Solitary wave solutions for the general KDV equation by Adomian decomposition method

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Abstract

In recent publications [Chaos, Solitons Fractals 12 (2001) 2283; Int. J. Appl. Math. 3 (4) (2000) 361], we have dealt with the numerical solutions of the Korteweg–De-Vries (KDV) and modified Korteweg–De-Vries (MKDV) equations. We extend this study to a more general nonlinear equation, which is the general Korteweg–De-Vries (GKDV) equation, in which the previous studies is a special case of it. The method applied here is Adomian decomposition method, which has been developed by George Adomian [Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Academic Publishers, Boston, MA, 1994]. Numerical examples are tested to illustrate the pertinent feature of the proposed algorithm.

1. Introduction

The type of equations we are handling is attracting many researches, and a great deal of work has already been done in some of these types, specially problems involving both convection and diffusion, leading to a family of nonlinear partial differential equations, in which numerical schemes are the
most available method to study its solutions. These equations such as Korteweg–De-Vries (KDV) equation [4,5], RLW equation [6], EWE equation [7] and the general KDV (GKDV) equation [4], the general form of this equation shown in the following section, in which all the above equation [1,2] are special cases, is our interest here, to be studied.

2. The general KDV equation

The GKDV equation in the $x$-direction has the form [4]

$$u_t + e u^p u_x + \mu u_{xxx} = 0, \quad a \leq x \leq b, \quad (1)$$

where $p = 1, 2, 3, \ldots$ is positive integer, $e$ and $\mu$ are positive parameters and the subscripts $t$ and $x$ denote differentiation with respect to time and space respectively, with the initial condition

$$u(x, 0) = f(x)$$

and the boundary conditions $u$, $u'' \to 0$ as $x \to \pm \infty$ if we put

$$u(x, t) = f(X); \quad X = x - ct, \quad (2)$$

where $c$ represents the arbitrary constant velocity of the wave traveling in the positive direction $x$-axis. Substitution of (2) into (1) leads to the ordinary differential equation:

$$-cf' + e f^p f'' + \mu f''' = 0, \quad (3)$$

where a prime denotes differentiation with respect to $X$. If we integrate (3) and use the boundary conditions we get

$$u(x, t) = \left[ \frac{c(p + 1)(p + 2)}{2e} \sec h^2 \left[ \frac{p}{2} \sqrt{c/\mu} (x - x_0 - ct) \right] \right]^{1/p}. \quad (4)$$

In this paper we apply the Adomian decomposition method (ADM) for the GKDV equation. It is well known in the literature [3,9–13] that ADM provides the solution of differential and integral equations is a rapidly convergent series. The obtained series may provide the solution in a closed form. However, for concrete problems, the $n$-term approximate $u_n$ defined by

$$\phi_n = \sum_{k=0}^{n-1} u_k(x, t) \quad (5)$$

can be used for numerical approximations. It was shown in [1,2,8–12] among others that the effectiveness of the method could be dramatically improved by determining further components of the solution $u(x, t)$. The reliability of Adomian method gives it a wider applicability in handling evolution models.
3. Adomian decomposition method

We begin with the equation

\[ Lu + R(u) + F(u) = g(t), \]  

(6)

where \( L \) is the operator of the highest-ordered derivatives with respect to \( t \) and \( R \) is the remainder of the linear operator. The nonlinear term is represented by \( F(u) \). Thus we get

\[ Lu = g(t) - R(u) - F(u). \]  

(7)

The inverse \( L^{-1} \) is assumed an integral operator given by

\[ L^{-1} = \int_0^t (\cdot) \, dt \]  

(8)

the operating with the operator \( L^{-1} \) on both sides of Eq. (7) we have

\[ u = f_0 + L^{-1}(g(t) - R(u) - F(u)), \]  

(9)

where \( f_0 \) is the solution of homogeneous equation

\[ Lu = 0 \]  

(10)

involving the constants of integration. The integration constants involved in the solution of homogeneous equation (10) are to be determined by the initial or boundary condition according as the problem is initial-value problem or boundary-value problem.

