Some Types of Separation Axioms in Topological Spaces *

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Abstract

In this paper, we introduce some types of separation axioms via \( \omega \)-open sets, namely \( \omega \)-regular, completely \( \omega \)-regular and \( \omega \)-normal space and investigate their fundamental properties, relationships and characterizations. The well-known Urysohn’s Lemma and Tietze Extension Theorem are generalized to \( \omega \)-normal spaces. We improve some known results. Also, some other concepts are generalized and studied via \( \omega \)-open sets.

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1. Introduction

Throughout this work, a space will always mean a topological space, \((X, \mathcal{I})\) and \((Y, \sigma)\) will denote spaces on which no separation axioms are assumed unless explicitly stated. The notations \(T_{\text{dis}}\), \(T_{\text{ind}}\) denote the discrete and indiscrete topologies and \(\varnothing\) denotes the usual topology for the set of all real numbers \(R\). For a subset \(A\) of a space \((X, \mathcal{I})\), the closure and the interior of \(A\) will be denoted by \(\text{Cl}_X A\) and \(\text{Int}_X A\) (or simply \(\text{Cl} A\) and \(\text{Int} A\)), respectively. A point \(x \in X\) is called a condensation point of \(A\) [13, pp. 90] if for each \(G \in \mathcal{I}\) with \(x \in G\), the set \(G \cap A\) is uncountable. \(A\) is called \(\omega\)-closed [8] if it contains all its condensation points. The complement of an \(\omega\)-closed set is called \(\omega\)-open. It is well known that a subset \(U\) of a space \((X, \mathcal{I})\) is \(\omega\)-open if and only if for each \(x \in U\), there exists \(G \in \mathcal{I}\) such that \(x \in G\) and \(G - U\) is countable. The family of all \(\omega\)-open subsets of a space \((X, \mathcal{I})\) is denoted by \(\mathcal{I}\omega\), forms a topology on \(X\) finer than \(\mathcal{I}\). The \(\omega\)-closure and \(\omega\)-interior, which are defined in the same way as \(\text{Cl} A\) and \(\text{Int} A\), and they are denoted by \(\omega\text{Cl} A\) and \(\omega\text{Int} A\), respectively. Several characterizations of \(\omega\)-closed subsets were provided in [3, 4, and 5]. A subset \(A\) of a space \(X\) is called \(\omega\)-dense [2] if \(\omega\text{Cl} A = X\). Authors in General topology used the notation of \(\omega\)-open sets to define some other types of sets, mappings and spaces, till Al-Hawary [1] and Rao and et al [6] used the nation of \(\omega\)-open sets in fuzzy and bitopological spaces, respectively. So we recall the following results and notions:

**Theorem 1.1.** [4] If \(U\) is \(\omega\)-open subset of \(X\), then \(U - C\) is \(\omega\)-open for every countable subset \(C\) of \(X\).

**Theorem 1.2.** [4 and 5] For any space \((X, \mathcal{I})\) and any subset \(A\) of \(X\),

1. \(\mathcal{I}\omega = (\mathcal{I}\omega)^\omega = \mathcal{I}\omega\).
2. \((\mathcal{I} A)^\omega = (\mathcal{I}\omega)_A\).

**Definition 1.3.** [5] A space \((X, \mathcal{I})\) is said to be locally-countable if each point of \(X\) has a countable open neighborhood.
It is easy to see that

**Theorem 1.4.** Let \((X, \mathcal{S})\) be a space. Then \(\mathcal{S}^\omega = T_{dis}\) if and only if the space \((X, \mathcal{S})\) is locally countable.

**Definition 1.5.** [5] A space \(X\) is said to be anti-locally countable if each non-empty open subset of \(X\) is uncountable.

Note that a space \((X, \mathcal{S})\) is anti-locally-countable if and only if \((X, \mathcal{S}^\omega)\) is so.

**Lemma 1.6.** [5] If a space \((X, \mathcal{S})\) is anti-locally-countable, then

1. \(\omega Cl A = Cl A\), for every \(\omega\)-open subset \(A\) of \(X\).
2. \(\omega Int A = Int A\), for every \(\omega\)-closed subset \(A\) of \(X\).

Al-Zoubi in [4] has improved part (1) of the above result, by proving the following lemma:

**Lemma 1.7.** [4] If \((A, \mathcal{S}_A)\) is an anti-locally countable subspace of a space \((X, \mathcal{S})\), then \(\omega Cl A = Cl A\).

**Definition 1.8.** [2] A space \(X\) is said to be an \(\omega\)-space if every \(\omega\)-open set is open.

**Definition 1.9.** A function \(: (X, \mathcal{S}) \rightarrow (Y, \rho)\) is called

1. \(\omega\)-continuous [8] if \(f^{-1}(U)\) is \(\omega\)-open in \(X\), for each open subset \(U\) of \(Y\),
2. \(\omega\)-irresolute [2] if \(f^{-1}(U)\) is \(\omega\)-open in \(X\), for each \(\omega\)-open subset \(U\) of \(Y\),
3. almost \(\omega\)-continuous [10] if for each \(x \in X\), and each open subset \(V\) of \(Y\) containing \(f(x)\), there exists an \(\omega\)-open subset \(U\) of \(X\) that containing \(x\) such that \(f(U) \subseteq Int_Y Cl_Y V\),
4. almost \(\omega\)-continuous [2] if for each \(x \in X\), and each open subset \(V\) of \(Y\) containing \(f(x)\), there exists an \(\omega\)-open subset \(U\) of \(X\) containing \(x\) such that \(f(U) \subseteq \omega Int_Y Cl_Y V\),
5. almost weakly-\(\omega\)-continuous [2] if for each \(x \in X\), and each open subset \(V\) of \(Y\) containing \(f(x)\), there exists an \(\omega\)-open subset \(U\) of \(X\) that containing \(x\) such that \(f(U) \subseteq Cl_Y V\),
6. pre-$\omega$-open [2] if image of every $\omega$-open set is $\omega$-open.

We use the almost $\omega$-continuous mapping in the sense of Nour for a mapping that satisfies part (3) and almost-$\omega$-continuous in the sense of Omari and Noorani for mappings that satisfy part (4) of Definition 1.9. Simply, they are same if $Y$ is an anti-locally countable space and it is clear from the fact that $IntA \subseteq \omega IntA$, the almost $\omega$-continuity in sense of Nour implies the almost-$\omega$-continuity in the sense of Omari and Noorani. But the converse is not true in general. For example, the mapping $f : (R, \wp^\omega) \rightarrow (Y, \rho)$ defined by $f(x) = 1$ if $x \in Q$ and $f(x) = 3$ if $x \in Irr$ is almost-$\omega$-continuity in the sense of Omari and Noorani, but not in the sense of Nour, where $\rho = \{\phi, \{1\}, \{2\}, \{1, 2\}, X\}$ and $Q$ and $Irr$ denote the set of all rational and irrational numbers.

**Theorem 1.10.** [4] Let $f : (X, \Im) \rightarrow (Y, \rho)$ be a mapping from an anti-locally countable space $(X, \Im)$ onto a regular space $(Y, \rho)$. Then the following are equivalent:

1. $f$ is continuous,
2. $f$ is $\omega$-continuous,
3. $f$ is almost $\omega$-continuous mapping in the sense of Nour,
4. $f$ is almost-$\omega$-continuous in the sense of Omari and Noorani,
5. $f$ is almost weakly $\omega$-continuous.

**Theorem 1.11.** [9] Let $A \subseteq X$ and $f : (X, \Im) \rightarrow (Y, \rho)$ be an $\omega$-continuous mapping. Then $f_A : (A, \Im_A) \rightarrow (Y, \rho)$ is $\omega$-continuous.

**Lemma 1.12.** [4] The open image of an $\omega$-open set is $\omega$-open.

### 2. More Properties of $\omega$-open Sets and Some Other Results

It is easy to see that:

**Theorem 2.1.** Let $(A, \Im_A)$ be any subspace of a space $(X, \Im)$. Then for any $B \subseteq A$, we have:
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1. \( \omega \text{Cl}_A B = (\omega \text{Cl}_X B) \cap A \),
2. \( \omega \text{Int}_X B = \omega \text{Int}_A B \cap \omega \text{Int}_X A \),
3. \( \omega b_A (B) \subseteq (\omega b_X (B)) \cap A \),
4. \( \omega b_A (B) = \omega \text{Cl}_X B \cap \omega \text{Cl}_X (A - B) \cap A \).

