On the Attitude Recovery of an Underactuated Spacecraft Using Two Control Moment Gyroscopes

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Abstract—In recent years there has been a growing interest in development and utilization of small satellites (smallsats) for scientific missions using single or multiple spacecraft (SC). Due to their small size and low mass, it is not generally possible to equip smallsats with redundant actuators. In case of a failure in the SC’s attitude control actuators, one has to reconfigure the control laws in order to maintain the SC’s attitude by using only the remaining healthy actuators. In this work, an algorithm that is based on nonholonomic control theory and hybrid control systems is presented for recovery of an underactuated axis-symmetric SC (having only two control torque channels). Our goal is that the recovered SC, in addition to being stabilized, can slew to a desired attitude. Using Lyapunov analysis, the asymptotic stability of the system is guaranteed by employing the proposed control law. Simulations are performed to demonstrate the performance capabilities of the recovered system.

1. INTRODUCTION

The problem of re-orientation and stabilization of spacecraft (SC) has been studied extensively in the literature in the past two decades. For example, necessary and sufficient conditions for controllability of the SC was provided in [1]. The control of SC rest-to-rest maneuvers under slew rate and control constraints have also been studied and global asymptotic stability was guaranteed in [2]. In recent years, there has been a great deal of interest in the development of small satellites (smallsats) for space exploration, Earth observation and other missions. Examples include MOST (Microvariability and Oscillations of STars), Canada’s first and only space telescope (it is also the smallest space telescope in the world), launched on June 30, 2003 [3]; BilSat-1, Turkey’s low-cost, agile, Earth observation microsatellite [4]; UoSAT-12, a mini-satellite for low cost Earth observation [5], among others.

Due to the low mass and volume of smallsats, it is not feasible to use redundant actuators in the attitude control subsystem of these SC. Therefore, in case of an actuator failure, the SC may be left with less than three independent actuators. In other words, the SC behaves as an underactuated system with only two available control inputs (this has actually occurred in the UoSAT-12 and many other SC [6], [7]).

Design of stabilizing feedback controllers for underactuated SC has been a subject of numerous studies in the past two decades. Specifically, it was shown in [8] that there exists no continuous time-invariant feedback control law that can stabilize an underactuated rigid body system. The authors proposed a feedback control law which locally asymptotically drives the rigid body to a motion around one of the axes. Recently, [9] proposed a switched control strategy for attitude stabilization of an underactuated SC, which guarantees the global stability of the closed-loop system. While the above-mentioned studies are concerned with SC actuated by thrusters (external torques), the study of an underactuated SC with momentum exchange actuators is not extensive in the literature. The following studies discuss the stabilization of an underactuated SC with reaction wheels [7], [10], [11], [12].

In contrast to reaction wheels and thrusters, control moment gyroscopes (CMGs) have very high torque capacity, which can be used to increase the maneuverability and decrease the mass of a smallsat. The dynamics and control of SC with four CMGs in pyramidal configuration was presented in [13]. In a very recent study, the effects of failure of one of the CMGs in the pyramidal configuration were studied in [18], where the authors have shown that failure of one of the actuators will not make the SC an underactuated system and three-axis controller design was still possible.

In this paper, a control law for attitude recovery of an underactuated symmetric SC is presented. By recovery, we mean that the resulting underactuated SC, in addition to being stabilized, can slew to a desired attitude. It is shown in this paper that there exists no continuous time-invariant controller for global asymptotic stabilization of underactuated SC (with two control channels). Thus, a control algorithm for attitude recovery of the SC based on nonholonomic control theory and hybrid systems theory is presented. The global asymptotic stability of the attitude error of the closed-loop system is guaranteed by invoking Lyapunov analysis. In addition, the dynamics of the CMGs is considered in the controller design for recovery of the SC attitude. The above represents our main contributions, which have not been developed in previous studies in the literature. It is shown by simulations that the responses of the closed-loop system as well as the control efforts are quite smooth and possess no oscillations.
II. DYNAMICS OF A CONTROL MOMENT GYROSCOPE (CMG)

In this section, dynamic equations of a single gimbal control moment gyro (CMG) is developed. The plant, shown in Fig. 1, consists of a high inertia rotor suspended in an assembly with four angular degrees of freedom \( (\chi_1, \chi_2, \chi_3, \chi_4) \). The rotor spin torque \( (\tau_1) \) is provided by a dc motor (motor \#1). The first transverse gimbal assembly (body C) is driven by another dc motor (motor \#2) that applies a torque to axis 2 \((\tau_2)\). In this work, we shall assume that the gimbal axis 3 is locked so that bodies A and B become one rigid body. We now state our first assumption:

Assumption 1: The third gimbal axis is locked at \( \chi_3 = 0 \). Consequently, we have \( \dot{\xi}_3 = 0 \) and \( \ddot{\xi}_3 = 0 \).

In view of the above assumption, the resulting plant is useful for demonstration of gyroscopic torque action where the position and the rate \( \chi_4 \) and \( \dot{\xi}_4 \), respectively may be controlled by the second rotating angle \( \chi_2 \) through motor \#2 while the rotor is spinning.

