Performance analysis of a modified spatial smoothing technique for direction estimation
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Abstract
The statistical performance analysis of ESPRIT, root-MUSIC, minimum-norm methods for direction estimation, due to finite data perturbations, using the modified spatially smoothed covariance matrix, is developed. Expressions for the mean-squared error in the direction estimates are derived based on a common framework. Based on the analysis, the use of the modified smoothed covariance matrix improves the performance of the methods when the sources are fully correlated. Also, the performance is better even when the number of subarrays is large unlike in the case of the conventionally smoothed covariance matrix. However, the performance for uncorrelated sources deteriorates due to an artificial correlation introduced by the modified smoothing. The theoretical expressions are validated using extensive simulations. © 1999 Elsevier Science B.V. All rights reserved.

Zusammenfassung
Es wird die Analyse der statistischen Eigenschaften von ESPRIT, root-MUSIC und der Minimum-Norm Methoden zur Richtungsschätzung entwickelt, wobei aufgrund der Beeinträchtigung der Verfahren durch endliche Datenmengen die modifiziert räumlich geglättete Kovarianzmatrix verwendet wird. Basierend auf allgemeinen Methoden werden Ausdrücke für den mittleren quadratischen Fehler der Richtungsschätzungen abgeleitet. Aus der Analyse ergibt sich, daß die Verwendung der modifiziert geglätteten Kovarianzmatrix die Leistung der Methoden verbessert, wenn vollständig korrellierte Quellen vorliegen. Im Unterschied zur Verwendung der konventionell geglätteten Kovarianzmatrix zeigt sich auch dann eine bessere Leistung, wenn die Zahl der Sensoruntergruppen groß ist. Die Leistung bei unkorrellierten Quellen verschlechtert sich allerdings aufgrund einer künstlichen Korrelation, die durch das modifizierte Glätten herbeigeführt wird. Die theoretischen Ausdrücke werden anhand von ausgedehnten Simulationen bestätigt. © 1999 Elsevier Science B.V. All rights reserved.

Résumé
Nous développons une analyse des performances statistiques des méthodes ESPRIT, root-MUSIC, et à norme minimale pour l’estimation de direction, due à des perturbations de données finies. Les expressions de l’erreur quadratique moyenne des estimations de direction sont déduites sur la base d’un schéma commun. Sur base de l’analyse, l’utilisation de la matrice de covariance lissée modifiée améliore les performances des méthodes lorsque les sources sont entièrement corrélées. De même, les performances sont meilleures même lorsque le nombre de sous-tableaux est grand,

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1. Introduction

The problem of direction-of-arrival (DOA) estimation, has found wide applications and has been of considerable interest to researchers in the recent past. The recently developed sub space methods like multiple signal classification (MUSIC) [2,18], minimum-norm method [8,16], estimation of signal parameters via rotational invariance techniques (ESPRIT) [17] have been shown to perform well. In this paper, the special case of a uniform linear array (ULA) is considered. The subspace methods operate on the estimated covariance matrix formed from the data vectors containing the output of all the sensors at a time instant called as a snapshot.

The popular and simple approach to obtain an estimate of the covariance matrix, given N snapshots, is the Forward only approach. This approach fails when the signals are fully correlated to each other [1]. Hence, various spatial smoothing schemes (which are applicable only to the ULA) were proposed for tackling the problem of correlated sources. They are forward smoothing [5,19] and forward–backward smoothing [9,22]. Recently, a modified spatial smoothing technique (MSS), which defines both the modified forward smoothed and modified forward–backward smoothed covariance matrices and fully utilizing the cross correlations of the subarray outputs, has been proposed in [4]. For the forward–backward case, this newly proposed MSS has been shown to produce a better MUSIC spectrum when compared with the conventional scheme for the case of three fully correlated sources [4].

Statistical performance analysis of the subspace methods, using the conventional covariance matrix has been carried out in [11–13,20,21] and for the spatial smoothing case in [3,9,14,15]. In this paper, a performance analysis of the subspace methods (ESPRIT, MUSIC and minimum-norm method), due to finite data, for the case of modified forward smoothing and modified forward–backward smoothing is done and compared with the existing spatial smoothing schemes [15]. Theoretical expressions for the calculation of the mean-squared error in the DOA estimate using the subspace methods have been developed and they have been substantiated by extensive simulation studies. A comparative performance study of the MSS and the conventional schemes highlights the strengths and weaknesses of the MSS technique.