Note that the following example shows that the particular case of part (iii) of [8, Theorem 3.1] is not true. It also shows that the general case of [8, Corollary 3.2] is not true:

**Example 2.2.** Let \( X \) be an uncountable set equipped with the topology \( \Sigma = \{ \phi, A, B, X \} \), where \( A \) and \( B \) are uncountable disjoint subsets of \( X \) such that \( X = A \cup B \). Then \( X \) is a hereditary Lindelöf space and it is easy to see that a subset \( G \) of \( X \) is \( \omega \)-open if and only if \( G = X - C, G = A - C \) or \( G = B - C \), for some countable subset \( C \) of \( X \). Hence a subset \( F \) of \( X \) is \( \omega \)-closed if and only if \( G = C, G = A \cup C \) or \( G = B \cup C \), for some countable subset \( C \) of \( X \). But there is no \( \omega \)-open subset of \( X \) which is a \( G_\delta \)-set, except for the open sets \( \phi, A, B \) and \( X \). Al-Zoubi [4] proved that the conditions that \( X \) is anti-locally-countable and \( Y \) is regular are essential in Theorem 1.2.16. But we can improve his result by dropping the condition that \( f \) is surjection. For this, we need to prove the following lemma:

**Lemma 2.3.** Let \((X, \Sigma)\) be an anti-locally countable space and let \( A \) be a subset of \( X \). If for a point \( x \in A \), there exists an open subset \( G \) of \( X \) which contains \( x \) and \( G - A \) countable, then \( \text{Cl} G \subseteq \text{Cl} A \).

**Proof.** Let \( x \in A \) and \( G \) be an open set in \( X \) such that \( x \in G \) and \( G - A \) is countable. Suppose that \( y \in \text{Cl} G - \text{Cl} A \), then there exists an open set \( V \) containing \( y \) such that \( V \cap A = \phi \). Since \( y \in \text{Cl} G \), \( \phi \neq V \cap G \subseteq G - A \). This is a contradiction.

**Remark 2.4.** The converse inclusion of Lemma 2.3 is not true in general. As a simple example in the usual space \((R, \mathcal{P}^\omega)\), taking \( A = \text{Irr} \), since \( \sqrt{2} \in A \), \( \sqrt{2} \in (1,2) \in \phi \) and \( (1,2) - A \) is countable. But \( R = \text{Cl} A \nsubseteq [1,2] = \text{Cl} (1,2) \).

As an immediate consequence of Lemma 2.3, we have the following corollary:
Corollary 2.5. Let \((X, \mathcal{S})\) be an anti-locally countable space and \(A\) be an \(\omega\)-open subset of \(X\). Then for each point \(x \in A\), there exists an open subset \(G\) of \(X\) containing \(x\) such that \(\text{Cl} G \subseteq \text{Cl} A\).

The following theorem is an improvement version of Theorem 1.10.

Theorem 2.6. Let \(f\) be a mapping from an anti-locally countable space \((X, \mathcal{S})\) into a regular space \((Y, \rho)\). Then the following statements are equivalent:

1. \(f\) is continuous,
2. \(f\) is \(\omega\)-continuous,
3. \(f\) is almost \(\omega\)-continuous in the sense of Nour,
4. \(f\) is almost \(\omega\)-continuous in the sense of Omari and Noorani,
5. \(f\) is almost weakly \(\omega\)-continuous.

Proof. In general the implications \((1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)\) follows from their definitions and the facts that \(\mathcal{S} \subseteq \mathcal{S}^\omega\) and \(\rho \subseteq \rho^\omega\), see [4]. To show the implication \((5) \Rightarrow (1)\), let \(x \in X\) and \(V\) be any open subset of \(Y\) with \(f(x) \in V\). By regularity of \(Y\), we can choose two open sets \(V_1\) and \(V_2\) in \(Y\) such that \(f(x) \in V_1 \subseteq \text{Cl} V_1 \subseteq V_2 \subseteq \text{Cl} V_2 \subseteq V\). Since \(f\) is almost weakly \(\omega\)-continuous, there exists an \(\omega\)-open subset \(U\) in \(X\) containing \(x\) such that \(f(U) \subseteq \text{Cl} V_1\). Consequently, \(U \subseteq f^{-1}(\text{Cl} V_1)\). Since \(x \in U \in \mathcal{S}^\omega\), there exists an open set \(G\) in \(X\) with \(x \in G\) and \(G - U\) is countable. So by Lemma 1.6 and Lemma 2.3, we have \(\text{Cl} X G \subseteq \text{Cl} X U = \omega \text{Cl} X U\). Hence \(G \subseteq \omega \text{Cl} X U \subseteq \omega \text{Cl} X (f^{-1}(\text{Cl} Y V_1)) \subseteq (\omega \text{Cl} X f^{-1}(V_2))\). Now, we have to show that \(\omega \text{Cl} X f^{-1}(V_2) \subseteq f^{-1}(\text{Cl} Y V_2)\). Let \(u \in \omega \text{Cl} X f^{-1}(V_2)\). Suppose that \(u \notin f^{-1}(\text{Cl} Y V_2)\). Then \(f(u) \notin \text{Cl} Y V_2\). This implies that there exists an open set \(W\) in \(Y\) containing \(f(u)\) such that \(W \cap V_2 = \phi\). Hence \((\text{Cl} Y W) \cap V_2 = \phi\). Since \(f(u) \in W \in \rho\), by hypothesis there exists an \(\omega\)-open subset \(H\) in \(X\) containing \(u\) such that \(f(H) \subseteq \text{Cl} Y W\). Since \(u \in \omega \text{Cl} X f^{-1}(V_2), H \cap f^{-1}(V_2) \neq \phi,\) and hence \(f(H) \cap V_2 \neq \phi\). This implies that \(\text{Cl} Y W \cap V_2 \neq \phi\) which is impossible. Thus, \(\omega \text{Cl} X f^{-1}(V_2) \subseteq f^{-1}(\text{Cl} Y V_2)\). Hence \(G \subseteq f^{-1}(\text{Cl} Y V_2)\). Therefore, \(f(G) \subseteq \text{Cl} Y V_2\). Hence \(f\) is continuous.

In a similar way as continuity, it is easy to prove the following results:
Theorem 2.7. Every constant mapping from \((X, \mathcal{S})\) into \((R, \varnothing)\) is \(\omega\)-continuous. Moreover, if \(f\) and \(g\) from \((X, \mathcal{S})\) into \((R, \varnothing)\) are \(\omega\)-continuous mappings, then the following statements are true:

1. \(f \pm g, fg, |f|, \min\{f, g\}\) and \(\max\{f, g\}\) are \(\omega\)-continuous mappings,

2. If \(g(x) \neq 0\) for all \(x \in X\), then \(\frac{f}{g}\) is \(\omega\)-continuous.

Theorem 2.8. Let \(f_n : (X, \mathcal{S}) \to (R, \varnothing)\) be \(\omega\)-continuous mappings for all \(n \in \mathbb{N}\). If \(f : (X, \mathcal{S}) \to (R, \varnothing)\) is a mapping such that the series \(\sum_{n=0}^{\infty} f_n(x)\) is uniformly convergent to \(f(x)\), then \(f\) is an \(\omega\)-continuous mapping.

Definition 2.9. A subset \(A\) of a space \((X, \mathcal{S})\) is said to be an \(\omega\)-zero-set of \(X\) if there exists an \(\omega\)-continuous mapping \(f : (X, \mathcal{S}) \to (R, \varnothing)\) such that \(A = \{x \in X; f(x) = 0\}\) and a subset is called co\(\omega\)-zero-set if it is the complement of an \(\omega\)-zero-set. Furthermore, if \(f : (X, \mathcal{S}) \to (R, \varnothing)\) is an \(\omega\)-continuous mapping, then the set \(\omega Z(f) = \{x \in X; f(x) = 0\}\) is called the \(\omega\)-zero-set of \(f\).

Remark 2.10. 1. Every \(\omega\)-zero-set of a space is \(\omega\)-closed and hence every co\(\omega\)-zero-set is an \(\omega\)-open set,

2. Every zero-set of any space is an \(\omega\)-zero-set.

The following examples show that the converse of neither parts of Remark 2.10 is true:

Example 2.11. Consider an \(\omega\)-closed subset \(Q\) of the space \((R, \varnothing)\). We have to show the set \(Q\) is not \(\omega\)-zero-set. Suppose that \(Q\) is an \(\omega\)-zero-set. Then there exists an \(\omega\)-continuous mapping \(f : (X, \mathcal{S}) \to (R, \varnothing)\) such that \(A = \{x \in X; f(x) = 0\}\) = \(Q\). Therefore, \(f(x) = 0\) if and only if \(x \in Q\). Since \(f\) is \(\omega\)-continuous, \(f\) is a continuous mapping \{by Theorem 2.6\}. Hence \(Q\) is a zero-set. Consequently, \(Q\) is a closed subset of \(R\), which is a contradiction.