Taking the coordinate frame definition from Fig. 1 and considering Assumption 1, a nonlinear model for the considered CMG can be obtained as follows:

\[
\begin{align*}
\dot{\xi}_1 &= \frac{J_D \sin(\chi_2)}{\beta_1 + \beta_2 \sin^2(\chi_2)} \xi_1 + \left( \frac{\beta_3 \sin^2(\chi_2)}{\beta_1 + \beta_2 \sin^2(\chi_2)} - 1 \right) \xi_4 \cos(\chi_2) \xi_2 \\
&\quad + \frac{1}{J_D + \beta_1 + \beta_2 \sin^2(\chi_2)} \tau_1 \\
\dot{\xi}_2 &= \frac{J_D}{I_c + I_D} \cos(\chi_2) \xi_1 \xi_4 \\
&\quad + \frac{I_c \sin(\chi_2) - K_C - I_D \sin(\chi_2) - J_D \cos(\chi_2)}{I_c + I_D} \xi_2^2 \\
&\quad + \tau_2 \frac{I_c + I_D}{I_c} \\
\dot{\xi}_4 &= -\frac{J_D \cos(\chi_2)}{\beta_1 + \beta_2 \sin^2(\chi_2)} \xi_1 \xi_2 - \frac{\beta_1 \sin(\chi_2)}{\beta_1 + \beta_2 \sin^2(\chi_2)} \cos(\chi_2) \xi_2 \xi_4 \\
&\quad - \frac{\sin(\chi_2)}{\beta_1 + \beta_2 \sin^2(\chi_2)} \tau_1
\end{align*}
\]

where \( \tau_i \ (i = 1, 2) \) is the torque applied by the \( ith \) motor; \( I_n, J_n, K_n \ (n = A, B, C, \text{and } D) \) are constant scalar moments of inertia about the system’s different axis for bodies A, B, C, and D. Additionally,

\[ \beta_1 = I_D + K_A + K_B + K_C, \quad \beta_2 = J_C - I_D - K_C, \]

and

\[ \beta_3 = [2(K_C - J_C + I_D) - J_D]. \]

The following assumptions are now stated for the remainder of the paper.

Assumption 2: The input torque \( \tau_1 \) is used to regulate the angular speed of the rotor \( \dot{\xi}_1 \). Therefore, the rotor angular speed is considered to be constant and equal to \( \Omega \) and the effect of the input torque \( \tau_1 \) on \( \dot{\xi}_4 \) is considered as a disturbance.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{Control Moment Gyroscope (CMG) system with 2 inputs and 4 outputs.}
\end{figure}

Assumption 3: The gimbal rate \( \dot{\xi}_2 \) is small compared to the rotor angular speed such that it does not contribute to the total angular momentum of the SC [20].

Based on the assumptions above, the linearized model for the CMG about \( \chi_2 = 0 \) equilibrium point is obtained as:

\[
\begin{align*}
\dot{\xi}_1 &= \frac{1}{J_D} \tau_1 \\
\frac{\dot{\xi}_4}{\dot{\xi}_2} &= \left[ \begin{array}{cc}
0 & -I_D \Omega \\
I_C + K_A + K_B + K_C & 0
\end{array} \right] \frac{\xi_4}{\xi_2} + \left[ \begin{array}{c}
\frac{1}{I_c + I_D}
\end{array} \right] \tau_2
\end{align*}
\]

III. MODELING OF THE SPACECRAFT WITH TWO CONTROL MOMENT GYROS

A. Dynamics of the Spacecraft

Assuming a rigid SC with momentum exchange actuators, such as CMGs, the rotational dynamic equations of motion are governed by [13]:

\[ \mathbf{H}_s + \omega^\times \mathbf{H}_s = \mathbf{T}_{ext} \]

where \( \mathbf{H}_s \) is the angular momentum vector of the system expressed in the SC body-fixed control axes and \( \mathbf{T}_{ext} \) is the external torque vector (e.g. gravity gradient, solar pressure and aerodynamic torques) expressed in the same body axes; and \( \omega = (\omega_1, \omega_2, \omega_3) \) is the SC angular velocity vector. Additionally, \( \omega^\times \) represents the cross product matrix, i.e.:

\[ \omega^\times = \begin{bmatrix}
0 & -\omega_3 & \omega_2 \\
\omega_3 & 0 & -\omega_1 \\
-\omega_2 & \omega_1 & 0
\end{bmatrix} \]

The total angular momentum vector consists of the SC main body angular momentum and the CMGs angular momentum:

\[ \mathbf{H}_s = \mathbf{J} \omega + \mathbf{h} \]

where \( \mathbf{J} \) is the inertia matrix of the SC including CMGs and \( \mathbf{h} = (h_1, h_2, h_3) \) is the total CMG momentum vector, all expressed in the SC body-fixed control axis [13].