2. Mathematical data model for the ULA

Consider a ULA of M identical omnidirectional sensors. Let \( d \) be narrowband plane waves, centered at \( \omega_0( = 2\pi f_0) \) impinge on the array from directions \( \theta_1, \ldots, \theta_d \). Let \( d_1 \) be the distance between any two adjacent sensors and \( c_0 \) the velocity of the wavefront in the medium. Using complex signal representation, the received signal at the \( l \)th sensor at time instant \( n \) can be expressed as

\[
y_l(n) = \sum_{i=1}^{d} p_i(n) e^{j(\omega_0 - 1)h_0} + n_l(n),
\]

where \( p_i(n) \) is the response of the sensor to the \( i \)th source at time instant \( n \), \( n_l(n) \) is the noise amplitude vector of the \( l \)th sensor at time instant \( n \) and \( \omega_j = (\omega_0 d_1 / c_0) \sin(\theta_j) = 2\pi (d_1 / \lambda_0) \sin(\theta_j) \), \( \lambda_0( = c_0 / f_0) \) denoting the wavelength of the source. The data from all the sensors in a vector is called a snapshot and can be represented as

\[
Y(n) = V_s p(n) + N(n),
\]

The notation used in this paper is that all boldfaced letters denote vectors and matrices. Superscripts T, *, H denote transpose, complex conjugate and complex-conjugate transpose of matrices, respectively. \( \hat{x} \) denotes the expectation of \( x \).
where \( Y(n) = [y_1(n), \ldots, y_M(n)] \), \( p(n) = [p_1(n), \ldots, p_d(n)] \) and \( N(n) = [n_1(n), \ldots, n_d(n)] \). \( V_s \) is the array steering matrix of the ULA which has a Vandermonde structure, with the \( i \)th column represented by \( V(\omega) \), the normalized array steering vector given by \( \sqrt{M} V(\omega) = [1, e^{j\omega}, \ldots, e^{j(M-1)\omega}] \), where \( \omega = (\omega_0 d_1/c_0) \sin(\theta) \). The term DOA will refer to \( \omega_i \) or \( \theta_i \) depending on the context. The noise process, \( N(i) \), satisfies \( N(i)N^H(s) = \sigma_n^2 I_{1s} \) and \( N(i)N^H(s) = 0 \). The signals are uncorrelated with the additive noise and assumed to satisfy \( p(i)p^H(s) = P_{\delta_{is}} \) and \( p(i)p^H(s) = 0 \). The covariance matrix of the data vector is given as

\[
R = \overline{Y(n)Y^H(n)} = V_sPV_s^H + \sigma_n^2 I = R_s + \sigma_n^2 I. \tag{3}
\]

The eigendecomposition of \( R \) is given by \( R = \sum_{k=1}^{N} \lambda_k S_k S_k^H = E \Lambda E^H \), where \( E \) contains the orthonormal eigenvectors, \( S_k, \ldots, S_M \) and \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_M) \) where \( \lambda_i \) denotes the \( i \)th eigenvalue. Also, the eigenvalues are assumed to be arranged in nonincreasing order. Hence \( R \) can be written as

\[
R = E_s \Lambda_s E_s^H + \sigma_n^2 I, \quad E_s = [S_1, S_2, \ldots, S_M],
\]

\[
\Lambda_s = \text{diag}(\lambda_1^s, \lambda_2^s, \ldots, \lambda_d^s).
\]

Also,

\[
E_n = [S_{d+1}, S_{d+2}, \ldots, S_M] \quad \text{and} \quad E_n^H E_s = 0.
\]

Using these properties, subspace methods like MUSIC, minimum-norm and ESPRIT have been proposed.

3. Covariance matrix – the conventional estimators

The statistics of the estimated subspaces, depend upon the approach used to estimate the covariance matrix and the assumptions made on the data. Since the eigen-based subspace methods use the properties of the signal and noise subspaces, estimates of \( R, \hat{R} \) are needed which preserve the rank and structure of the subspaces.

3.1. Forward only smoothing (FS) approach

Using \( R \) (Eq. (3)) in a coherent signal environment (when \( P \) becomes singular) yields poor results. To overcome this difficulty, a spatial smoothing technique was proposed for the ULA [8]. In this technique, the array is divided into \( K \) smaller sub-arrays each containing \( L \) sensors where \( K = M - L + 1 \). It is important to note that each sub-array corresponds to the case of a ULA with \( L \) sensors. Let \( Y_p(n) \) be the output vector of the \( p \)th subarray. The covariance matrices of the sub-arrays are averaged to obtain the smoothed estimate \( \hat{R}_{fs} \), i.e.

\[
\hat{R}_{fs} = \frac{1}{K} \sum_{p=1}^{K} \hat{R}_p, \quad \text{where} \quad \hat{R}_p = \frac{1}{N} \sum_{n=1}^{N} Y_p(n)Y_p^H(n). \tag{5}
\]

It has been shown in [19] that this method builds the rank of \( R \) and preserves the structure of the subspaces, in the case of coherent sources.