Example 2.12. Let \(f : (X, \mathcal{S}) \to (R, \varnothing)\) be a mapping defined by \(f(a) = 0\) and \(f(b) = 1 = f(c)\), where the \(X = \{a, b, c\}\) and \(\mathcal{S} = \{\varnothing, X, \{a\}\}\). Then \(f\) is \(\omega\)-continuous, but not a continuous function. Hence the set \(\{a\}\) is an \(\omega\)-zero-set which is not zero-set.

Lemma 2.13. If \(A\) is an \(\omega\)-zero-set of a space \(X\), then there exists an \(\omega\)-continuous mapping \(f : X \to R\) such that \(f \geq 0\) and \(A = \omega Z(f)\).
Proof. Since \( A = \omega Z(g) \) for some \( \omega \)-continuous mapping \( g : X \to \mathbb{R} \), by Theorem 2.7, the mapping \( f = |g| \geq 0 \) is \( \omega \)-continuous and \( A = \omega Z(f) \).

Lemma 2.14. The intersection and union of any finite number of \( \omega \)-zero-sets is also an \( \omega \)-zero-set. If \( \omega Z(f) \) and \( \omega Z(g) \) are \( \omega \)-zero-sets of \( f \) and \( g \), then \( \omega Z(f) \cup \omega Z(g) = \omega Z(fg) \), \( \omega Z(f) \cap \omega Z(g) = \omega Z(h) \), where \( h = f + g \).

Proof. By Theorem 2.7, it follows that both \( fg \) and \( h = f + g \) are \( \omega \)-continuous. Therefore, \( \omega Z(f) \cup \omega Z(g) = \omega Z(fg) \), \( \omega Z(f) \cap \omega Z(g) = \omega Z(h) \) are \( \omega \)-zero-sets.

Lemma 2.15. If \( \alpha \in \mathbb{R} \) and \( f : X \to \mathbb{R} \) is an \( \omega \)-continuous mapping, then the set \( A = \{ x \in X; f(x) \geq \alpha \} \) as well as \( B = \{ x \in X; f(x) \leq \alpha \} \) are \( \omega \)-zero-sets, and hence the sets \( \{ x \in X; f(x) < \alpha \} \) and \( \{ x \in X; f(x) > \alpha \} \) are \( \omega \) cozero-sets.

Proof. By using Theorem 2.7, it is easy to see that \( A = \omega Z(\min\{f(x)-\alpha, 0\}) \) and \( B = \omega Z(\max\{f(x)-\alpha, 0\}) \) are \( \omega \)-zero-sets.

Lemma 2.16. If \( A \) and \( B \) are disjoint \( \omega \)-zero-sets the space \( X \), then there exist disjoint \( \omega \) cozero-sets \( U \) and \( V \) containing \( A \) and \( B \), respectively.

Proof. Let \( A = \omega Z(f) \) and \( B = \omega Z(g) \). Then the mapping \( h : X \to \mathbb{R} \) given by \( h(x) = \frac{f(x)}{f(x)+g(x)} \) is well-defined and in view of Theorem 2.7 it is \( \omega \)-continuous, \( h(A) = \{0\} \) and \( h(B) = \{1\} \). Then by Lemma 2.15, the sets \( \{ x \in X; h(x) > \frac{1}{2} \} \) and \( \{ x \in X; h(x) < \frac{1}{4} \} \) are the required \( \omega \) cozero (hence \( \omega \) open) sets.

Corollary 2.17. If \( X \) is anti-locally countable, then every \( \omega \)-zero-set of \( X \) is a zero-set.

Proof. It follows from Theorem 2.6.

Now, we recall the following known definition.

Definition 2.18. [8] A space \((X, \mathfrak{I})\) is said to be \( \omega-T_1 \) (resp. \( \omega-T_2 \)) if for each pair of distinct points \( x \) and \( y \) of \( X \), there exist \( \omega \)-open sets \( U \) and \( V \) containing \( x \) and \( y \), respectively such that \( y \notin U \) and \( x \notin V \) (resp. \( U \cap V = \emptyset \)).
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Since each countable subset of any space is $\omega$-closed, it is easy to see that each space is an $\omega$-$T_1$ space. Therefore, the results (Theorem 3.12 and Corollary 3.13 of [2]) are trivial and they do not need to $Y$ satisfy any separating axiom. Note that every $T_2$-space is an $\omega$-$T_2$-space but not conversely.

**Theorem 2.19.** Let $X$ be an anti-locally countable space. Then $X$ is an $\omega$-$T_2$ space if and only if $X$ is a $T_2$-space.

**Proof.** Let $X$ be an anti-locally countable space. It is enough to prove that $X$ is a $T_2$-space if $X$ is an $\omega$-$T_2$-space. For this, let $x \neq y$ in an $\omega$-$T_2$-space $X$. Then it is easy to see that, there is an $\omega$-open set $U$ containing $x$ such that $y \notin \omega Cl U$. Since $U$ is an $\omega$-open set, there exists an open set $G$ containing $x$ such that $G - U$ is countable. In virtue of Lemma 1.6 and Corollary 2.5, we have $\omega Cl U = Cl U$ and $Cl G \subseteq Cl U$. Thus $G$, $X - Cl G$ are disjoint open sets in $X$ containing $x$ and $y$, respectively. Hence $X$ is a $T_2$-space.

### 3. $\omega$-Regular and Completely $\omega$-Regular Space

**Definition 3.1.** A space $(X, \mathcal{S})$ is called an $\omega$-regular space, if for each $\omega$-closed subset $H$ of $X$ and a point $x$ in $X$ such that $x \notin H$, there exist disjoint $\omega$-open sets $U$ and $V$ containing $x$ and $H$, respectively.

That is, a space $(X, \mathcal{S})$ is $\omega$-regular if and only if the space $(X, \mathcal{S}^\omega)$ is regular. Now, we have the following results:

**Theorem 3.2.** A space $X$ is $\omega$-regular if and only if for each point $x$ in $X$ and each $\omega$-open set $G$ containing $x$, there exists an $\omega$-open set $U$ such that $x \in U \subseteq \omega Cl U \subseteq G$.

**Proposition 3.3.** Every locally-countable space is an $\omega$-regular space.

The following example shows that the converse of Proposition 3.3 is not true in general.

**Example 3.4.** Consider the closed ordinal space $X = [0, \Omega]$, where $\Omega$ is the first uncountable ordinal and the subspace $[0, \Omega)$ of $X$ (see [12, Example 43, p. 68]). Since $X$ is an $\omega$-space and regular space, it is $\omega$-regular. Since any $\omega$-open set which contains $\Omega$ is uncountable, $X$ is not locally-countable.
Theorem 3.5. If each point of a space \((X, \mathcal{I})\) contained in some \(\omega\)-open subset \(G\) such that \(\omega \text{Cl}G\) is an \(\omega\)-regular subspace of \(X\), then \((X, \mathcal{I})\) is \(\omega\)-regular.

Proof. Let \(x \in G \in \mathcal{I}\). Then by hypothesis, there exists an \(\omega\)-open set \(V\) containing \(x\) such that \((H, \mathcal{I}_H)\) is an \(\omega\)-regular subspace of \(X\), where \(H = \omega \text{Cl}V\). Since \(x \in G \cap H \in \mathcal{I}_H\), by \(\omega\)-regularity of \((H, \mathcal{I}_H)\), there exists an \(\omega\)-open subset \(U\) of \(H\) such that \(x \in U \subseteq \omega \text{Cl}H U \subseteq G \cap H \subseteq G\). Since \(H\) is \(\omega\)-closed and \(x \in V \subseteq H\), \(x \in \omega \text{Int}_X H\). Thus by Theorem 2.1, we have \(x \in \omega \text{Int}_X U \subseteq \omega \text{Cl}_X(U) = \omega \text{Cl}_X(U) \cap H = \omega \text{Cl}_H U \subseteq G\). Hence \(X\) is an \(\omega\)-regular space.

Theorem 3.6. If a space \(X\) is an anti-locally countable \(\omega\)-regular space, then \(X\) is a regular and \(\omega\)-space.

Proof. Let \(G\) be any open set in \(X\) and let \(x\) be a point in \(X\) such that \(x \in G\). Then by Theorem 3.2, there exists an \(\omega\)-open set \(U\) in \(X\) such that \(x \in U \subseteq \omega \text{Cl}U \subseteq G\). Since \(x \in U\), there exists an open set \(V\) such that \(x \in V\) and \(V - U\) is countable. Hence by Lemma 1.6 and Corollary 2.5, we have \(\text{Cl}U = \omega \text{Cl}U\) and \(\text{Cl}V \subseteq \text{Cl}U\). Thus, \(x \in V \subseteq \text{Cl}V \subseteq G\). Therefore, \(X\) is regular.