Combining (6) and (7) we obtain:

\[ (\mathbf{J} \dot{\omega} + \dot{\mathbf{h}}) + \omega^\times (\mathbf{J} \omega + \mathbf{h}) = \mathbf{T}_{ext} \]
We introduce the control torque vector generated by the CMGs as \( u = (u_1, u_2, u_3) \). Consequently, (8) can be re-written as:

\[
\begin{align*}
J \dot{\omega} + \omega \times J \omega &= u + T_{ext} \\
h + \omega \times h &= -u
\end{align*}
\]

(9) \hspace{10cm} (10)

**B. Kinematic Model of the Spacecraft**

We will use the Gibbs vector (also known as the Rodriguez Parameters (RP))\(^1\) for presentation of the attitude kinematics. The Gibbs vector is defined as [13]:

\[
g = \frac{1}{2} \left[ 1 + g \times + g \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right] \omega
\]

\[
= \frac{1}{2} \begin{bmatrix}
1 + g_1^2 & g_1g_2 - g_3 & g_1g_3 + g_2 \\
g_1g_2 + g_3 & 1 + g_2^2 & g_2g_3 - g_1 \\
g_1g_3 - g_2 & g_2g_3 + g_1 & 1 + g_3^2
\end{bmatrix} \omega
\]

(12)

where \( g = (g_1, g_2, g_3) \); \( e = (e_1, e_2, e_3) \) is the Euler’s eigenaxis vector, and \( \theta \) is the associated rotational angle. Kinematic differential equations of the Gibbs vector can be described as [13]:

\[
g = \frac{1}{2} \left[ 1 + g \times + g \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right] \omega
\]

(11)

where \( I \) is the identity matrix having proper dimensions.

**C. Governing SC Equations for the Tracking Error**

To develop the equations governing the tracking error, we need to define three reference frames. Let \( \mathbb{N} \) be the inertial frame, \( \mathbb{B} \) be the body frame, and \( \mathbb{C} \) be the command frame (commanded motion). The reference Gibbs vector is denoted by \( g_c \) and the angular acceleration and its derivative in this frame is denoted by \( \omega_c \) and \( \dot{\omega}_c \). We represent unit vector triads in these frames by \( \hat{n}, \hat{b}, \hat{c} \), respectively. Furthermore, the direction cosine matrices among these frames are parameterized by \( g_b, g_c, \) and \( g_e \) as follows:

\[
\begin{align*}
\mathbb{N} \rightarrow \mathbb{B} & \Rightarrow \hat{b} = \mathcal{C}(g_b)\hat{n} \\
\mathbb{N} \rightarrow \mathbb{C} & \Rightarrow \hat{c} = \mathcal{C}(g_e)\hat{n} \\
\mathbb{C} \rightarrow \mathbb{B} & \Rightarrow \hat{b} = \mathcal{C}(g_b)\hat{c}
\end{align*}
\]

(13)

where \( \mathcal{C} \) is the direction cosine matrix on the Gibbs vector, defined as [13]:

\[
\mathcal{C}(g) = \left[ 1 - g \times \right]^{-1}
\]

(14)

To find the relation between the transformations given above, we write:

\[
\hat{b} = \mathcal{C}(g_c)\hat{c} = \mathcal{C}(g)\mathcal{C}(g_c)\hat{n}
\]

\[
= \mathcal{C}(g_b)\hat{n}
\]

Consequently, we have:

\[
\mathcal{C}(g_b) \equiv \mathcal{C}(g)\mathcal{C}(g_c)
\]

and by noting \( \mathcal{C}^T = \mathcal{C}^{-1} \), we obtain:

\[
\mathcal{C}(g_b) = \mathcal{C}(g_c)\mathcal{C}^{-1}(g_c)
\]

(15)

The attitude error is parameterized by \( g \); although despite the fact that \( g \neq g_b - g_c \), it was shown in [14], [15] that \( g_{bc} \rightarrow g_c \) also implies \( g \rightarrow 0 \). Thus, for the purpose of representing the attitude error, both vectors \( g \) and \( g_b - g_c \) are equivalently valid. However, the use of the vector \( g \) greatly simplifies the mathematics and accompanying algebra required for construction of the controller.

The angular velocity error, \( \omega_{e} \), is defined as:

\[
\omega_e = \omega - \omega_{e}^b = \omega - \mathcal{C}(g)\omega_c
\]

(16)

where \( \omega^b \) is the commanded angular velocity in the body frame and \( \omega_c \) is the commanded angular velocity on the command frame.

The first time derivative of \( \omega_e \) is required in the development of the attitude tracking controller. It is defined as:

\[
\dot{\omega}_e = \dot{\omega} - \mathcal{C}(g)^{\times}\omega_c + \omega_{e}^c\mathcal{C}(g)\dot{\omega}_c
\]

(17)

which is obtained by nothing that \(^2\):

\[
\mathcal{C}(g)\dot{\omega}_c = -\omega_{e}^c\mathcal{C}(g)\omega_c
\]

(18)

Considering (9), (12), (17) and ignoring the effects of the external torques, the governing equations for the attitude error vector \( g \) and the angular velocity error \( \omega_e \) are given as follows:

\[
\dot{g} = -\omega_{e}^{\times}\mathcal{C}(g)\omega_c
\]

(19)

where the input torque \( u \) is defined in (10).

**IV. FEEDBACK CONTROL DESIGN**

The design of stabilizing and tracking controllers for the SC using the Gibbs parameters and modified rodriguez parameters has been the subject of several papers in the past ten years; e.g. see [14], [15], [16]. In all of these works, the authors have assumed that the SC is a fully actuated system, and the designed controllers are derived based on this assumption. However, in case of an attitude actuator failure the SC will no longer remain fully actuated. Consequently, the controller must be redesigned when an actuator fails. The stabilization of an underactuated SC using the Euler angles presentation is studied in [8] and [10], using quaternion presentation in [9], using Gibbs vector in [7], [11], and using Tsiotras-Longusky representation in [12]. However, design of an attitude recovery in the sense that the underactuated SC has to slew to a desired attitude expressed in the Gibbs vector, which is the subject of this paper, has not been addressed in the literature before.