The ADM assumes that the unknown function \( u(x,t) \) can be expressed by an infinite series of the form

\[ u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \]  

(11)

and the nonlinear operator \( F(u) \) can be decomposed by an infinite series of polynomials given by

\[ F(u) = \sum_{n=0}^{\infty} A_n, \]  

(12)

where \( u_n(x,t) \) will be determined recurrently, and \( A_n \) are the so-called polynomials of \( u_0, u_1, \ldots, u_n \) defined by

\[ A_n = \frac{1}{n!} \frac{d^n}{dx^n} \left[ F \left( \sum_{i=0}^{\infty} \lambda^i u_i \right) \right]_{\lambda=0}, \quad n = 0, 1, 2, \ldots \]  

(13)

It is now well known in the literature that these polynomials can be constructed for all classes of nonlinearity according to algorithms set by Adomian.
[3, 8] and recently developed by an alternative approach in [1, 2, 11, 12]. Applying the inverse operator $L^{-1}$ on both sides of Eq. (1) and using the initial condition we find

$$u(x, t) = f(x) - L^{-1}(\varepsilon u^p u_x + \mu u_{xxx})$$

by using (11) and (12) into the functional equation (14) gives

$$\sum_{n=0}^{\infty} u_n(x, t) = f(x) - L^{-1}\left(\varepsilon \sum_{n=0}^{\infty} A_n + \mu \left(\sum_{n=0}^{\infty} u_n\right)_{xxx}\right).$$

Identifying the zeros component $u_0(x, t)$ by $f(x)$, the remaining components where $n \geq 1$ can be determined by using the recurrence relation

$$\begin{align*}
\begin{aligned}
   u_0(x, t) &= f(x), \\
   u_{n+1}(x, t) &= -L^{-1}(\varepsilon A_n + \mu (u_{n})_{xxx}), \quad n \geq 0,
\end{aligned}
\end{align*}$$

then other polynomials can be generated in a similar way. The first few components of $u_n(x, t)$ follows immediately upon setting:

$$\begin{align*}
\begin{aligned}
   u_0(x, t) &= f(x), \\
   u_1(x, t) &= -L^{-1}(\varepsilon A_0 + \mu (u_0)_{xxx}), \\
   u_2(x, t) &= -L^{-1}(\varepsilon A_1 + \mu (u_1)_{xxx}), \\
   u_3(x, t) &= -L^{-1}(\varepsilon A_2 + \mu (u_2)_{xxx}), \\
   u_4(x, t) &= -L^{-1}(\varepsilon A_3 + \mu (u_3)_{xxx}),
\end{aligned}
\end{align*}$$

where $A_n$ are Adomian polynomials that represent the nonlinear term $(\varepsilon u^p u_x)$ and given by

$$\begin{align*}
\begin{aligned}
   A_0 &= u_0^p u_{0x}, \\
   A_1 &= p u_0^{p-1} u_1 u_{0x} + u_0^p u_{1x}, \\
   A_2 &= u_0^p u_{2x} + p u_0^{p-1} u_1 u_{1x} + p u_0^{p-1} u_{0x} u_2 + \frac{1}{2} p(p - 1) u_0^{p-2} u_1^2 u_{0x}, \\
   A_3 &= u_0^p u_{3x} + p u_0^{p-1} u_2 u_{1x} + p u_0^{p-1} u_{0x} u_3 + p(p - 1) u_0^{p-2} u_{0x} u_1 u_2 \\
   &\quad + \frac{1}{6} \left[2 p u_0^{p-1} u_1 u_{2x} + 3 p(p - 1) u_0^{p-2} u_1^2 u_{1x} + 4 p u_0^{p-1} u_1 u_{2x} \right] + p(p - 1)(p - 2) u_0^{p-3} u_1^3 u_{0x}. \quad (18)
\end{aligned}
\end{align*}$$