3.2. Forward–Backward smoothing (FBS) approach

Using the property of \( V_s \) for the ULA, the FBS estimator of the covariance matrix is given as

\[
\hat{R}_{fbs} = \frac{\hat{R}_{fs} + J\hat{R}_{fs} J}{2}, \tag{6}
\]

which preserves the structure of the subspaces of the covariance matrix.

4. The modified spatial smoothing technique

The currently known spatial smoothing schemes FS and FBS do not consider the cross correlations of the subarray outputs. A modified spatial smoothing technique (MSS) was proposed in [4], defining both the modified forward only smoothed and modified forward–backward smoothed covariance matrices, which fully utilizes the cross correlations of the subarray outputs.

The \( L \times L \) cross covariance matrix of the \( i \)th and \( j \)th subarrays is given by [4]

\[
R^{ij} = V_s^H \Psi^{i-1} P(\Psi^{j-1})^H V_s^H + \sigma_n^2 I_{\delta_{ip}}, \tag{7}
\]

where \( J \) is the \( M \times M \) exchange matrix with 1s on the antidiagonal. Note that \( J = J', J' = I \).
where $\Psi = \text{diag}(e^{j\omega_1}, e^{j\omega_2}, \ldots, e^{j\omega_n})$. Note that the noises from different subarrays are assumed to be uncorrelated. However, when subarrays are overlapped, this assumption does not hold. In this case the noise covariance can be approximately removed using its estimate or its impact on this algorithm simply ignored. For $M \gg L$ this choice seems to be yielding reasonably good simulation results [4]. But due to the use of higher-order moments, the consistency of the estimates would be affected.

To develop expressions for backward smoothing define the $L \times L$ cross covariance matrix

$$\tilde{R}^{ij} = J(R^{ij})^* J.$$

The newly proposed MSS [4] defines the modified forward only smoothed matrix as

$$R_{\text{mf}s} = \frac{1}{K} \sum_{i=1}^{K} \sum_{j=1}^{K} R^{ij} R^{ji}, \tag{8}$$

$$\tilde{R}_{\text{mf}s} = \frac{1}{K} \sum_{i=1}^{K} \sum_{j=1}^{K} \tilde{R}^{ij} \tilde{R}^{ji}$$

and the modified forward–backward smoothed covariance matrix as

$$R_{\text{mf}bs} = \frac{1}{2K} \sum_{i=1}^{K} \sum_{j=1}^{K} [R^{ij} R^{ji} + \tilde{R}^{ij} \tilde{R}^{ji}], \tag{9}$$

$$\tilde{R}_{\text{mf}bs} = \frac{1}{2K} \sum_{i=1}^{K} \sum_{j=1}^{K} [\tilde{R}^{ij} \tilde{R}^{ji} + \tilde{R}^{ij} \tilde{R}^{ji}]$$

$$\quad \quad \quad = \frac{1}{2K} \sum_{i=1}^{K} \sum_{j=1}^{K} [\tilde{R}^{ij} \tilde{R}^{ji}] + \frac{1}{2K} \sum_{i=1}^{K} [\tilde{R}^{ij} \tilde{R}^{ji}], \tag{9}$$

where

$$\tilde{R}^{ij} = \frac{1}{N} \sum_{n=1}^{N} Y_i(n) Y_j^H(n),$$

$$\tilde{R}^{ij} = J(R^{ij})^* J, \quad \tilde{R}^{ij} = J(R^{ij})^* J.$$

Using Eq. (7) in Eq. (8),

$$R_{\text{mf}s} = V_s \tilde{S} V_s^H + \sigma_n^2 I, \tag{10}$$

where $\tilde{S}$, the $d \times d$ smoothed signal covariance matrix, is given by

$$\tilde{S} = \frac{1}{K} \sum_{i=1}^{K} \Psi_i^{-1} S_i (\Psi_i^{-1})^H$$

and

$$S_i = \sum_{j=1}^{K} [P(\Psi^{-1})^j V_s^H V_s \Psi_j^{-1} P] + 2 \sigma_n^2 P.$$

Using Eq. (7) in Eq. (9),

$$R_{\text{mf}bs} = V_s \tilde{S}_b V_s^H + \sigma_n^2 I, \tag{11}$$

where

$$\tilde{S}_b = \frac{\tilde{S} + J\tilde{S}^* J}{2}.$$

The above simplification for $R_{\text{mf}s}$ and $R_{\text{mf}bs}$ has been possible only under the assumption that the noises from different subarrays are uncorrelated.