We are now in a position to state our assumptions:

\(^1\)For more information on the Gibbs vector and its relation to the quaternion representation of the SC attitude refer to the Appendix.

\(^2\)A proof of this fact is provided in the Appendix.
**Assumption 4:** Without loss of generality, the SC initially has three actuators (e.g. CMGs) aligned with its three principal axis and the failed actuator is aligned with the SC 2-axis; thus, the torque components of the underactuated SC are being projected along the x and y-axes. Based on this assumption we have:

\[ u_3 = 0 \]

**Assumption 5:** The initial angular momentum of the SC is zero. If the initial momentum is high, one needs to perform some independent ‘de-tumbling’ manoeuvre, e.g. using magnetorquers [7]. Specifically, this assumption implies that:

\[ [\omega_1(0), \omega_2(0), \omega_3(0)] = [0, 0, 0] \]

**Assumption 6:** The inertia matrix of the SC is diagonal, i.e. \( J = \text{diag}(J_1, J_2, J_3) \), where \( J_1 = J_2 = J \). The main significance of this assumption is that:

\[ \dot{\omega}_3 = 0 \]

**Assumption 7:** A rest-to-rest attitude maneuver is considered. This implies that the commanded angular velocity, \( \omega_c \) and its derivative, \( \dot{\omega}_c \) is zero, given that we are interested to slew to the commanded attitude position \( \omega_c \).

By invoking Assumption 4 to Assumption 7 and based on equations (10), (18) and (19), the following error dynamics is obtained, which describes the relationship between the SC input torque and the error attitude, namely:

\[
\begin{align*}
\dot{\mathbf{g}} &= \mathbf{\bar{R}}(\mathbf{g}) \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \\
\dot{\omega}_1 &= u_1/J \\
\dot{\omega}_2 &= u_2/J
\end{align*}
\]  
(20)

where

\[
\mathbf{\bar{R}}(\mathbf{g}) = \frac{1}{2} \begin{bmatrix}
1 + \frac{g_1^2}{g_2 + g_3} & g_1 g_2 - g_3 \\
g_1 g_2 - g_3 & 1 + \frac{g_2^2}{g_1 + g_3}
\end{bmatrix}
\]

(23)

For the commanded control torque input \( \mathbf{u} = [u_1, u_2]^T \), the CMG rate command \( \mathbf{h} = [h^x, h^y]^T \) is chosen as:

\[
\begin{align*}
\dot{h}^x &= -u_1 \\
\dot{h}^y &= -u_2
\end{align*}
\]  
(24)

(25)

where \( h^x \) is the angular momentum of the CMG aligned with the x-axis and \( h^y \) is the angular momentum of the CMG aligned with the y-axis. The angular momentum of the CMGs are determined by using the following equations:

\[
\begin{align*}
h^x &= K_A \xi_4^x \\
h^y &= K_A \xi_4^y
\end{align*}
\]  
(26)

(27)

where \( \xi_4 \) can be found from (5) for each actuator in either the x or the y axis.

In the remainder of this section we consider the problem of designing a state feedback controller that asymptotically stabilizes the equilibrium of the origin of the system (20)-(22). It will be shown that the system cannot be asymptotically stabilized to a desired equilibrium by using any time-invariant continuous feedback in the sense of Lyapunov.

**A. Stabilization of the Kinematic Error Equations**

A feedback scheme for control of an underactuated SC is introduced in this section. As the first step, we introduce the following coordinate transformation in order to simplify the SC attitude kinetics, namely

\[
\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} = \frac{2}{1 + \frac{g_1^2}{g_2 + g_3} + \frac{g_2^2}{g_1 + g_3}} \begin{bmatrix} 1 + \frac{g_1^2}{g_2 + g_3} & g_1 - g_3 g_2 \\
-(g_1 g_2 + g_3) & 1 + \frac{g_2^2}{g_1 + g_3} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}
\]

(28)

where \( \nu = [v_1, v_2]^T \) is a new input vector. Consequently, the SC kinematics (20) reduces to:

\[
\begin{align*}
\dot{g}_1 &= v_1 \\
\dot{g}_2 &= v_2 \\
\dot{g}_3 &= g_1 v_2 - g_3 v_1
\end{align*}
\]  
(29)

(30)

(31)

The system given above is a well-known Brockett’s non-holonomic (double) integrator [21]. This system has very interesting properties. For example, no matter what control law is used, whenever \( g_2 \) and \( g_3 \) are both zero, \( \dot{g}_1 \) will also be zero and \( g_1 \) will remain constant. Therefore, no continuous time-invariant state-feedback control law can make the null solution asymptotically stable in the sense of Lyapunov. Furthermore, whenever \( g_2 \) and \( g_3 \) are both “small”, only “large” control signals will be able to produce significant changes in \( \dot{g}_1 \).

The following definition will be used in the remainder of this paper.

**Definition 1** [22]: The system (29)-(31) is said to be globally asymptotically stable (GAS) by using a state-feedback controller:

\[
\nu = (v_1, v_2) = f(g_1, g_2, g_3)
\]

(32)

if it is stable at \( (g_1, g_2, g_3) = (0, 0, 0) \) and all the solutions \( (\bar{g}_1(t), \bar{g}_2(t), \bar{g}_3(t)) \) of the closed-loop system are bounded and well-defined over \([0, +\infty)\), where \( f \) is a nonlinear function. In addition,

\[
\lim_{t\to\infty} \bar{g}_i(t) = (0, 0, 0)
\]

(33)

**Remark 1:** It is important to note that the above definition of asymptotic stability does not quantify the speed of convergence of the trajectories to the origin.