If \(G\) is an arbitrary \(\omega\)-open set in \(X\) and \(x\) is any point of \(G\), then by the above argument, we can prove that \(G\) is an open set. This implies that \(X\) is an \(\omega\)-space.

The following result gives the relationship between \(\omega\)-regular and an \(\omega\)-\(T_2\)-space:

Proposition 3.7. Every \(\omega\)-regular space is an \(\omega\)-\(T_2\) space.

Proof. Obvious.

The following examples show that the converse of Proposition 3.7 is not true in general.

Example 3.8. Consider the Smirnov’s Deleted Sequence Topology [12, Example 64, pp. 88] \(\eta\) on the set of all real number \(R\), which is defined as: if \(A = \{\frac{1}{n}; n \in N\}\), then \(\eta = \{U \subseteq R; U = G - B, G \in \varnothing\text{ and } B \subseteq A\}\). Since this topology is finer than the usual topology \(\varnothing\), \((R, \eta)\) is a \(T_2\)-space. Hence, \((R, \eta)\) is an \(\omega\)-\(T_2\) space. Since \((R, \eta)\) is a non regular anti locally-countable space, \((R, \eta)\) is not \(\omega\)-regular \{by Theorem 3.6\}. 

The following proposition gives a partial converse of Proposition 3.7 and another relationship between regularity and $\omega$-regularity:

**Proposition 3.9.**
1. Every $\omega$-compact $\omega$-$T_2$ space is an $\omega$-regular space,
2. Every $\omega$-compact $T_2$ space is both regular and $\omega$-regular space.

**Proof.** Straightforward.

The following theorem shows that the property of $\omega$-regularity is a hereditary property:

**Theorem 3.10.** Every subspace of an $\omega$-regular space is also $\omega$-regular.

**Proof.** Obvious.

**Definition 3.11.** A space $(X, \mathcal{S})$ is said to be a completely $\omega$-regular space if for every $\omega$-closed subset $F$ of $X$ and every point $x \in X - F$, there exists an $\omega$-continuous mapping $f : (X, \mathcal{S}) \to (I, \wp(I))$ (simply, $f : X \to I$), such that $f(x) = \{0\}$ and $f(F) = \{1\}$.

That is, a space $(X, \mathcal{S})$ is completely $\omega$-regular if and only if $(X, \mathcal{S}^\omega)$ is completely regular.

Now, it is easy to show the following results:

**Theorem 3.12.** A space $(X, \mathcal{S})$ is a completely $\omega$-regular space if and only if for every $\omega$-open subset $G$ of $X$ and every point $x \in G$, there exists an $\omega$-continuous mapping $f : X \to I$ such that $f(x) = 0$ and $f(y) = 1$ for all $y \notin G$.

**Proof.** Obvious.

**Proposition 3.13.** Every completely $\omega$-regular space is an $\omega$-regular space.

**Proof.** Obvious.

**Proposition 3.14.** Every locally-countable space is completely $\omega$-regular.

**Proof.** Obvious.

The converse of the Proposition 3.14 is not true; see Example 3.4.

**Question:** Is the converse of Proposition 3.13 true?

**Theorem 3.15.** A space $(X, \mathcal{S})$ is completely $\omega$-regular if and only if the collection of all $\omega$-$\omega$-zero-sets of $X$ form a base for $\mathcal{S}^\omega$. 
Proof. Let $V$ be any $\omega$-open set in a completely $\omega$-regular space $X$ and let $v \in V$. Then by Theorem 3.12, there exists an $\omega$-continuous mapping $g : X \to I$ such that $g(v) = 1$ and $g(X - V) = \{0\}$. Set $U = \{x \in X; g(x) \geq \frac{2}{3}\}$ and $G = \{x \in X; g(x) > \frac{2}{3}\}$. By Lemma 2.15, $U$ is an $\omega$-zero set and $G$ is a $co\omega$-zero set such that $x \in G \subseteq U \subseteq V$.

Conversely, suppose that the condition of theorem holds. Let $a \in X$ and $H$ be an $\omega$-closed set in $X$ such that $a \notin H$. Then by hypothesis, there exists an $\omega$-zero set, say $\omega Z(h)$ such that $a \in X - \omega Z(h) \subseteq X - H$, where $h : X \to I$ is an $\omega$-continuous mapping. Hence we have $h(a) = t > 0$. We define $f : X \to I$ by putting $f(x) = \min \{1, \frac{|h(x)|}{t}\}$. Then by Theorem 2.7, $f$ is an $\omega$-continuous mapping. Consequently, we have $f(a) = 1$ and $x \in \omega Z(h)$ for each $x \in H$. Therefore, $f(x) = 0$ for each $x \in H$. Hence $X$ is completely $\omega$-regular.

The following examples show that completely regularity and completely $\omega$-regularity are independent topological concepts:

Example 3.16. Let $X = \{a, b, c\}$ and $\mathcal{S} = \{\emptyset, X, \{a\}\}$. Then by Proposition 3.14, $(X, \mathcal{S})$ is a completely $\omega$-regular, but not completely regular because it is not regular.

Example 3.17. The usual space $(\mathbb{R}, \wp)$ is completely regular but not a completely $\omega$-regular space because it is not $\omega$-regular.

The following theorem shows that the property of $\omega$-regularity is a hereditary property:

Theorem 3.18. Every subspace of a completely $\omega$-regular space is also a completely $\omega$-regular space.

Proof. Let $(X, \mathcal{S})$ be a completely $\omega$-regular space and let $(Y, \mathcal{S}_Y)$ be a subspace of $(X, \mathcal{S})$. Suppose that $A$ is any $\omega$-closed set in $Y$ and $y$ is a point of $Y$ such that $y \notin A$. Since $A$ is an $\omega$-closed subset of $Y$, by Theorem 1.2, there exists an $\omega$-closed subset $H$ of $X$ such that $A = H \cap Y$. Since $y \in Y$ and $y \notin A$, $y \notin H$. By completely $\omega$-regularity of $X$, there exists an $\omega$-continuous mapping $f : X \to I$ such that $f(y) = \{0\}$ and $f(A) = \{1\}$. Hence by Theorem 1.11, the restriction mapping $f_Y : Y \to I$ is an $\omega$-continuous mapping and $f_Y(y) = 0 = f(y)$. Since $A \subseteq H$ and $A \subseteq Y$, $f_Y(A) \subseteq f(H) = \{1\}$. Thus $f_Y(A) = \{1\}$.
The following theorem gives a relationship between completely \(\omega\)-regularity and completely regularity:

**Theorem 3.19.** If \(X\) is an anti-locally countable and completely \(\omega\)-regular space, then \(X\) is completely regular and \(X\) is an \(\omega\)-space.

**Proof.** Let \(X\) be an anti-locally-countable completely \(\omega\)-regular space. Let \(H\) be a closed set in \(X\) and suppose that \(x\) is an arbitrary point in \(X\) such that \(x \notin H\). Since every closed set is \(\omega\)-closed and by completely \(\omega\)-regularity of a space \(X\), there exists an \(\omega\)-continuous mapping \(f : X \to I\) such that \(f(y) = \{0\}\) and \(f(H) = \{1\}\). Since \(X\) is anti locally-countable and \(I\) is a regular space, by Theorem 2.6, \(f : X \to I\) is a continuous mapping. Hence \(X\) is completely regular. Since \(X\) is completely \(\omega\)-regular, by Theorem 2.6, it is \(\omega\)-regular and then by Theorem 3.6, it is \(\omega\)-space.

4. \(\omega\)-Normal Space

**Definition 4.1.** A space \(X\) is called an \(\omega\)-normal space if for each pair of disjoint \(\omega\)-closed sets \(A\) and \(B\) in \(X\), there exist disjoint \(\omega\)-open sets \(U\) and \(V\) such that \(A \subseteq U\) and \(B \subseteq V\).

That is, a space \((X, \mathcal{S})\) is an \(\omega\)-normal space if and only if \((X, \mathcal{S}_\omega)\) is a normal space.

It is not difficult to prove the following characterizations of an \(\omega\)-normal space:

**Theorem 4.2.** A space \(X\) is an \(\omega\)-normal space if for each pair of \(\omega\)-open sets \(U\) and \(V\) in \(X\) such that \(X = U \cup V\), there exist \(\omega\)-closed sets \(A\) and \(B\) which are contained in \(U\) and \(V\), respectively and \(X = A \cup B\).