**Theorem 1 [4].** When the number of subarrays $K$ satisfies $K \gg d$, then the modified forward only smoothed signal covariance matrix $\tilde{S}$ is nonsingular.

This result indicates that the covariance matrices retain the structure used by the subspace methods.

5. **Case study**

It was observed in [4], that the MSS covariance matrix resulted in sharper peaks of the MUSIC spectrum for coherent sources. Since spatial smoothing reduces the effective correlation of the sources, a study of the extent of reducing correlation using the MSS technique is useful.

The MSS and the conventional smoothing schemes are compared with various parameters like signal-to-noise ratio (SNR), subarray length ($K$), source correlation coefficient ($\rho$) and DOA separation.

Considering the one source and two source cases and simplified expressions for $\tilde{S}$ appearing in the expressions for $R_{\text{mf}s}$ and $R_{\text{mf}bs}$ are presented.

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3 $R_{\text{RF}}, R_{\text{RFb}}, R_{\text{mf}s}, R_{\text{mf}bs}$ denote the forward smoothing (RF), forward–backward smoothing (RFB), modified forward smoothing (RMF), modified forward–backward smoothing (RMFB) covariance matrices, respectively.
5.1. One source case

Note that the source covariance matrix $\mathbf{P}$ for this case is a scalar. All the properties concerned with uncorrelated sources apply to the one source case and thus,

$$
S_1 = \sum_{j=1}^{K} [\mathbf{P}^{j-1} ]^{H} \mathbf{V}^{H} \mathbf{V}^{j-1} \mathbf{P} + 2\sigma_n^2 \mathbf{P} \\
= KP^2 L + 2\sigma_n^2 \mathbf{P},
$$

$$
\tilde{S} = \frac{1}{K} \sum_{i=1}^{K} \mathbf{P}^{i-1} \mathbf{S}_i (\mathbf{P}^{i-1})^H = S_1.
$$

The modified forward only smoothed covariance matrix is given by

$$
\mathbf{R}_{\text{mfs}} = \mathbf{V}_s \tilde{\mathbf{S}} \mathbf{V}_s^H + \sigma_n^4 \mathbf{I} \\
= (KP^2 L + 2\sigma_n^2 \mathbf{P}) \mathbf{V}_s \mathbf{V}_s^H + \sigma_n^4 \mathbf{I}. 
$$

From the definition of $\mathbf{R}_{\text{mfs}}$ in Eq. (9) it can be proved that $\mathbf{R}_{\text{mfs}} = \mathbf{R}_{\text{mfs}}$ and

$$
\mathbf{R}_s^+ = \frac{1}{(KP^2 L + 2\sigma_n^2 \mathbf{P})L^2} \mathbf{V}_s \mathbf{V}_s^H.
$$

5.2. Two source case

The two source scenario provides an interesting case to study the resolution capabilities of the subspace methods using the modified and conventional covariance estimators. The source covariance matrix is assumed to be of the form

$$
\mathbf{P} = \mathbf{P}_1 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix},
$$

where $\rho$ is the source correlation coefficient (which is assumed to be real). The sources are assumed to be equi-powered of power $\mathbf{P}_1$. The DOAs of the two sources are denoted by $\omega_1$ and $\omega_2$. The array steering matrix is given by

$$
\mathbf{V}_s = [\mathbf{V}_{\omega_1} \mathbf{V}_{\omega_2}].
$$

Like the one source case, $\mathbf{S}_1$ corresponding to $\mathbf{R}_{\text{mfs}}$ can be simplified as

$$
\mathbf{S}_1 = \sum_{j=1}^{K} [\mathbf{P}^{j-1} ]^{H} \mathbf{V}^{H} \mathbf{V}^{j-1} \mathbf{P} + 2\sigma_n^2 \mathbf{P} \\
= \mathbf{P}_1^2 \begin{bmatrix} b & c^* \\ c & b \end{bmatrix} + 2\sigma_n^2 \mathbf{P},
$$

where $b$ and $c$ are given by

$$
b = KL(1 + \rho^2) + 2\rho \text{Re}(a),
$$

$$
c = a + \rho^2 d^* + 2KL\rho,
$$

$a$ is given by

$$
a = \frac{(e^{j(\omega_1 - \omega_2)K} - 1)(e^{j(\omega_1 - \omega_2)L} - 1)}{(e^{j(\omega_1 - \omega_2)} - 1)^2},
$$