One of main goals of this paper is to obtain the controller (32), which globally asymptotically stabilizes (regulates) the system (29)-(31). For this purpose, we first design a time-invariant feedback controller for the case when the states of the system is initially not equal to zero, i.e. \( \bar{g}_i(0) \) and \( \bar{g}_3(0) \) are not zero. Later on, the results will be extended to the case when this assumption does not hold.

**B. A Smooth Time-Invariant Feedback Control Strategy**

In this subsection, we focus on the design of the control signal \( \nu(t) = (v_1(t), v_2(t)) \) subject to \( \bar{g}_1(0) + \bar{g}_3(0) \neq 0 \). The case where this assumption is not valid will be considered in the next subsection.
Let us consider the following state and control coordinate transformations:
\[
\begin{align*}
g_1 &= r \cos(\varphi) \\
g_2 &= r \sin(\varphi) \quad (34) \\
\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix} \begin{bmatrix} \ddot{u} \\ \ddot{v} \end{bmatrix} \quad (35)
\end{align*}
\]
where \( \ddot{u} \) and \( \ddot{v} \) are the new control inputs, and for the new coordinate \((r,z)\) we have
\[
\begin{align*}
r(t) &= \sqrt{g_1^2(t) + g_2^2(t)} \\
z(t) &= g_3(t) \quad (36)
\end{align*}
\]
This transformation is used only when \( r^2(0) = g_1^2(0) + g_2^2(0) \neq 0 \) (this will be used in the proofs). In the new coordinates, the transformed equations (29)-(31) are represented by:
\[
\begin{align*}
\dot{r} &= \ddot{u} \quad (37) \\
\dot{\varphi} &= \ddot{v} \quad (38) \\
\dot{z} &= r^2 \ddot{v} \quad (39)
\end{align*}
\]
The convergence of the system (29)-(31) to zero is now equivalent to the convergence of \( r \) and \( z \) to zero. It is clear that \((r,z) = (0,0)\) implies \((g_1,g_2,g_3) = (0,0,0)\).

The following feedback control law is proposed for stabilizing the kinematic model (37) and (39), namely:
\[
\begin{align*}
\ddot{u} &= \bar{k}_1 \ r \ (k_2^2 z^2 - r^2) \quad (40) \\
\ddot{v} &= -\bar{k}_3 \ z \quad (41)
\end{align*}
\]
where \( \bar{k}_1, \bar{k}_2, \bar{k}_3 > 0 \) are design parameters which are chosen according to the following relation:
\[
\bar{k}_3 = 2\bar{k}_1 (1 + \bar{k}_2) \quad (42)
\]
The resulting closed-loop system is now governed by:
\[
\begin{align*}
\dot{r} &= \bar{k}_1 \ r \ (k_2^2 z^2 - r^2) \quad (43) \\
\dot{\varphi} &= -\bar{k}_3 \ z \quad (44) \\
\dot{z} &= -\bar{k}_3 \ r^2 \ z \quad (45)
\end{align*}
\]
The actual controls for the system, \( v_1 \) and \( v_2 \) are related to the new control variables \( \ddot{u} \) and \( \ddot{v} \) via (35).

Remark 2: The controller proposed above cannot stabilize the states of the system to the origin from the points where \( r(0) = 0 \). However, the system states arbitrary close to, but not exactly on, the manifold of equilibria \( r(t) = 0 \) can be asymptotically converged to the equilibrium state.

Remark 3: The controller (40), (41) is smooth, time-invariant and continuous on the subset \( \mathbb{R}^3 - \{(0,0,0)\} \subseteq \mathbb{R} \).

The following theorem is now presented to guarantee stability of the transformed SC kinematics (37) and (39) by using the control law (40) and (41).

**Theorem 1:** The transformed underactuated SC kinematics (37) and (39) under the control law (40) and (41) is stable in \( \mathbb{R}^3 - \{(0,0,0)\} \subseteq \mathbb{R} \).

**Proof:** Let us define a state-space \( \mathcal{B} \) such that \( \mathcal{B} = \{ (r,z) | r \in \mathbb{R}^+ - \{0\}, z \in \mathbb{R} \} \cup \{ (0,0) \} \). In this space define a Lyapunov function candidate \( \dot{V}(r,z) \) as follows:
\[
\dot{V} = \frac{1}{2} (r^2 + z^2) \quad (46)
\]
It is clear that \( \dot{V} \) is positive definite in \( \mathcal{B} \). \( \dot{V} = 0 \) implies \( (r,z) = (0,0) \). In addition \( ||(r,z)|| \to +\infty \) implies \( \dot{V} \to +\infty \). Consequently, \( \dot{V} \) is a radially unbounded function.