**Theorem 4.3.** If \(X\) is any space, then the following statements are equivalent:

1. The space \(X\) is \(\omega\)-normal,

2. For each \(\omega\)-closed set \(A\) in \(X\) and each \(\omega\)-open set \(G\) in \(X\) containing \(A\), there is an \(\omega\)-open set \(U\) such that \(A \subseteq U \subseteq \omega\text{Cl}U \subseteq G\).

3. For each \(\omega\)-closed set \(A\) and each \(\omega\)-open set \(G\) containing \(A\), there exist \(\omega\)-open sets \(\{U_n, n \in N\}\) such that \(A \subseteq \bigcup\{U_n; n \in N\}\) and \(\omega\text{Cl}U_n \subseteq G\) for each \(n \in N\).
Now, we can establish the following Urysohn’s type lemma of $\omega$-normality which is an important characterization of the $\omega$-normal space:

**Theorem 4.4.** Let $X$ be any space. Then the following statements are equivalent:

1. $X$ is an $\omega$-normal space,

2. For each $\omega$-closed subset $A$ and $\omega$-open subset $B$ of $X$ such that $A \subseteq B$, there exists an $\omega$-continuous mapping $f : X \rightarrow I$ such that $f(A) = \{0\}$ and $f(X - B) = \{1\}$,

3. For each pair of disjoint $\omega$-closed subsets $F$ and $H$ of $X$, there exists an $\omega$-continuous mapping $f : X \rightarrow I$ such that $f(F) = \{0\}$ and $f(H) = \{1\}$.

**Proof.** $(1) \Rightarrow (2)$: Suppose that $B$ is an $\omega$-open subset of an $\omega$-normal space $X$ containing an $\omega$-closed subset $A$ of $X$. Then by Theorem 4.3, there exists an $\omega$-open set which we denote by $U_1$ such that $A \subseteq U_2 \subseteq \omega Cl U_3 \subseteq B$. Then $U_2$ and $B$ are $\omega$-open subsets of $X$ containing the $\omega$-closed sets $A$ and $\omega Cl U_3$, respectively. In the same way, there exist $\omega$-open sets, say $U_1$ and $U_3$ such that $A \subseteq U_4 \subseteq \omega Cl U_5 \subseteq U_6$ and $\omega Cl U_7 \subseteq U_8 \subseteq \omega Cl U_9 \subseteq B$. Continuing in this process, for each rational number in the open interval $(0, 1)$ of the form $t = \frac{m}{2^n}$, where $n = 1, 2, \ldots$ and $m = 1, 2, \ldots, 2^n - 1$, we obtain $\omega$-open sets of the form $U_t$ such that for each $s < t$ then $A \subseteq U_s \subseteq \omega Cl U_s \subseteq U_t \subseteq \omega Cl U_t$. We denote the set of all such rational numbers by $\Psi$, and define $f : X \rightarrow I$ as follows:

$$f(x) = \begin{cases} 
1 & \text{if } x \in X - B \\
\inf \{t; t \in \Psi \text{ and } x \in U_t\} & \text{if } x \in U_t 
\end{cases}$$

$f(X - B) = \{1\}$ and if $x \in A$, then $x \in U_t$ for all $t \in \Psi$. Therefore, by the definition of $f$, we have $f(x) = \inf f \Psi = 0$. Hence $f(B) = \{0\}$ and $f(x) \in I$ for all $x \in X$. It remains only to show that $f$ is an $\omega$-continuous mapping since the intervals of the form $[0, a)$ and $(b, 1]$ where $a, b \in (0, 1)$ form an open subbase of the space $I$. If $x \in U_t$ for some $t < a$, then $f(x) = \inf \{s; s \in \psi \text{ and } x \in U_s\} = r \leq t < a$. Thus $0 \leq f(x) < a$. If $f(x) = 0$, then $x \in U_t$ for all $t \in \Psi$. Hence $x \in U_t$ for some $t < a$. If $0 < f(x) < a$, by definition of $f$, we have $f(x) = \{s; s \in \Psi \text{ and } x \in U_s\} < a$ (Since $a < 1$). Thus $f(x) = t$ for some $t < a$, and hence $x \in U_t$ for some $t < a$. Therefore, we conclude that $0 \leq f(x) < a$ if and only if $x \in U_t$ for some $t < a$. Hence $f^{-1}([0, a)) = \cup \{U_t; t \in \Psi \text{ and } x \in U_t\}$.
which is an $\omega$-open subset of $X$. Also it is easy to assert that: $0 \leq f(x) \leq b$ if and only if $x \in U_t$ for all $t > b$. Let $x \in X$ such that $0 \leq f(x) \leq b$. It is evident that $f(x) < t$ for all $t > b$ which implies that $x \in U_t$ for all $t > b$. For the converse, let $x \in U_t$ for all $t > b$. Then $f(x) \leq t$ for all $t > b$. Thus $f(x) \leq b$ and it is clear from the definition of $f$, that $f(x) \geq 0$. This proves our assertion. Since for all $t > b$, there is $r \in \Psi$ such that $t > r > b$. Then $\omega Cl \cup_r \subseteq U_r$. Consequently we have $\cap \{U_t; t \in \Psi \text{ and } t > b\} = \cap \{\omega Cl U_r; r \in \Psi \text{ and } r > b\}$. Therefore, $f^{-1}([0,b]) = \{x; 0 \leq f(x) \leq b\} = \cap \{U_t; t \in \Psi \text{ and } t > b\} = \cap \{\omega Cl U_r; r \in \Psi \text{ and } r > b\}$. Since $f^{-1}((0,1]) = f^{-1}(I - [0,b]) = X - f^{-1}([0,b]) = \cup \{X - \omega Cl U_r; r \in \Psi \text{ and } r > b\}$ which is $\omega$-open, and this completes the proof of this part.

(2) $\Rightarrow$ (3) : Obvious.

(3) $\Rightarrow$ (1) : Let $A$ and $B$ be two disjoint $\omega$-closed subsets of $X$. Then by hypothesis, there exists an $\omega$-continuous mapping $f : X \to I$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Then the disjoint open sets $[0,\frac{1}{2})$ and $(\frac{1}{2},1]$ in $I$ containing $f(A)$ and $f(B)$, respectively. The $\omega$-continuity of $f$ gives that $f^{-1}([0,\frac{1}{2}])$ and $f^{-1}((\frac{1}{2},1])$ are disjoint $\omega$-open sets in $X$ containing $A$ and $B$, respectively. It completes the proof.

**Corollary 4.5.** Every $\omega$-normal space is completely $\omega$-regular and hence it is $\omega$-regular.

**Question:** Is the converse of corollary 4.5 true?

Recalling that the space $(X, \mathfrak{T})$ is lindelöf if and only if $(X, \mathfrak{T})$ is lindelöf [9].

**Theorem 4.6.** Every $\omega$-regular lindelöf space $X$ is $\omega$-normal.

**Proof.** Let $F$ be any $\omega$-closed and $U$ be any $\omega$-open subset of an $\omega$-regular lindelöf space $X$ such that $F \subseteq U$. Then by Theorem 3.2, for each $x \in F$, there exists an $\omega$-open set $V_x$ such that $x \in V_x \subseteq \omega Cl V_x \subseteq U$. Since $F$ is $\omega$-closed, by part (i) of [8, Theorem 3.3], $F$ is also a lindelöf subspace. Therefore, the cover $\{V_x; x \in F\}$ of $F$ has a countable subcover, say $\{V_n; n \in N\}$. Thus $F \subseteq \{V_n; n \in N\}$ and $\omega Cl V_n \subseteq U$ for each $n \in N$. Hence by Theorem 4.3, $X$ is $\omega$-normal.

In virtue of Theorem 2.8, Theorem 4.4 and the fact that every bounded closed intervals of $R$ are homeomorphic, we can generalize the Tietze Extension Theorem to $\omega$-normality which is also an important characterization of $\omega$-normal space.
**Theorem 4.7.** A space $X$ is $\omega$-normal if and only if every $\omega$-continuous mapping $g$ on an $\omega$-closed subset of $X$ into any closed interval $[a, b]$ has an $\omega$-continuous extension $f$ over $X$ into $[a, b]$.

The following result contains the relationship between $\omega$-normal and an $\omega$-zero-set:

**Proposition 4.8.** Let $X$ be a space. Then

1. An $\omega$-zero-set of $X$ is $\omega$-closed and it is the intersection of many countable $\omega$-open sets,

2. Let $H$ be an $\omega$-closed subset of $X$ which is the intersection of many countable $\omega$-open sets. If $X$ is $\omega$-normal, then $H$ is an $\omega$-zero-set.