$\tilde{S}$ can now be simplified as

$$
\tilde{S} = \frac{1}{K} \sum_{i=1}^{K} \mathbf{P}^{i-1} \mathbf{S}_i (\mathbf{P}^{i-1})^H = \begin{bmatrix} e & f^* \\ f & e \end{bmatrix},
$$

where

$$
e = \mathbf{P}_1^2 b + 2\sigma_n^2 P_1, \\
f = \mathbf{P}_1^2 h + 2\sigma_n^2 P_1 g,
$$

$$
h = \frac{p^2}{K}, \\
g = \frac{q^2 h}{K^2}, \\
t = \frac{e^{j(\omega_1 - \omega_2)L} - 1}{e^{j(\omega_1 - \omega_2)} - 1}.
$$

From the definition of $\mathbf{R}_{\text{mfs}}$, $\mathbf{S}_b$ can be simplified as

$$
\mathbf{S}_b = \begin{bmatrix} e & f^* \\ f & e \end{bmatrix}, \\
f_b = \frac{(e + f * (e^{j(\omega_1 - \omega_2)K} - 1))}{2}.
$$

It is worthwhile to examine two typical cases: $\rho = 0$ and $\rho = 1$. The latter case is important because the main idea of spatial smoothing is to tackle coherent sources. In order to study the effectiveness of MSS, a measure of the relative cross-term magnitude, defined as $r = |f/e|$, is considered. The sources would be highly correlated if $r$ is close to 1.

Case 1. Here, $\rho = 0$ and the main diagonal term $e$ and the cross term $f$ of $\mathbf{S}$ can be simplified as

$$
e = KL\mathbf{P}_1^2 + 2\sigma_n^2 \mathbf{P}, \\
f = qc\mathbf{P}_1^2, \\
q = \frac{t}{K}.
$$

Note that the cross term $f$ is nonzero for $\rho = 0$. Thus, there is an artificial correlation introduced at $\rho = 0$ in the case of $\mathbf{R}_{\text{mfs}}$. This is not so in the case of $\mathbf{R}_{\text{c}}$. Similarly, $\mathbf{S}_b = \mathbf{S}$ which indicates the same behaviour.

Case 2. For $\rho = 1$, $r = q$.

Numerical evaluation of $r$ as a function of $K$, for some values of $\rho$, results in the following observations:

- For $\rho = 0$, RMF and RMFB are identical (as expected) but both of them introduce an artificial cross term or in other words an artificial correlation.
For $\rho = 0.5$, RMFB has a small $r$ for all $K$. For $9 \leq K \leq 13$ RFB has an $r$ which is even smaller than that of RMFB. For $K = 8$ and $\rho = 0.5$, RFB and RMFB have almost identical values of $r$.

For $\rho = 1$, RMF and RF have the same Relative cross term magnitudes. RMFB has a relative cross term magnitude which is much smaller than RFB for almost all $K$.

6. Statistical performance analysis

6.1. Expressions for the mean-squared error in DOA for the subspace methods

A unified framework for the performance analysis of the signal and noise subspace methods was developed in [7,11–15]. Based on this framework, the statistics of the error in the DOA for the various subspace methods is presented below. Define the following vectors:

$$
\beta = R^s_v V(\omega_i),
$$

$$
\mu^H = V^H(\omega_i)P_n,
$$

$$
\nu^H = V^H(\omega_i)P_p,
$$

$$
\varepsilon^H = -\beta^H V^{-\dagger} (W' - z^H W')P_n,
$$

where, $\beta^H, \nu^H$ and $\mu^H$ belong to the noise subspace, and $\beta$ belongs to the signal subspace. Also,

$$
P_g = \begin{bmatrix} 1 \\ \mathbf{g} \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{g}^H \end{bmatrix},
$$

where $[1 \mathbf{g}]$ is the minimum-norm vector in the noise subspace [8]. $W', W'$ are matrices which, when pre-multiplied, select the first $L - 1$ and the last $L - 1$ rows of a matrix, respectively.

$$
R^s_v = E_s A_s^{-1} E_s^H,
$$

$$
V_i(\omega) = \frac{\partial V(\omega)}{\partial \omega},
$$

Let $\alpha, \beta, \gamma, \delta$ be arbitrary vectors of the same dimension and compatible with the dimension of $\hat{R}$. Then

$$
\Gamma_{x\beta, \gamma\delta} = (\alpha^H \hat{R})((\beta^H \hat{R}^\dagger)\gamma^\delta).
$$

For example, $\Gamma_{\mu\mu, \beta\beta} = (\mu_1^H \hat{R}_1^p)(\mu_2^H \hat{R}_2^p)\beta^\beta$.