Differentiating the Lyapunov function candidate above along the trajectories of the closed-loop system (43) and (45), yields:
\[
\begin{align*}
\dot{V} &= r \dot{r} + z \dot{z} \\
&= r (\bar{k}_1 \ r (k_2^2 z^2 - r^2)) + z (-\bar{k}_3 \ r^2 z) \\
&= \bar{k}_1 (k_2^2 z^2 - r^2) - \bar{k}_3 r^2 z^2 \\
&= (\bar{k}_1 k_2^2 - \bar{k}_3) r^2 z^2 - \bar{k}_1 r^4
\end{align*}
\]
From (42), and after some algebraic manipulations we obtain:
\[
\dot{\dot{V}} = -(2\bar{k}_1 + \bar{k}_1 k_2^2) r^2 z^2 - \bar{k}_1 r^4 \quad (48)
\]
Thus, \( \dot{\dot{V}} \) is negative semi-definite in \( \mathcal{B} \). Using the Lyapunov theory we can conclude that \( \forall (r(0),z(0)) \in \mathcal{B} \) the closed-loop system (43) and (45) is stable. This completes the proof of the theorem.

**V. MAIN RESULTS**

In the previous section, we assumed \( r(0) \neq 0 \), and based on this assumption we developed a recovery procedure for an underactuated SC. In this subsection, we will deal with the situation when \( r(0) = 0 \).

A. Switching Strategy and Spacecraft Kinematics Stabilization

According to [23], the simplest way to deal with the situation when \( r(0) = 0 \) is to first use a nonzero constant input action \( v_1 = v_1^* \neq 0 \) to derive \( g_3(r) \) state away from zero. Then, at the time instant \( t = T_s > 0 \) switch \( v = (v_1, v_2) \) to a stabilizing controller which is based on the control law (40), (41). The time response of the system (29)-(31) under the control law \( v = (v_1^*,0) \), when \( r(0) = 0 \) is given by:
\[
\begin{align*}
g_3(r) &= v_1^* t, \quad (49) \\
g_1(r) &= g_2(r) = 0, \\
g_3(r) &= g_3^0
\end{align*}
\]
Consequently, it takes \( T_s = \frac{\sqrt{2}}{k_3} \) sec for the system to move from \( g_3(0) = 0 \) to \( g_3(T_s) = g_3^0 \).

Based on the discussion given above, the following switched controller is now proposed for stabilization of the underactuated SC kinematics, namely:
\[
\begin{align*}
v_1(t) &= \begin{cases} 
  v_1^* 
  & \text{if } r(0) = 0 \text{ and } 0 \leq t < T_s \\
  \bar{v}_1(t) & \text{else}
\end{cases} \quad (50) \\
v_2(t) &= \begin{cases} 
  0 
  & \text{if } r(0) = 0 \text{ and } 0 \leq t < T_s \\
  \bar{v}_2(t) & \text{else}
\end{cases} \quad (51)
\end{align*}
\]
where
\[ \bar{v}_1 = r \left( \bar{k}_1 \left( \bar{k}_2^2 z^2 - r^2 \right) \cos \varphi + \bar{k}_3 z \sin \varphi \right) \] (52)
\[ \bar{v}_2 = r \left( \bar{k}_1 \left( \bar{k}_2^2 z^2 - r^2 \right) \sin \varphi - \bar{k}_3 z \cos \varphi \right) \] (53)

Now we are ready to present the main result and theorem of this paper.

**Theorem 2:** Under the switched control law presented in (50) and (51) the underactuated SC error kinematics (29)-(31) is GAS in the sense that all the system’s trajectories satisfy the requirements of Definition 1.

**Proof:** In order to prove this theorem, we first consider the case when \( r(0) \neq 0 \). Without loss of generality, it is assumed that \( t \geq T_0 \). Consider the following Lyapunov function candidate:

\[ V' = \frac{1}{2} (g_1^2 + g_2^2 + g_3^2) \] (54)

It is clear that the function given above is a radially unbounded positive definite function and \( V' = 0 \) implies \( (g_1, g_2, g_3) = (0, 0, 0) \). The time derivative of the above function along the trajectories of the closed-loop system (29)-(31), (34), (36), (45), (52) and (53) is given by:

\[ V' = g_1 \bar{v}_1 + g_2 \bar{v}_2 + g_3 \bar{v}_3 \]
\[ = r^2 \left[ \bar{k}_1 \left( \bar{k}_2^2 z^2 - r^2 \right) \cos \varphi + \bar{k}_3 z \sin \varphi \right] \cos \varphi \]
\[ + \left[ \bar{k}_1 \left( \bar{k}_2^2 z^2 - r^2 \right) \sin \varphi - \bar{k}_3 z \cos \varphi \right] \sin \varphi \]
\[ - \bar{k}_3 r^2 z^2 \]
\[ = \bar{k}_1 r^2 \left( \bar{k}_2^2 z^2 - r^2 \right) - \bar{k}_3 r^2 z^2 \]
\[ = \bar{k}_1 \bar{k}_2^2 z^2 - \bar{k}_1 r^4 - \bar{k}_3 r^2 z^2 \]

Using (42) we obtain:

\[ V' = \bar{k}_1 \bar{k}_2^2 z^2 - \bar{k}_1 r^4 - 2\bar{k}_1 (1 + \bar{k}_2^2) r^2 z^2 \]
\[ = - \bar{k}_1 \bar{k}_2^2 z^2 - 2\bar{k}_1 r^2 z^2 - \bar{k}_1 r^4 \]
\[ = - \bar{k}_1 \bar{k}_2^2 z^2 + 2 z^2 + r^2 \leq 0 \] (56)