The scheme in (17) can easily determine the components $u_n(x, t)$, $n \geq 0$. It is in principle, possible to calculate more components in the decomposition series to enhance the approximation. Consequently, can recursively determine every term of the series $\sum_{n=0}^{\infty} u_n(x, t)$, and hence the solution $u(x, t)$ is readily obtained in a series form. It is interesting to note that we obtained the solution by using the initial condition only. We note that the papers [1, 2] are special cases of our study.
4. The conservation laws for GKDV equation

It is important to discuss the conservation laws for our problem; the GKDV equation possesses many invariant polynomials can be derived easily as shown in the following cases [4,13]:

For $p = 1$, the conservative quantities $I_i$ ($i = 1, \ldots, 4$) can be written as:

\[ I_1 = \int_{-\infty}^{\infty} u(x,t) \, dx, \]
\[ I_2 = \int_{-\infty}^{\infty} u^2(x,t) \, dx, \]
\[ I_3 = \int_{-\infty}^{\infty} \left( u^3(x,t) - \frac{3\mu}{\varepsilon} u_x^2(x,t) \right) \, dx, \]
\[ I_4 = \int_{-\infty}^{\infty} \left( u^4(x,t) - \frac{12\mu}{\varepsilon} u(x,t)u_x^2(x,t) + \frac{36\mu^2}{5\varepsilon^2} u_{xx}^2(x,t) \right) \, dx. \]

For $p = 2$, the conservative quantities $I_i$ ($i = 1, \ldots, 4$) can be written as:

\[ I_1 = \int_{-\infty}^{\infty} u(x,t) \, dx, \]
\[ I_2 = \int_{-\infty}^{\infty} u^2(x,t) \, dx, \]
\[ I_3 = \int_{-\infty}^{\infty} \left( u^4(x,t) - \frac{6\mu}{\varepsilon} u_x^2(x,t) \right) \, dx, \]
\[ I_4 = \int_{-\infty}^{\infty} \left( u^6(x,t) - \frac{30\mu}{\varepsilon} u^2(x,t)u_x^2(x,t) + \frac{18\mu^2}{\varepsilon^2} u_{xx}^2(x,t) \right) \, dx. \]

For $p > 2$, there are only three conservation laws, the conservative quantities $I_i$ ($i = 1, 2, 3$) can be written as:

\[ I_1 = \int_{-\infty}^{\infty} u(x,t) \, dx, \]
\[ I_2 = \int_{-\infty}^{\infty} u^2(x,t) \, dx, \]
\[ I_3 = \int_{-\infty}^{\infty} \left( u^{p+2}(x,t) - \frac{\mu(p+1)(p+2)}{2\varepsilon} u_x^2(x,t) \right) \, dx. \]

5. Solution of GKDV equation by using ADM

Computations are for different values $p = 1, 2$ and 3. It is well known that GKDV equation has the single soliton solution
\[ u(x,t) = \left[ \frac{c(p+1)(p+2)}{2e} \sec^2 k(x - x_0 - ct) \right]^{1/p}, \quad (19) \]

where \( k = (p/2)\sqrt{c/\mu} \).

When we put \( p = 1 \) we get the KdV equation in [1] and by ADM where \( c = 1.3, \varepsilon = 3, \) and \( x_0 = 7.5\sqrt{c} \) we get

\[ u_0(x,t) = 1.3 \text{sech}^2[8.55132 - 0.570088x], \]

\[ u_1(x,t) = t(-1.9269 \text{sech}^4[8.55132 - 0.570088x] \tanh[8.55132 - 0.570088x] - 1.9269 \text{sech}^2[8.55132 - 0.570088x] \tan h[8.55132 - 0.570088x]), \]