**Proof.** (1) Let $F$ be an $\omega$-zero-set of a space $X$. Then by Remark 2.10, $F$ is an $\omega$-closed subset of $X$. Then by Lemma 2.13, there exists an $\omega$-continuous mapping $f : X \to \mathbb{R}$ such that $f \geq 0$ and $F = \omega Z(f)$. Hence $F = \cap\{U_n; n \in Z^+\}$, where $U_n = \{x \in X; f(x) < \frac{1}{n}\}$.

(2) Let $H$ be an $\omega$-closed subset of an $\omega$-normal space $X$ such that $H = \cap\{U_n; n \in Z^+\}$, where $U_n$ is an $\omega$-open set for each $n \in Z^+$. Since $H \subseteq U_n$ for each $n \in Z^+$ and $X$ is an $\omega$-normal space, for each $n \in Z^+$, there exists an $\omega$-continuous mapping $f_n : X \to [0, \frac{1}{3^n}]$ such that $f_n(H) = \{0\}$ and $f_n(X - U_n) = \{\frac{1}{3^n}\}$ {by Theorem 4.4}. Since $\sum_{n=0}^{\infty} f_n(x) \leq \sum_{n=0}^{\infty} \frac{1}{3^n}$ and the series $\sum_{n=0}^{\infty} \frac{1}{3^n}$ is absolutely convergent, the mapping $f : X \to \mathbb{R}$ given by $f(x) = \sum_{n=0}^{\infty} f_n(x)$ for each $x \in X$ is an $\omega$-continuous mapping and $H = \omega Z(f)$ {by Theorem 2.8}.

**Corollary 4.9.** Every locally-countable space is $\omega$-normal.

The converse of Corollary 4.9 is not true; see Example 3.4 and [12, Example 43, p. 68]. Now, since the usual topological space $(\mathbb{R}, \varphi)$ is not $\omega$-regular and by Corollary 4.5, $(\mathbb{R}, \varphi)$ is not $\omega$-normal. However, it is a normal space. This means that the $\omega$-normality is not implied by normality. The following example shows that the $\omega$-normality does not imply normality and hence this means that the $\omega$-normality and normality are independent topological concepts.
Example 4.10. Let $X = \{a, b, c\}$ and $\mathcal{I} = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$. Then by Corollary 4.9, the space $(X, \mathcal{I})$ is $\omega$-normal, but not a normal space because there are no disjoint open sets containing the disjoint closed sets $\{b\}$ and $\{c\}$, respectively.

The following theorem shows a relationship between $\omega$-normality and normality:

**Theorem 4.11.** Let $X$ be an anti-locally countable $\omega$-normal space. Then $X$ is a normal space and it is an $\omega$-space.

**Proof.** Let $F$ and $H$ be two disjoint closed subsets of an anti-locally countable $\omega$-normal space $X$. Then there exist $\omega$-open sets $U$ and $V$ such that $F \subseteq U$, $H \subseteq V$ and $U \cap V = \phi$. This implies that $\omega Cl U \cap V = \phi$ and $U \cap \omega Cl V = \phi$. Since $X$ is anti-locally-countable, we get $Cl U \cap V = \phi$ and $U \cap Cl V = \phi$ (by Lemma 1.6). Since $Int Cl U \subseteq Cl U$ and $Int Cl V \subseteq Cl V$, $Int Cl U \cap V = \phi$, $U \cap Int Cl V = \phi$. This implies that $Int Cl U \cap Cl V = \phi$ and $Cl U \cap Int Cl V = \phi$. Hence $Int Cl U$ and $Int Cl V$ are disjoint open sets in $X$ containing $F$ and $H$, respectively. This implies that $X$ is a normal space. By Corollary 4.5 and Theorem 3.6, it follows that $X$ is an $\omega$-space.

Recalling that a space $X$ is said to be a separable space [7, Definition 8.7.1, p. 175], if it contains a countable dense subset. Then it is easy to obtain the following results:

**Proposition 4.12.** There is no anti-locally countable separable space which is $\omega$-regular (and hence $\omega$-normal).

**Proof.** Obvious.

Note that Proposition 4.12, gives another way to proving that $(R, \phi)$ is neither $\omega$-regular nor $\omega$-normal. However, the space $(R, \phi)$ is an anti-locally-countable regular separable space.

**Proposition 4.13.** There is no uncountable space $(X, \mathcal{I})$ for which $(X, \mathcal{I}^{\omega})$ is separable.

**Proof.** Obvious.

**Theorem 4.14.** Every $\omega$-closed subspace of an $\omega$-normal space is also an $\omega$-normal space.
Proof. Obvious.

We recall that a topological space is said to be completely normal [11, Definition 1.4.1, p. 27] if every subspace of the space is normal. The following example shows that the property that being an $\omega$-normal of a space is not hereditary.

Example 4.15. Consider the Tychonoff Plank space $X = [0, \Omega] \times [0, \Omega_0]$ [12, Example 86, p. 106] and [7, Example 3.4, p. 145], where $\Omega$ and $\Omega_0$ denoted the first uncountable and first infinite countable ordinals. Since both ordinal spaces $[0, \Omega]$ and $[0, \Omega_0]$, are $\omega$-spaces, $X = [0, \Omega] \times [0, \Omega_0]$ is also an $\omega$-space. Since this space is normal, it is $\omega$-normal. Since this space is not a completely normal space, it is not hereditary normal. Hence it is not hereditary $\omega$-normal.

5. Some Covering and Characterizations of $\omega$-Normal Space

We begin this section with the following definition:

Definition 5.1. The family $\{A_\lambda; \lambda \in \Lambda\}$ of subsets of a space $(X, \mathcal{S})$ is called:

1. $\omega$-locally-finite if for each $x \in X$, there exists an $\omega$-open set $G$ containing $x$ such that the set $\{\lambda \in \Lambda; G \cap A_\lambda \neq \emptyset\}$ is finite,

2. $\omega$-discrete if for each point $x \in X$, there is an $\omega$-open set $G$ containing $x$ such that the set $\{\lambda \in \Lambda; G \cap A_\lambda \neq \emptyset\}$ has at most one member.

Proposition 5.2. Every locally-finite family of subsets of any space $(X, \mathcal{S})$ is $\omega$-locally-finite.

Proof. It follows from the fact that $\mathcal{S} \subseteq \mathcal{S}^\omega$.

The following example shows that the converse implication of Proposition 5.2 is not true in general.

Example 5.3. Consider the set $X = N$ equipped with the indiscrete topology $T_{ind}$. Then the family $\{\{n\}; n \in X\}$ is an $\omega$-discrete (and hence $\omega$-locally-finite) but not locally-finite (and hence not discrete).
The following example shows that the arbitrary union of \( \omega \)-closed sets need not be \( \omega \)-closed. That is, the union of \( \omega \)-closure of sets does not equal to the \( \omega \)-closure of their union as well as it show that the unions of closed sets need not be closed.

**Example 5.4.** Consider the usual topological space \((R, \varnothing)\). Then \(\{x\}; x \in (0, 1)\) is a family of \( \omega \)-closed subsets of \( R \). Thus \( \cup \{\omega Cl \{x\}; x \in (0, 1)\} = \cup \{\{x\}; x \in (0, 1)\} = (0,1) \) which is not \( \omega \)-closed. But \( \omega Cl (0, 1) = [0,1] \).

**Proposition 5.5.** If \( \{A_\lambda; \lambda \in \Lambda\} \) is an \( \omega \)-locally-finite family of subsets of a space \( X \), then \( \{\omega Cl A_\lambda; \lambda \in \Lambda\} \) is also \( \omega \)-locally-finite and \( \omega Cl (\cup A_\lambda; \lambda \in \Lambda) = \cup \{\omega Cl A_\lambda; \lambda \in \Lambda\} \).

