

[2] F. M. Atici and P. W. Eloe, Fractional q-Calculus on a Time Scale, *J. Nonlinear Mathematical Physics.* **14** 3 (2007) 333–344.

[3] F. M. Atici and P. W. Eloe, A transform method in discrete fractional calculus, *International Journal of Difference Equations,* **2** 2 (2007) 165–176.

[4] F. M. Atici and P. W. Eloe, Initial value problems in discrete fractional calculus, *Proc. Amer. Math. Soc.* **137** 3 (2009) 981–989.

[5] F. M. Atici and P. W. Eloe, Discrete fractional calculus with the Nabla Operator, *Electronic J. Qual. Theory Diff. Equ.* **1** (2009) 1–12.

[6] M. Bohner and G. Guseinov, The h-Laplace and q-Laplace transforms, *J. Math. Anal. Appl.* **365** (2010) 75–92.

[7] M. Bohner and A. Peterson, *Dynamic Equation on Time Scales: An Introduction with Applications,* Birkhäuser, Boston, 2001.

[8] M. Bohner and A. Peterson, Editors, *Advances in Dynamic Equations on Time Scales,* Birkhäuser, Boston, 2003.

[9] C. S. Goodrich, Solutions to a discrete right-focal boundary value problem, *Int. J. Difference Equ.* **5** (2010), 195–216.

[10] C. S. Goodrich, Continuity of solutions to discrete fractional initial value problems, *Comput. Math. Appl.* **59** (2010) 3489–3499.

[11] C. S. Goodrich, Some new existence results for fractional difference equations, *Int. J. Dyn. Syst. Differ. Equ.* **3** (2011) 145–162.

[12] C. S. Goodrich, On a discrete fractional three-point boundary value problem, *J. Difference Equ. Appl.* doi: 10.1080/10236198.2010.503240

[13] C. S. Goodrich, A comparison result for a discrete fractional boundary value problem with general boundary conditions, *Int. J. Difference Equ.* **6** (2011), in press.

[14] C. S. Goodrich, Existence and uniqueness of solutions to a fractional difference equation with nonlocal conditions, *Comput. Math. Appl.* **61** (2011) 191–202.

[15] C. S. Goodrich, On positive solutions to nonlocal fractional and integer-order difference equations, *Appl. Anal. Discrete Math.* **5** (2011) 122–132.

[16] C. S. Goodrich, Existence of a positive solution to a system of discrete fractional boundary value problems, *Appl. Math. Comput.* **217** (2011) 4740–4753.

[17] C. S. Goodrich, Existence of a positive solution to a first-order p-Laplacian BVP on a time scale, *Nonlinear Anal.* **74** (2011) 1926–1936.

[18] H. Gray and N. Zhang, On a new definition of the fractional difference, *Mathematics of Computation* **50** (182) (1988) 513–529.

[19] M. Holm, Sum and difference compositions in discrete fractional calculus, *CUBO Mathematical Journal,* **13** (3), (2011).

[20] M. Holm, Solutions to a discrete, fractional order, (N-1,1) boundary value problem, *International Journal of Dynamical Systems and Differential Equations* **3** (1-2) (2011) 267–287.

[21] M. Holm, The Laplace transform in discrete fractional calculus, *Computers and Mathematics with Applications* (2011), DOI: 10.1016/j.camwa.2011.04.019.

[22] W. Kelley and A. Peterson, *Difference Equations: An Introduction with Applications,* Second Edition. Harcourt/Academic Press 2001.

[23] D. Knuth, Two notes on notation, *American Mathematical Monthly* **99** (1992) 403–422.

[24] K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations,* John Wiley & Sons, Inc., New York, 1993.

[25] K. S. Miller and B. Ross, *Fractional Difference Calculus,* Proceedings of the International Symposium on Univalent Functions, Fractional Calculus and their Applications, Nihon University, Koriyama, Japan, 1988 139–152.

[26] J. D. Munkhammar, Fractional Calculus and the Taylor-Riemann Series, *Rose-Hulman Undergraduate Mathematics Journal* **6** (2005).

[27] K. Oldham and J. Spanier, *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order,* Dover Publications, Inc., Mineola, New York, 2002.

[28] I. Podlubny, *Fractional Differential Equations,* Academic Press, New York, 1999.

[29] I. Podlubny, *The Laplace Transformation Method for Linear Differential Equations of the Fractional Order.* arXiv:funct-an/9710005v1.