\[ u_2(x,t) = 0.0223133r^2 \text{sech}^8[8.55132 - 0.570088x](-16 - 2.40891 \times 10^8 \cosh[1.14018x]) - 13.9984 \cosh[2.28035x] + 1.9175 \times 10^{22} \cosh[3.42053x] + 2.40891 \times 10^8 \sinh[1.14018x] + 13.9984 \sinh[2.28035x] - 1.9175 \times 10^{22} \sinh[3.42053x]), \]

\[ u_3(x,t) = -0.00378966r^3 \text{sech}^{11}[8.55132 - 0.570088x](-2.11645 \times 10^8 \cosh[0.570088x] + 2.02992 \times 10^6 \cosh[0.570088x] - 4.02833 \times 10^{12} \cosh[1.71026x] - 5.39105 \times 10^{19} \cosh[2.85044x] - 1.80369 \times 10^{36} \cosh[3.99061x] + 9.6554 \times 10^{32} \cosh[5.13079x] + 2.11645 \times 10^6 \sinh[0.570088x] - 2.02992 \times 10^6 \sinh[0.570088x] + 4.02833 \times 10^{12} \sinh[1.71026x] + 5.39105 \times 10^{19} \sinh[2.85044x] + 1.80369 \times 10^{36} \sinh[3.99061x] - 9.6554 \times 10^{32} \sinh[5.13079x]), \]

\[ u_4(x,t) = 0.0578456r^4 \text{sech}^{14}[8.551315688243534 - 0.5700877125495691x] \times \text{sech}^{12}[8.551315688243537 - 0.5700877125495689x] \times (\cosh[22334x] + \sinh[22334x])(-7.39866 \times 10^{-8} \cosh[22332.9x] + \cosh[22334x] + 2.34491 \times 10^7 \cosh[22335.2x] + 4.15927 \times 10^{14} \cosh[22336.3x] + 5.2526 \times 10^{21} \cosh[22337.5x] + 3.73804 \times 10^{28} \cosh[22338.6x] - 1.7707 \times 10^{35} \cosh[22339.7x] - 7.33238 \times 10^{42} \cosh[22340.9x] - 1.06437 \times 10^{50} \cosh[22342x] - 9.06476 \times 10^{56} \cosh[22343.2x] - 4.7598 \times 10^{63} \cosh[22344.3x] - 1.34212 \times 10^{70} \cosh[22345.4x] - 8.16424 \times 10^{75} \cosh[22346.6x] + 3.64203 \times 10^{82} \cosh[22347.7x] + 7.39866 \times 10^{-8} \sinh[22332.9x] - \sinh[22334x] - 2.34491 \times
\[ x \times 10^7 \sinh[22335.2x] - 4.15927 \times 10^{14} \sinh[22336.3x] \\
- 5.2526 \times 10^{21} \sinh[22337.5x] - 3.73804 \times 10^{28} \sinh[22338.6x] \\
+ 1.7707 \times 10^{35} \sinh[22339.7x] + 7.33238 \times 10^{42} \sinh[22340.9x] \\
+ 1.06437 \times 10^{50} \sinh[22342.2x] + 9.06476 \times 10^{56} \sinh[22343.2x] \\
+ 4.7598 \times 10^{63} \sinh[22344.3x] + 1.34212 \times 10^{70} \sinh[22345.4x] \\
+ 8.16424 \times 10^{75} \sinh[22346.6x] - 3.64203 \times 10^{82} \sinh[22347.7x], \]

and make comparison between the exact solution and ADM where

\[
u_{\text{Approximation}}(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + u_4(x,t),
\]

\[ t = 0.01 \text{ and } p = 1 \text{ in the following Table 1.} \]

From Table 1, we find that the ADM has a small error.

The conservation laws for GKDV where \( p = 1 \) (\( I_1, I_2, I_3 \) and \( I_4 \)) are given in Table 2.

In Fig. 1 the graph shows the approximation solution of KDV equation (GKDV, \( p = 1 \)) for \( t = 0 \), and \( 0 \leq x \leq 30 \).