**Proof.** Let \( x \in X \). Since \( \{A_\lambda; \lambda \in \Lambda\} \) is \( \omega \)-locally-finite, there exists an \( \omega \)-open set \( G \) containing \( x \) such that the set \( \{\lambda \in \Lambda; G \cap A_\lambda = \varnothing\} \) is finite. Since \( G \cap A_\lambda = \varnothing \) if and only if \( G \cap \omega Cl A_\lambda = \varnothing \), \( \{\lambda \in \Lambda; G \cap \omega Cl A_\lambda = \varnothing\} \) is finite. \( \{\omega Cl A_\lambda; \lambda \in \Lambda\} \) is also \( \omega \)-locally-finite. Since \( \cup \{\omega Cl A_\lambda; \lambda \in \Lambda\} \subseteq \omega Cl (\cup \{A_\lambda; \lambda \in \Lambda\}) \), we have only to prove that \( \omega Cl (\cup \{A_\lambda; \lambda \in \Lambda\}) \subseteq \cup \{\omega Cl A_\lambda; \lambda \in \Lambda\} \). Let \( x \notin \cup \{\omega Cl A_\lambda; \lambda \in \Lambda\} \). Since \( \{\omega Cl A_\lambda; \lambda \in \Lambda\} \) is \( \omega \)-locally-finite, there exists an \( \omega \)-open set \( U \) containing \( x \) such that \( \Lambda_0 = \{\lambda \in \Lambda; U \cap \omega Cl A_\lambda = \varnothing\} \) is finite. Set \( V = U \cap (\cup \{X - \omega Cl A_\lambda; \lambda \in \Lambda_0\}) \) an \( \omega \)-open subsets of \( X \) containing \( x \) such that \( V \cap (\cup \{A_\lambda; \lambda \in \Lambda\}) = \cup \{V \cap A_\lambda; \lambda \in \Lambda\} = \varnothing \). Thus \( x \notin \omega Cl (\cup \{A_\lambda; \lambda \in \Lambda\}) \). This completes the proof.

**Corollary 5.6.** If \( \{A_\lambda; \lambda \in \Lambda\} \) is a locally-finite family of subsets of a space \( X \), then \( \{\omega Cl A_\lambda; \lambda \in \Lambda\} \) is also locally-finite and \( \omega Cl (\cup A_\lambda; \lambda \in \Lambda) = \cup \{\omega Cl A_\lambda; \lambda \in \Lambda\} \).

**Proof.** It follows from Proposition 5.2 and Proposition 5.5.

It is easy to see that for any subset \( A \) and any \( \omega \)-open subset \( G \) of any space \( X \), \( G \cap A = \varnothing \) if and only if \( G \cap \omega Cl A = \varnothing \). But the following example shows that with this fact, the \( \omega \)-locally-finiteness of the \( \omega \)-closure of a family does not imply the \( \omega \)-locally-finiteness of the family. Also, it shows that the locally-finiteness of the closure of a family does not imply the locally-finiteness of the family.

**Example 5.7.** Consider the set \( X = R \) equipped with the topology \( \rho = \{\varnothing, X, Irr\} \). Then the family \( \{(p, p+1); p \in \mathbb{Z}\} \) is neither \( \omega \)-locally-finite nor
Proposition 5.8. Let \( \{A_\lambda; \lambda \in \Lambda\} \) be a family of subsets of a space \( X \) and \( \{B_\gamma; \gamma \in \Gamma\} \) be an \( \omega \)-locally-finite \( \omega \)-shrinkable of \( X \) such that for each \( \gamma \in \Gamma \), the set \( \{\lambda \in \Lambda; B_\gamma \cap A_\lambda \neq \emptyset\} \) is finite. Then there exists an \( \omega \)-locally-finite family \( \{G_\lambda; \lambda \in \Lambda\} \) of \( \omega \)-open sets of \( X \) such that \( A_\lambda \subseteq G_\lambda \) for each \( \lambda \in \Lambda \).

Proof. For each \( \lambda \), let \( G_\lambda = X - (\cup \{B_\gamma; B_\gamma \cap A_\lambda = \emptyset\}) \). So it is easy to see that \( A_\lambda \subseteq G_\lambda \) and by Proposition 5.6, \( G_\lambda \) is \( \omega \)-open for each \( \lambda \). Let \( x \in X \). Since \( \{B_\gamma; \gamma \in \Gamma\} \) is \( \omega \)-locally-finite, there is an \( \omega \)-open set \( U \) containing \( x \) such that the set \( \Gamma_0 = \{\gamma \in \Gamma; U \cap B_\gamma \neq \emptyset\} \) is finite. Thus \( U \cap B_\gamma = \emptyset \) for each \( \gamma \not\in \Gamma_0 \). Therefore, \( U \subseteq \cup \{B_\gamma; \gamma \in \Gamma_0\} \). Also, since for each \( \gamma \in \Gamma_0 \), \( G_\lambda \cap B_\gamma = \emptyset \) if and only if \( A_\lambda \cap B_\gamma = \emptyset \), the finiteness of \( \{\lambda \in \Lambda; B_\gamma \cap A_\lambda \neq \emptyset\} \) implies the finiteness of \( \{\lambda \in \Lambda; U \cap G_\lambda \neq \emptyset\} \) and this completes the proof.

Definition 5.9. An \( \omega \)-open covering \( \{U_\lambda; \lambda \in \Lambda\} \) of a space \( X \) is said to be \( \omega \)-shrinkable if there exists an \( \omega \)-open covering \( \{V_\lambda; \lambda \in \Lambda\} \) of \( X \) such that \( \omega ClV_\lambda \subseteq U_\lambda \) for each \( \lambda \in \Lambda \).

Theorem 5.10. Let \( X \) be a space. Then the following statements are equivalent:

1. \( X \) is \( \omega \)-normal,
2. Each point-finite \( \omega \)-open covering of \( X \) is \( \omega \)-shrinkable,
3. Each finite \( \omega \)-open covering of \( X \) has a locally-finite \( \omega \)-closed refinement,
4. Each finite \( \omega \)-open covering of \( X \) has an \( \omega \)-locally-finite \( \omega \)-closed refinement.

Proof. (1) \( \Rightarrow \) (2) : Let \( \{U_\lambda; \lambda \in \Lambda\} \) be a point-finite \( \omega \)-open covering of an \( \omega \)-normal space \( X \). Assume that \( \Lambda \) is well-ordered. We shall construct the \( \omega \)-shrinkings to \( \{U_\lambda; \lambda \in \Lambda\} \) by the transfinite induction. For this; let \( \mu \) be an element of \( \Lambda \) and suppose that for each \( \lambda < \mu \), we have an \( \omega \)-open set \( V_\lambda \) such that \( \omega ClV_\lambda \subseteq U_\lambda \) and for each \( v < \mu \), \( [\cup \{V_\lambda; \lambda < v\}] \cup [\cup \{U_\lambda; \lambda \geq v\}] = X \). Let \( x \in X \). Since \( \{U_\lambda; \lambda \in \Lambda\} \) is point-finite, there is the largest element, say
\[\xi \in \Lambda \text{ such that } x \in U_\xi. \text{ If } \xi \geq \mu, \text{ then } x \in \bigcup\{U_\lambda; \lambda \geq \mu\}. \text{ However, if } \xi < \mu, \text{ then } x \in \bigcup\{V_\lambda; \lambda < \mu\}. \] Hence \[\bigcup\{V_\lambda; \lambda < \mu\} \cup \bigcup\{U_\lambda; \lambda \geq \mu\} = X. \] Thus \(U_\mu\) contains the complement of an \(\omega\)-open set \(\bigcup\{V_\lambda; \lambda < \mu\} \cup \bigcup\{U_\lambda; \lambda > \mu\}\).

Since \(X\) is an \(\omega\)-normal space, there exits an \(\omega\)-open set, say \(V_\mu\) such that \((X - \bigcup\{V_\lambda; \lambda < \mu\}) \cup \bigcup\{U_\lambda; \lambda > \mu\}) \subseteq V_\mu \subseteq \omega Cl V_\mu \subseteq U_\mu\) (by Theorem 4.3).

Hence \[\bigcup\{V_\lambda; \lambda \leq \mu\} \cup \bigcup\{U_\lambda; \lambda \geq \mu\} = X. \] Hence the construction of the \(\omega\)-shrinking of \(\{U_\lambda; \lambda \in \Lambda\}\) is completed by transfinite induction.

\((2) \Rightarrow (3)\) : Obvious.

\((3) \Rightarrow (4)\) : Follows from Proposition 5.2.

\((4) \Rightarrow (1)\) : Let \(X\) be a space which satisfies condition (4) and let \(U\) and \(V\) be two \(\omega\)-open subsets of \(X\) such that \(U \cup V = X\). Then \(\{U, V\}\) is a finite \(\omega\)-open covering of \(X\). Then by hypothesis, this covering has an \(\omega\)-locally-finite \(\omega\)-closed refinement, say \(\Psi\). Let \(F\) and \(H\) be the union of these members of \(\Psi\) which is contained in \(U\) and \(V\), respectively. Then by Proposition 5.5, \(F\) and \(H\) are \(\omega\)-closed subsets of \(X\). Since \(\Psi\) is a cover of \(X\), in view of Theorem 4.2, \(X\) is \(\omega\)-normal.