The MSE of the error in DOA for MUSIC is given by

$$
(\Delta \omega_i^2)_{mu} = \frac{\mu_{\mu, \beta, \beta} + \text{Re} \Gamma_{\mu, \beta, \beta}}{2(V^H_1(\omega)P_p V_1(\omega))^2}.
$$

The MSE of the error in DOA for the minimum-norm method is given by

$$
(\Delta \omega_i^2)_{mn} = \frac{\nu_{\nu, \beta, \beta} + \text{Re} \Gamma_{\nu, \beta, \beta}}{2(V^H_1(\omega)P_p V_1(\omega))^2}.
$$

The MSE of the error in DOA for ESPRIT is given by

$$
(\Delta \omega_i^2)_{e} = \frac{1}{4} [\Gamma_{x, \beta, \beta} - \text{Re} \Gamma_{x, \beta, \epsilon, \beta}].
$$

It is clear that the statistics of the estimate of the covariance matrix estimate, due to finite data, is needed for obtaining the expressions.

6.2. Modified spatial smoothing – statistics of the estimated covariance matrix

In order to determine the statistics of the estimated covariance matrix the assumptions about signal and noise mentioned before, are used. Consider the case where $\hat{R}$ is equal to $\hat{R}_{mfs}$ and derive expressions for $\Gamma_{x\beta, \gamma\delta}$ which can be used for the calculation of the mean-squared error in the DOA using the subspace methods. In the calculation of the mean-squared error in the DOA using the subspace methods the terms $\Gamma_{\nu\nu, \beta\beta}$ and $\Gamma_{\nu\beta, \nu\beta}$ arise, where $\nu = \text{noise subspace vector and } \beta = \text{signal subspace vector. The cases of the modified forward only smoothed and modified forward–backward Smoothed covariance matrices are examined separately.}$

6.3. Statistics of the modified forward only smoothed covariance matrix

6.3.1. Case 1

Here $\Gamma_{x\beta, \gamma\delta} = \Gamma_{x\beta, \gamma\beta}$.

Recall the definitions of $\hat{R}^U$ and $\hat{R}_{mfs}$. We can now write

$$
\varepsilon^H \hat{R}_{mfs} \beta = \frac{1}{N^2} \sum_{K_p = 1}^{K} \sum_{q = 1}^{K} \sum_{n = 1}^{N} \sum_{m = 1}^{N} \varepsilon^H Y_p(n) Y_q^H(n) Y_q(m) Y_p^H(m) \beta.
$$
\((z^H \tilde{R}_{mfs})^T (z^H \tilde{R}_{mfs})\beta^* \)

\[ = \frac{1}{N^4 K^2} \sum_{p=1}^{K} \sum_{q=1}^{K} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{r=1}^{K} \sum_{s=1}^{K} \sum_{l=1}^{L} \sum_{a=1}^{L} x_u^* N_x \beta_a \]

\[ \sum_{k=1}^{N} \sum_{l=1}^{L} (z^H Y_p(i))(P^H(j)Y_q(j))(Y_p(i)\beta) \]

\[ \times (z^H Y_p(i))^* (P^H(k)Y_q(j))^* (Y_p(i)\beta)^* \]  

(17)

Recall that for the pth subarray

\[ Y_p(i) = V_x \Psi^T p + N_x \]

Since \( z \) belongs to the noise subspace it is orthogonal to the columns of \( V_x \) and we have

\[ z^H Y_p(i) = z^H N_x \]

Expanding the various terms in Eq. (17) in terms of the subarray components we have

\[ z^H Y_p(i) = z^H N_x = \sum_{a=1}^{L} z_u^* N_x \]

\[ Y_q(h)Y_q(j) = \sum_{b=1}^{L} Y_q(h)Y_q(j) \]

\[ Y_p(h)\beta = \sum_{c=1}^{L} \beta_c Y_p(h) \]

\[ (z^H Y_p(i))^* = (z^H N_x)^* = \sum_{a=1}^{L} z_u^* N_x \]

\[ (P^H(k)Y_q(j))^* = \sum_{e=1}^{L} Y_e(k)Y_e^*(l) \]

\[ (Y_p(i)\beta)^* = \sum_{j=1}^{L} \beta_j^* Y_p(i) \]

where \( Y_p(i) \) is the ith element of the pth subarray, \( z_u \) the uith element of the vector \( z \), and \( \beta_c \) the cth element of the vector \( \beta \).