In Fig. 2 the surface shows the approximation solution of KDV equation (GKDV, \( p = 1 \)) for \(-4 \leq t \leq 4 \) and \( 0 \leq x \leq 30 \).

When we put \( (p = 2) \) we get the modified Korteweg–De-Vries (MKDV) equation in [2] and by ADM where \( c = 1, \mu = 1, \varepsilon = 6, \) and \( x_0 = 0 \) we get

Table 1
Comparison between the exact solution and approximation solution (ADM) and the absolute error for both of them

<table>
<thead>
<tr>
<th>( x )</th>
<th>Exact solution</th>
<th>Approximation solution</th>
<th>Absolute error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( 1.91420146 \times 10^{-7} )</td>
<td>( 1.9420146 \times 10^{-7} )</td>
<td>( 1.1554853 \times 10^{-18} )</td>
</tr>
<tr>
<td>5</td>
<td>( 5.725819 \times 10^{-5} )</td>
<td>( 5.725842 \times 10^{-5} )</td>
<td>( 2.31629096 \times 10^{-10} )</td>
</tr>
<tr>
<td>10</td>
<td>( 0.01701572016 )</td>
<td>( 0.1701386996 )</td>
<td>( 1.8501999 \times 10^{-6} )</td>
</tr>
<tr>
<td>15</td>
<td>( 1.29992860011 )</td>
<td>( 1.30000000000 )</td>
<td>( 7.139988 \times 10^{-5} )</td>
</tr>
<tr>
<td>20</td>
<td>( 0.01752423257 )</td>
<td>( 0.01752264488 )</td>
<td>( 1.8680775 \times 10^{-6} )</td>
</tr>
<tr>
<td>25</td>
<td>( 5.898095 \times 10^{-5} )</td>
<td>( 5.8974536 \times 10^{-5} )</td>
<td>( 6.4149581 \times 10^{-9} )</td>
</tr>
<tr>
<td>30</td>
<td>( 1.971797 \times 10^{-7} )</td>
<td>( 1.97158212 \times 10^{-7} )</td>
<td>( 2.1447330 \times 10^{-11} )</td>
</tr>
</tbody>
</table>

Table 2
Computed quantities \( I_1, I_2, I_3 \) and \( I_4 \) for the GKDV (\( p = 1 \)) by ADM

<table>
<thead>
<tr>
<th>Time</th>
<th>( I_1 )</th>
<th>( I_2 )</th>
<th>( I_3 )</th>
<th>( I_4 )</th>
</tr>
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<tbody>
<tr>
<td>0.0</td>
<td>4.5607</td>
<td>3.95261</td>
<td>3.08303</td>
<td>65.8372</td>
</tr>
<tr>
<td>0.1</td>
<td>4.5607</td>
<td>3.95261</td>
<td>3.08303</td>
<td>65.8372</td>
</tr>
<tr>
<td>0.2</td>
<td>4.5607</td>
<td>3.95261</td>
<td>3.08303</td>
<td>65.8372</td>
</tr>
<tr>
<td>0.3</td>
<td>4.5607</td>
<td>3.95261</td>
<td>3.08303</td>
<td>65.8372</td>
</tr>
<tr>
<td>0.4</td>
<td>4.5607</td>
<td>3.95261</td>
<td>3.08303</td>
<td>65.8372</td>
</tr>
<tr>
<td>0.5</td>
<td>4.5607</td>
<td>3.95261</td>
<td>3.08303</td>
<td>65.8372</td>
</tr>
</tbody>
</table>
\[ u_0(x, t) = \text{sech}[x], \]

\[ u_1(x, t) = t \text{sech}[x] \tanh[x], \]

\[ u_2(x, t) = 0.25r^2 \text{sech}^3[x](-3 + \cosh[2x]), \]

\[ u_3(x, t) = (1/48)r^3 \text{sech}^5[x] \tanh[x](27 - 60 \cosh[x] - 68 \cosh[2x] + 12 \cosh[3x] + \cosh[4x]), \]

Fig. 1. The graph shows the approximation solution of KDV equation for \( t = 0 \) and \( 0 \leq x \leq 30 \).