**Theorem 5.11.** Let \(\{U_\lambda; \lambda \in \Lambda\}\) be an \(\omega\)-locally-finite family of an \(\omega\)-open set of an \(\omega\)-normal space \(X\), and let \(\{E_\lambda; \lambda \in \Lambda\}\) be a family of \(\omega\)-closed sets such that \(E_\lambda \subseteq G_\lambda\) for each \(\lambda \in \Lambda\). Then there exists a family \(\{V_\lambda; \lambda \in \Lambda\}\) of \(\omega\)-open sets such that \(E_\lambda \subseteq V_\lambda \subseteq \omega Cl V_\lambda \subseteq G_\lambda\) for each \(\lambda \in \Lambda\) and the families \(\{E_\lambda; \lambda \in \Lambda\}\) and \(\{\omega Cl V_\lambda; \lambda \in \Lambda\}\) are similar.

**Proof.** Assume that \(\Lambda\) is well-ordered. We shall construct a family \(\{V_\lambda; \lambda \in \Lambda\}\) of \(\omega\)-open sets such that \(E_\lambda \subseteq V_\lambda \subseteq \omega Cl V_\lambda \subseteq G_\lambda\) for each \(\lambda \in \Lambda\) by using the transfinite induction. First, we define the family \(\{A_\lambda^v; \lambda \in \Lambda\}\) by

\[
A_\lambda^v = \begin{cases} 
\omega Cl V_\lambda & \text{if } \lambda \leq \mu \\
E_\lambda & \text{if } \lambda > \mu
\end{cases}
\]

Suppose that \(\mu \in \Lambda\) and \(V_\lambda\) are defined for each \(v < \mu\) such that the family \(\{A_\lambda^v; \lambda \in \Lambda\}\) is similar to \(\{E_\lambda; \lambda \in \Lambda\}\). Let \(\{B_\lambda; \lambda \in \Lambda\}\) be the family given by

\[
B_\lambda = \begin{cases} 
\omega Cl V_\lambda & \text{if } \lambda \leq \mu \\
E_\lambda & \text{if } \lambda > \mu
\end{cases}
\]

To show \(\{B_\lambda; \lambda \in \Lambda\}\) is similar to \(\{E_\lambda; \lambda \in \Lambda\}\). Suppose that \(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k \in \Lambda\) such that \(\lambda_1 < \lambda_2 < \ldots < \lambda_j < \mu < \lambda_{j+1} < \ldots < \lambda_k\). Since \(\lambda_j < \lambda_\mu\), \(\{A_\lambda^j; \lambda \in \Lambda\}\) and \(\{E_\lambda; \lambda \in \Lambda\}\) are similar. Since \(\cap\{B_\lambda; i = 1, 2, 3, \ldots, k\} = \)
Theorem 4.3, there exists an $\omega \lambda \leq \cap \{ \}$

Suppose that $\mu = \cap \{ \}$ to show

$\lambda$

Let $\mu < \lambda$.

Corollary 5.11.

Proof. (1) ⇒ (2): It follows by putting $G_i = X$, for each $i = 1, 2, \ldots, k$ in Theorem 5.11.

(2) ⇒ (3) and (3) ⇒ (1) are obvious.

Corollary 5.12. Let $X$ be a topological space. Then the followings statements are equivalent:

1. $X$ is $\omega$-normal,

2. For each finite family $\{E_i; i = 1, 2, 3, \ldots, k\}$ of $\omega$-closed sets of $X$, there is a family $\{V_i; i = 1, 2, 3, \ldots, k\}$ of $\omega$-open sets such that $E_i \subseteq V_i$ for each $i = 1, 2, \ldots, k$, and the families $\{E_i; i = 1, 2, 3, \ldots, k\}$ and $\{\omega Cl V_i; i = 1, 2, 3, \ldots, k\}$ are similar,

3. For each pair $E_1$ and $E_2$ of disjoint $\omega$-closed sets of $X$, there is a pair $V_1$ and $V_2$ of $\omega$-open sets of $X$ such that $\omega Cl V_1$ and $\omega Cl V_2$ are disjoint.

Proof. (1) ⇒ (2): It follows by putting $G_i = X$, for each $i = 1, 2, \ldots, k$ in Theorem 5.11.

(2) ⇒ (3) and (3) ⇒ (1) are obvious.

Corollary 5.13. Let $(X, \mathcal{S})$ be a space and $B \subseteq A \subseteq X$. Then the following statements are true:

1. Let $A \in \mathcal{S}^\omega$. Then $B \in \mathcal{S}_A^\omega$ if and only if $B \in \mathcal{S}^\omega$,
2. Let \( A \) be \( \omega \)-closed. Then \( B \) is \( \omega \)-closed in \( A \) if and only if it is \( \omega \)-closed in \( X \).

**Proof.** It follows from Theorem 1.2.

**Theorem 5.14.** Let \( E \) be an \( \omega \)-closed subset of an \( \omega \)-normal \( X \) and let \( \{G_\lambda; \lambda \in \Lambda\} \) be an \( \omega \)-locally-finite family of an \( \omega \)-open set of \( X \) such that \( E \subseteq \cup \{G_\lambda; \lambda \in \Lambda\} \). Then there exists a family \( \{V_\lambda; \lambda \in \Lambda\} \) of \( \omega \)-open sets of \( X \) such that \( \omega Cl V_\lambda \subseteq G_\lambda \) for each \( \lambda \in \Lambda \), \( E \subseteq \cup \{G_\lambda; \lambda \in \Lambda\} \) and \( \{\omega Cl V_\lambda; \lambda \in \Lambda\} \) are similar to \( \{E \cap G_\lambda; \lambda \in \Lambda\} \).

**Proof.** Let \( \Gamma \) be the family of finite subsets of \( \Lambda \) such that \( E \cap (\cap \{G_\lambda; \lambda \in \gamma\}) \neq \phi \) for each \( \gamma \in \Gamma \). The family \( \{E \cap (\cap \{G_\lambda; \lambda \in \gamma\}); \gamma \in \Gamma\} \) of non-empty \( \omega \)-open subsets of \( E \) is \( \omega \)-locally-finite and from Theorem 4.14, \( E \) is \( \omega \)-normal. Hence by Corollary 4.5, \( E \) is \( \omega \)-regular. Therefore, for each \( \gamma \in \Gamma \), there exists a non-empty \( \omega \)-closed set \( D_\gamma \) of \( E \) such that \( D_\gamma \subseteq E \cap (\cap \{G_\lambda; \lambda \in \gamma\}) \). Since \( \{G_\lambda; \lambda \in \Lambda\} \) is \( \omega \)-locally-finite and \( E \) is an \( \omega \)-closed subset of \( X \), the family \( \{D_\gamma; \gamma \in \Gamma\} \) consists of \( \omega \)-closed subsets of \( X \) and it is \( \omega \)-locally-finite in \( X \) {by Corollary 5.13}. Since \( E \) is \( \omega \)-normal and each \( \omega \)-locally-finite family is point-finite, it follows that there is an \( \omega \)-locally-finite \( \omega \)-closed covering \( \{H_\lambda; \lambda \in \Lambda\} \) of \( E \) such that \( H_\lambda \subseteq E \cap G_\lambda \) for each \( \lambda \) {by Theorem 5.10}. Let \( F_\lambda = E_\lambda \cup \{D_\gamma; \gamma \in \Gamma\} \) for each \( \lambda \). Then by Proposition 5.5, \( F_\lambda = E_\lambda \cup \{D_\gamma; \gamma \in \Gamma\} \) is \( \omega \)-closed in both \( E \) and \( X \) and \( F_\lambda = E_\lambda \cap G_\lambda \) for each \( \lambda \). Also \( \{F_\lambda; \lambda \in \Lambda\} \) is an \( \omega \)-closed covering of \( E \). Furthermore, the families \( \{F_\lambda; \lambda \in \Lambda\} \) and \( \{E \cap G_\lambda; \lambda \in \Lambda\} \) are similar. For if \( \gamma \in \Gamma \), then \( D_\gamma \subseteq \cap \{F_\lambda; \lambda \in \gamma\} \). Hence \( \cap \{F_\lambda; \lambda \in \Lambda\} = \phi \). Since \( X \) is an \( \omega \)-normal space, there exists a family \( \cup \{V_\lambda; \lambda \in \Lambda\} \) of \( \omega \)-open subsets of \( X \) such that \( F_\lambda \subseteq V_\lambda \subseteq \omega Cl V_\lambda \subseteq G_\lambda \) for each \( \lambda \) {by Theorem 5.11}. Therefore, \( \{\omega Cl V_\lambda; \lambda \in \Lambda\} \) and \( \{E \cap G_\lambda; \lambda \in \Lambda\} \) are similar. This completes the proof.
References