Using the above equations in Eq. (17) we get

\[ \Gamma_{x^p y^p} = (z^H \tilde{R}_{mfs}^T)(z^H \tilde{R}_{mfs})\beta^* \]

\[ = \frac{1}{N^4 K^2} \sum_{p=1}^{K} \sum_{q=1}^{K} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{r=1}^{K} \sum_{s=1}^{K} \sum_{l=1}^{L} \sum_{a=1}^{L} \sum_{c=1}^{L} z_u^* \beta_c z_u \beta_c^* \]

\[ \sum_{k=1}^{N} \sum_{l=1}^{L} (Y_p(i))Y_q(j)Y_p(j)Y_q(j)Y_e(k)Y_e(l)N_x \]

(18)

So, from Eq. (18) the expectation of eight Gaussian random variables are needed. Note that \( Y_{pa}(i) \), \( Y_{pa}(j) \), etc. are all zero mean complex Gaussian random variables. Using the expectation of eight Gaussian random variables and the aforementioned assumptions on the statistics of the signal and noise vectors, the above equation can be simplified. There are totally 24 nonvanishing terms, each being the product of four quantities, each quantity being the expected value of the product of two random variables [10]. Typically, we have

\[ \sum_{p=1}^{K} \sum_{q=1}^{K} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{r=1}^{K} \sum_{s=1}^{K} \sum_{l=1}^{L} \sum_{a=1}^{L} \sum_{c=1}^{L} \]

\[ \sum_{k=1}^{N} \sum_{l=1}^{L} (Y_p(i))Y_q(j)Y_p(j)Y_q(j)Y_e(k)Y_e(l)N_x \]

(19)

where \( V_x \) is the ath row of \( V_x \).

6.3.2. Case 2

Here we have \( \Gamma_{x^p y^p} = \Gamma_{x^p y^p} = (z^H \tilde{R}_{mfs}^T)(z^H \tilde{R}_{mfs})\beta^* \) which is just \( \Gamma_{x^p y^p} \) in which the second term \( (z^H \tilde{R}_{mfs}^T)\beta^* \) is replaced by \( (z^H \tilde{R}_{mfs})\beta^* \). Following the steps as in the derivation of Eq. (18) from Eq. (17) we have

\[ \Gamma_{x^p y^p} = (z^H \tilde{R}_{mfs}^T)(z^H \tilde{R}_{mfs})\beta^* \]

\[ = \frac{1}{N^4 K^2} \sum_{p=1}^{K} \sum_{q=1}^{K} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{r=1}^{K} \sum_{s=1}^{K} \sum_{l=1}^{L} \sum_{a=1}^{L} \sum_{c=1}^{L} \]

\[ \sum_{k=1}^{N} \sum_{l=1}^{L} (Y_p(i))Y_q(j)Y_p(j)Y_q(j)Y_e(k)Y_e(l)N_x \]

(20)

Remark. The expressions are quite messy. Some simplification is possible by noting that \( \beta \) is same for all the three subspace methods and that there is a similarity between Eq. (18) for the calculation of \( \Gamma_{x^p y^p} \) and Eq. (20) for the calculation of \( \Gamma_{x^p y^p} \). These can be utilized to calculate \( \Gamma_{x^p y^p} \) and \( \Gamma_{x^p y^p} \) and hence the MSE in the DOA for all the three subspace methods, using equations in Section 6.1.

6.4. Statistics of the modified forward–backward smoothed covariance matrix

Recall the definition of \( \tilde{R}_{mfs} \),

\[ \Gamma_{x^p y^p} = t_1 + t_2 + t_3 + t_4 \]  

(21)
where $t_1 = \frac{1}{4}E[(\beta^H \tilde{\mathbf{R}}_{\text{msf}} \delta)(\beta^H \tilde{\mathbf{R}}_{\text{msf}} \delta)]$, $t_2 = \frac{1}{4}E[(\beta^H \tilde{\mathbf{R}}_{\text{msf}} \delta)(\beta^H \tilde{\mathbf{J}} \tilde{\mathbf{R}}_{\text{msf}} \mathbf{J}^T)\delta]$, $t_3 = \frac{1}{4}E[(\alpha^H \tilde{\mathbf{J}} \tilde{\mathbf{R}}_{\text{msf}} \mathbf{J}^T)\delta](\beta^H \tilde{\mathbf{R}}_{\text{msf}} \delta)^*$, and $t_4 = \frac{1}{4}E[(\beta^H \tilde{\mathbf{J}} \tilde{\mathbf{R}}_{\text{msf}} \mathbf{J}^T)\delta](\beta^H \tilde{\mathbf{J}} \tilde{\mathbf{R}}_{\text{msf}} \mathbf{J}^T)\delta]$. Here again, two cases arise.