First observe that since \( V' \) is radially unbounded, all solutions are globally bounded. Consider now the set \( \mathcal{H} = \{ (g_1, g_2, g_3) : V' = 0 \} \). Trajectories in \( \mathcal{H} \) satisfy \( (g_1, g_2, g_3) = (0, 0, 0) \), which by definition (11) implies \( \theta = 0 \), noting that \( (e_1, e_2) \) cannot be zero since this would result in \( r(0) = 0 \). Hence, from definition (11), we have \( g_3 = 0 \) because \( \theta = 0 \). The largest invariant set in \( \mathcal{H} \) is therefore the set \( \mathcal{H}_0 = \{ (g_1, g_2, g_3) = (0, 0, 0) \} \). By invoking LaSalle’s theorem [22], and since \( V' \) is radially unbounded, the system is globally asymptotically stable. In particular, all the trajectories of the system asymptotically approach to \( \mathcal{H}_0 \).

Based on the discussion earlier in this subsection, the time response of the system when \( r(0) = 0 \) is bounded by:

\[ |g_i(t)| \leq \mathcal{G}, \] (57)
\[ |g_i(t)| = 0, \] (58)
\[ |g_i(t)| = \mathcal{G}, \quad \forall t < T_0, \] (59)

Consequently, the states of the system are well-defined and bounded for \( t > 0 \) under the control laws (50) and (51), and asymptotically convergence to zero as \( t \to +\infty \). This satisfies the condition (29) and completes the proof of the theorem.

**Remark 4:** The stability of the error angular velocity, \( \bar{\omega} \), follows from stability of the error kinematics. From (29), (30) it is clear that \( (\bar{g}_1, \bar{g}_2, \bar{g}_3) = (0, 0, 0) \) implies \( (\bar{v}_1, \bar{v}_2) = (0, 0) \). This in fact, with reference to (28), implies \( (\bar{\omega}_1, \bar{\omega}_2) = (0, 0) \), which means that the error angular velocity approaches to zero as error kinematics approaches to zero.

**B. Stabilization of the SC Dynamics**

Let us denote the requested angular velocity from the controller by \( \omega_{\delta 1} \) and \( \omega_{\delta 2} \), which respectively correspond to the commanded angular velocity in the \( x \) and \( y \) axes, respectively. The commanded angular velocity must be realized by an actuator, i.e. CMGs in our case. The angular momentum of the CMG is given by (26) and (27). From (21), (22) and (24)-(27) we have the following relations between the SC angular velocity and the CMG angular velocity:

\[ \omega_{\delta 1} = -\frac{K_a}{J_i} \bar{\xi}_1 \] (60)
\[ \omega_{\delta 2} = -\frac{K_a}{J_i} \bar{\xi}_2 \] (61)

where the superscripts \( x, y \) denote that the CMG is aligned with the \( x \) and \( y \) axes, respectively. Let us introduce \( e^x = \bar{\xi}_1 - \bar{\xi}_1^* \) and \( e^y = \bar{\xi}_2 - \bar{\xi}_2^* \), which are the CMG angular velocity tracking errors in the \( x \) and \( y \) directions, respectively. By assuming \( \bar{\xi}_1^* = \bar{\xi}_2^* = 0 \), and considering (4) the open-loop tracking error dynamics is obtained as follows:

\[ \begin{bmatrix} e^x \\ e^y \end{bmatrix} = \begin{bmatrix} 0 \\ -J_o \Omega \end{bmatrix} \frac{J_o \Omega}{I_c + K_b + K_c} \begin{bmatrix} e^x \\ e^y \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{J_o}{I_c + J_o} \end{bmatrix} \tau_2 \] (62)

where \( i = x, y \). A state-feedback controller is employed to stabilize the dynamics of the CMG as given below:

\[ \tau_2 = -K \xi' \] (63)

where \( \xi' = [\bar{\xi}_1, \bar{\xi}_2] \). The state-feedback gain \( K \) is chosen such that the closed-loop dynamics has a faster dynamics than the dynamics of the kinematic equations. This guarantees the stability of the dynamics of the SC.

**VI. CLOSED-LOOP SIMULATION RESULTS**

The effectiveness of our proposed and developed control technique is demonstrated next through simulations. The following numerical values for the SC are used in the simulations, which are based on MOST, namely: \( J = J_1 = J_2 = 2.5625 \text{kg} \cdot \text{m}^2, J_3 = 4.2250 \text{kg} \cdot \text{m}^2 \). The initial attitude error is set to \( \bar{\xi}(0) = [0, 0, -0.8]^T \).

The parameters for the controllers are found by trial and error based on simulations to find a reasonably good response. The parameters for the control law are: \( \bar{k}_1 = 0.05, \bar{k}_2 = 1.5 \). Hence, the third gain is found as \( \bar{k}_3 = 0.3250 \) according to (42). The control signal \( \bar{v}_1 \) is found according to the relation \( \bar{v}_1 = \bar{\omega}_1 \bar{\eta} \), where \( \bar{\eta} \) and \( T_0 \) are set to \(-0.5 \) and 100 sec, respectively. The state feedback gain for the
by using the designed controller is depicted in Fig. 2. It is clear that the system has a smooth response after the switch to the equilibrium point (zero error). The SC angular velocities are completely executable by the available CMGs on the SC task.