Fig. 2. The surface shows the approximation solution of KDV equation for \( -4 \leq t \leq 4 \) and \( 0 \leq x \leq 30 \).
\[ u_4(x, t) = (1/3072)t^4 \sech^9(x)(189603 - 58080 \cosh(x) - 214888 \cosh(2x) + 54432 \cosh(3x) + 36508 \cosh(4x) - 7200 \cosh(5x) - 1368 \cosh(6x) + 96 \cosh(7x) + \cosh(8x)), \]

and make comparison between the exact solution and ADM where

\[ u_{\text{Approximation}}(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + u_3(x, t) + u_4(x, t), \]

\[ t = 0.01 \] and \( p = 2 \) in the following Table 3.

From Table 3, we find that the ADM has a small error.

The Conservation laws for GKDV where \( p = 2 \) (\( I_1, I_2, I_3 \) and \( I_4 \)) are given in Table 4.

In Fig. 3 the graph shows the approximation solution of MKDV equation (GKDV, \( p = 2 \)) for \( t = 0 \) and \(-10 \leq x \leq 10\).

In Fig. 4 the surface shows the approximation solution of MKDV equation (GKDV, \( p = 2 \)) for \(-4 \leq t \leq 4 \) and \(-10 \leq x \leq 10\).

When we put \( (p = 3) \) we get the GkDV equation and by ADM, where \( c = 0.6, \mu = 1, \varepsilon = 6, \) and \( x_0 = 15 \) we get

\[ u_0(x, t) = \sech^{2/3}\left(\frac{3}{2}0.6(x - 15)\right), \]

### Table 3

<table>
<thead>
<tr>
<th>( x )</th>
<th>Exact solution</th>
<th>Approximation solution</th>
<th>Absolute error</th>
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</thead>
<tbody>
<tr>
<td>(-10)</td>
<td>(8.98964 \times 10^{-5})</td>
<td>(8.98964 \times 10^{-5})</td>
<td>(8.00279 \times 10^{-15})</td>
</tr>
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<td>(-8)</td>
<td>(6.64249 \times 10^{-4})</td>
<td>(6.64249 \times 10^{-4})</td>
<td>(4.40016 \times 10^{-13})</td>
</tr>
<tr>
<td>(-4)</td>
<td>(0.0362548691467)</td>
<td>(0.036254867929)</td>
<td>(1.21889 \times 10^{-9})</td>
</tr>
<tr>
<td>(-2)</td>
<td>(0.2632512096)</td>
<td>(0.263251183311)</td>
<td>(2.62908 \times 10^{-8})</td>
</tr>
<tr>
<td>(0)</td>
<td>(0.99995000208)</td>
<td>(0.9999499708)</td>
<td>(4.99992 \times 10^{-9})</td>
</tr>
<tr>
<td>(2)</td>
<td>(0.2683760234)</td>
<td>(0.26837609578)</td>
<td>(2.33828 \times 10^{-8})</td>
</tr>
<tr>
<td>(4)</td>
<td>(0.0369867699)</td>
<td>(0.03698677117)</td>
<td>(1.25795 \times 10^{-9})</td>
</tr>
<tr>
<td>(8)</td>
<td>(6.776680898 \times 10^{-4})</td>
<td>(6.776680893 \times 10^{-4})</td>
<td>(4.57938 \times 10^{-13})</td>
</tr>
<tr>
<td>(10)</td>
<td>(9.1712413092 \times 10^{-7})</td>
<td>(9.17124130999 \times 10^{-7})</td>
<td>(8.33213 \times 10^{-15})</td>
</tr>
</tbody>
</table>