The rotor spinning rate, $\Omega$ (see (5)) is set to 400 rpm.

control law and the other two error angles remain bounded and converge asymptotically ($as t \to \infty$) to zero.

The reported simulations confirm the analytical results obtained in this paper and clearly show the effectiveness and performance of our introduced switched controller.

VII. CONCLUSIONS

The problem of attitude recovery of an underactuated spacecraft (SC) using only two control moment gyroscopes (CMGs) is considered in this paper. CMGs are commonly used for attitude control of small satellites (smallsats), which have a higher torque capability compared to other momentum exchange actuators, e.g. reaction wheels, momentum wheels, etc. As shown in this paper, the error kinematics of the SC with only two inputs (i.e. CMGs in our case) using the Gibbs vector parameterization can be transformed to the well-known Brockett’s nonholonomic double integrator form. It was shown in previous studies that this class of nonlinear systems cannot be stabilized by using a continuous time-invariant controller. Consequently, by using the theory of nonholonomic control and hybrid systems theory we have proposed and developed a controller for an underactuated SC using Lyapunov theory analysis. The designed controller shows in simulations very good performance for the required SC task.

REFERENCES


or commanded attitude quaternions \((q_{1c}, q_{2c}, q_{3c}, q_{4c})\) and the current attitude quaternions \((q_1, q_2, q_3, q_4)\) as follows:

\[
\begin{bmatrix}
q_{1c} \\
q_{2c} \\
q_{3c} \\
q_{4c}
\end{bmatrix} =
\begin{bmatrix}
q_{4c} & q_{3c} & -q_{2c} & -q_{1c} \\
-q_{3c} & q_{4c} & q_{1c} & -q_{2c} \\
q_{2c} & -q_{1c} & q_{4c} & -q_{3c} \\
q_{1c} & q_{2c} & q_{3c} & q_{4c}
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4
\end{bmatrix}
\]

(66)

The elements of Gibbs vector, \(g_i, \ i = 1, 2, 3\) can be expressed in terms of the four quaternion elements \(q_i, \ i = 1, 2, 3\) (or the error quaternion) as follows:

\[
g_i = \frac{q_i}{q_4}, \ i = 1, 2, 3
\]

(67)

It can be seen from the above equations that the Gibbs vector has a geometric singularity at \(\theta = \pm 180^\circ\), which corresponds to \(q_4 = 0\) in (67). Thus, any rotation less than a half revolution can be expressed by using these parameters. Note however that a single instance of half revolutions are generally not encountered in most attitude maneuvers.

Next, we intend to prove the following expression:

\[
\dot{\mathbf{\epsilon}}(\mathbf{g}) \mathbf{\omega}_e = -\omega \times \mathbf{\epsilon}(\mathbf{g}) \mathbf{\omega}_e
\]

From the analytical dynamics it is known that:

\[
\dot{\mathbf{\epsilon}}(\mathbf{g}_b) = -\omega \times \mathbf{\epsilon}(\mathbf{g}_b), \quad \dot{\mathbf{\epsilon}}(\mathbf{g}_e) = -\omega_e \times \mathbf{\epsilon}(\mathbf{g}_e)
\]

Based on (15), we can write:

\[
\dot{\mathbf{\epsilon}}(\mathbf{g}) = \frac{d}{dt}[\mathbf{\epsilon}(\mathbf{g}_b) \mathbf{\epsilon}_e^T(\mathbf{g}_e)]
\]

\[
= \dot{\mathbf{\epsilon}}(\mathbf{g}_b) \mathbf{\epsilon}_e^T(\mathbf{g}_e) + \dot{\mathbf{\epsilon}}(\mathbf{g}_e) \mathbf{\epsilon}_e^T(\mathbf{g}_b) - \omega \times \dot{\mathbf{\epsilon}}(\mathbf{g}_b) \mathbf{\epsilon}_e^T(\mathbf{g}_e) + \omega_e \times \dot{\mathbf{\epsilon}}(\mathbf{g}_e) \mathbf{\epsilon}_e^T(\mathbf{g}_b)
\]

\[
= -\omega \times \dot{\mathbf{\epsilon}}(\mathbf{g}_b) + \omega_e \times \dot{\mathbf{\epsilon}}(\mathbf{g}_e)
\]

(68)

Therefore, we obtain:

\[
\dot{\mathbf{\epsilon}}(\mathbf{g}) \mathbf{\omega}_e = -\omega \times \dot{\mathbf{\epsilon}}(\mathbf{g}) \mathbf{\omega}_e + \omega_e \times \dot{\mathbf{\epsilon}}(\mathbf{g}) \mathbf{\omega}_e
\]

(69)

**APPENDIX**

The Gibbs vector (or Rodriguez Parameters (RP)) is a recent addition to the family of attitude representations and is particularly well suited for describing very large attitudes. The RP are able to describe an orientation with only three parameters, instead of the four parameters required by the quaternions. The quaternion parameters are defined by:

\[
\mathbf{q} = \mathbf{e} \sin \frac{\theta}{2}
\]

\[
q_4 = \cos \frac{\theta}{2}
\]

where \(\mathbf{q} = (q_1, q_2, q_3)\). The fourth element, \(q_4\), is commonly referred to as the scalar component of the quaternion vector and indicates the principal angle. The attitude error quaternions \((q_{1c}, q_{2c}, q_{3c}, q_{4c})\) are computed by using the desired