### Table 4

<table>
<thead>
<tr>
<th>Time</th>
<th>(I_1)</th>
<th>(I_2)</th>
<th>(I_3)</th>
<th>(I_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.0)</td>
<td>3.14159</td>
<td>2</td>
<td>0.666667</td>
<td>0.126032</td>
</tr>
<tr>
<td>(0.1)</td>
<td>3.14159</td>
<td>1.9997</td>
<td>0.66632</td>
<td>0.126032</td>
</tr>
<tr>
<td>(0.2)</td>
<td>3.14159</td>
<td>1.99949</td>
<td>0.666076</td>
<td>0.126031</td>
</tr>
<tr>
<td>(0.3)</td>
<td>3.14159</td>
<td>1.99869</td>
<td>0.666066</td>
<td>0.126031</td>
</tr>
<tr>
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<td>3.14159</td>
<td>1.99838</td>
<td>0.666016</td>
<td>0.126030</td>
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<tr>
<td>(0.5)</td>
<td>3.14159</td>
<td>1.99809</td>
<td>0.666006</td>
<td>0.126092</td>
</tr>
</tbody>
</table>
\[ u_1(x,t) = \frac{3}{5} \sqrt[3]{0.6t} \operatorname{sech}^{5/3} \left( \frac{3}{2} \sqrt[3]{0.6(x - 15)} \right) \sinh \left( \frac{3}{2} \sqrt[3]{0.6(x - 15)} \right), \]

\[ u_2(x,t) = \frac{27}{500} \left( r^3 \operatorname{sech}^{8/3} \left( \frac{3}{2} \sqrt[3]{0.6(x - 15)} \right) \right) \left( 4 - \cosh \left( 3 \sqrt[3]{0.6(x - 15)} \right) \right), \]

\[ u_3(x,t) = \frac{27 \sqrt{0.6}}{20000} \left( r^7 \operatorname{sech}^{17/3} \left( \frac{3}{2} \sqrt[3]{0.6(x - 15)} \right) \right) \left( -758 \sinh \left( \frac{3}{2} \sqrt[3]{0.6(x - 15)} \right) \right) \]

\[ + 203 \sinh \left( \frac{9}{2} \sqrt[3]{0.6(x - 15)} \right) + \sinh \left( \frac{3}{2} \sqrt{15(x - 15)} \right) \].

Fig. 3. The graph shows the approximation solution of MKDV equation for \( t = 0 \) and \(-10 \leq x \leq 10\).

Fig. 4. The surface shows the approximation solution of MKDV equation for \(-4 \leq t \leq 4\) and \(-10 \leq x \leq 10\).
\[ u_4(x, t) = -\frac{243}{16000000} \left( t^4 \operatorname{sech}^{26/3} \left[ \frac{3}{2} \sqrt{0.6(x - 15)} \right] \right) \left( -2875665 + 3219906 \times \cosh \left[ 3\sqrt{0.6(x - 15)} \right] - 513232 \cosh \left[ 6\sqrt{0.6(x - 15)} \right] + 15198 \times \cosh \left[ 9\sqrt{0.6(x - 15)} \right] + \cosh \left[ 12\sqrt{0.6(x - 15)} \right] \right), \]

and make comparison between the exact solution and ADM where

\[ u_{\text{Approximation}}(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + u_3(x, t) + u_4(x, t), \]

\[ t = 0.01 \text{ and } p = 3 \text{ in the following Table 5.} \]

From Table 5, we find that the ADM has a small error.

The quantities \( I_1, I_2, \) and \( I_3 \) for \( p = 3 \) are given in Table 6.

The conservation laws for GKDV where \( p = 3 \) (\( I_1, I_2, \) and \( I_3 \)) are given in Table 6.

In Fig. 5 the graph shows the approximation solution of GKDV equation \((p = 3)\) for \( t = 0 \) and \(-10 \leq x \leq 30\).

In Fig. 6 the surface shows the approximation solution of GKDV equation \((p = 3)\) for \(-4 \leq t \leq 4 \) and \(-10 \leq x \leq 30\).

### Table 5

Comparison between the exact solution and approximation solution (ADM) and the absolute error for both of them

<table>
<thead>
<tr>
<th>( x )</th>
<th>Exact solution</th>
<th>Approximation solution</th>
<th>Absolute error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.42109253510941 \times 10^{-5}</td>
<td>1.42109253510944 \times 10^{-5}</td>
<td>2.558039 \times 10^{-19}</td>
</tr>
<tr>
<td>5</td>
<td>6.8334127534582 \times 10^{-4}</td>
<td>6.8334127534584 \times 10^{-4}</td>
<td>1.2034644 \times 10^{-17}</td>
</tr>
<tr>
<td>10</td>
<td>0.0328587003323</td>
<td>0.0328587003312</td>
<td>1.1029511 \times 10^{-12}</td>
</tr>
<tr>
<td>15</td>
<td>0.99998380026</td>
<td>0.99998377664</td>
<td>2.3619595 \times 10^{-8}</td>
</tr>
<tr>
<td>20</td>
<td>0.033165545569</td>
<td>0.033165545570</td>
<td>1.2801774 \times 10^{-12}</td>
</tr>
<tr>
<td>25</td>
<td>6.8972265387632 \times 10^{-4}</td>
<td>6.8972265387631 \times 10^{-4}</td>
<td>1.366095 \times 10^{-17}</td>
</tr>
<tr>
<td>30</td>
<td>1.43436339949599 \times 10^{-5}</td>
<td>1.43436339949597 \times 10^{-5}</td>
<td>2.5072175 \times 10^{-19}</td>
</tr>
</tbody>
</table>

### Table 6

Computed quantities \( I_1, I_2, \) and \( I_3 \) for the GKDV \((p = 3)\) by ADM

<table>
<thead>
<tr>
<th>Time</th>
<th>( I_1 )</th>
<th>( I_2 )</th>
<th>( I_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>3.62042</td>
<td>2.226629</td>
<td>0.857676</td>
</tr>
<tr>
<td>0.1</td>
<td>3.62042</td>
<td>2.226629</td>
<td>0.857674</td>
</tr>
<tr>
<td>0.2</td>
<td>3.62042</td>
<td>2.226628</td>
<td>0.857670</td>
</tr>
<tr>
<td>0.3</td>
<td>3.62042</td>
<td>2.226626</td>
<td>0.857599</td>
</tr>
<tr>
<td>0.4</td>
<td>3.62042</td>
<td>2.226625</td>
<td>0.857580</td>
</tr>
<tr>
<td>0.5</td>
<td>3.62042</td>
<td>2.226619</td>
<td>0.857509</td>
</tr>
</tbody>
</table>
Nonlinear GKDV equation appears in various fields such as solid-state physics, plasma physics, fluid physics and quantum field theory. The computations associated with the GKDV equation discussed above were performed by using Mathematica 4. The goal to obtain solitary wave solution for the GKDV equation by using ADM has been achieved. The method has been applied directly without using bilinear forms, Wronskian, or inverse scattering method. The solutions presented in this study can be effectively used to examine the related problems of scientific applications. The obtained results demonstrate the reliability of the algorithm and it a wider applicability to nonlinear evolution equations.

6. Conclusion

Nonlinear GKDV equation appears in various fields such as solid-state physics, plasma physics, fluid physics and quantum field theory. The computations associated with the GKDV equation discussed above were performed by using Mathematica 4. The goal to obtain solitary wave solution for the GKDV equation by using ADM has been achieved. The method has been applied directly without using bilinear forms, Wronskian, or inverse scattering method. The solutions presented in this study can be effectively used to examine the related problems of scientific applications. The obtained results demonstrate the reliability of the algorithm and it a wider applicability to nonlinear evolution equations.
References