6.4.1. Case 1

Here, $\Gamma_{x\beta,\gamma} = \Gamma_{x\beta\beta}$. From Eq. (18) we can write the term $t_1$ as

$$t_1 = \frac{1}{4}(\alpha^H \tilde{\mathbf{R}}_{\text{msf}} \beta)(\beta^H \tilde{\mathbf{R}}_{\text{msf}} \beta)^* = \frac{1}{4N^4K^2} \sum_{p=1}^{K} \sum_{q=1}^{K} \sum_{r=1}^{N} \sum_{s=1}^{N} \sum_{k=1}^{K} \sum_{l=1}^{K} \sum_{a=1}^{L} \sum_{b=1}^{L} \sum_{c=1}^{L} \sum_{d=1}^{L} \alpha^H \beta a \beta^H \beta^* c/1$$

$$\times N_{pa}(i)Y_{qb}(i)Y_{pb}(j)Y_{rc}(l)Y_{se}(k)N_{rd}(k).$$

(22)

For the forward–backward case it can be shown that [7]

$$\mathbf{J} \alpha^H = \alpha \mathbf{z}_i^{-L-1},$$

where $\alpha$ is either $\mu_i$ (MUSIC) or $\epsilon_i$ (ESPRIT) and $\mathbf{z}_i = e^{j\omega_i}$, $\nu_i$ (minimum-norm method) does not satisfy the above relationship except when the noise subspace is of dimension one (i.e. $M = d + 1$). $\beta$ equals $\beta_i$ for the $i$th DOA, $1 \leq i \leq d$. Unlike $\alpha$, $\beta_i$ does not depend on the method and satisfies

$$\mathbf{J} \beta_i^H = \beta_i \mathbf{z}_i^{-L-1}.$$  

Using the above relations we can write, for the case of ESPRIT and MUSIC,

$$(\beta^H \tilde{\mathbf{J}} \tilde{\mathbf{R}}_{\text{msf}} \beta)(\alpha^H \tilde{\mathbf{R}}_{\text{msf}} \beta)^* = (\alpha^H \tilde{\mathbf{R}}_{\text{msf}} \beta)(\beta^H \tilde{\mathbf{J}} \tilde{\mathbf{R}}_{\text{msf}} \mathbf{J}^T)\delta)\beta^*,$$

$$(\alpha^H \tilde{\mathbf{J}} \tilde{\mathbf{R}}_{\text{msf}} \mathbf{J}^T)\delta)(\beta^H \tilde{\mathbf{J}} \tilde{\mathbf{R}}_{\text{msf}} \mathbf{J}^T)\delta)\beta^* = (\beta^H \tilde{\mathbf{J}} \tilde{\mathbf{R}}_{\text{msf}} \mathbf{J}^T)\delta)(\alpha^H \tilde{\mathbf{J}} \tilde{\mathbf{R}}_{\text{msf}} \mathbf{J}^T)\delta)^*,$$

$$(\beta^H \tilde{\mathbf{J}} \tilde{\mathbf{R}}_{\text{msf}} \mathbf{J}^T)\delta)(\beta^H \tilde{\mathbf{J}} \tilde{\mathbf{R}}_{\text{msf}} \mathbf{J}^T)\delta) = (\beta^H \tilde{\mathbf{J}} \tilde{\mathbf{R}}_{\text{msf}} \mathbf{J}^T)\delta)(\beta^H \tilde{\mathbf{J}} \tilde{\mathbf{R}}_{\text{msf}} \mathbf{J}^T)\delta)^*.$$  

Using the relations in Eq. (23) for the calculation of $t_2$–$t_4$ it can be seen that $t_1 = t_4$ and $t_2 = t_3^*$. Here $t_1$ and $t_4$ are both real.

Hence, for ESPRIT and MUSIC,

$$\Gamma_{x\beta\beta} = t_1 + t_2 + t_3 + t_4 = 2(t_1 + \text{Re}(t_2)).$$

(24)

where $t_2$ is given by

$$t_2 = \frac{1}{4}(\alpha^H \tilde{\mathbf{R}}_{\text{msf}} \beta)(\beta^H \tilde{\mathbf{R}}_{\text{msf}} \beta)^* = \frac{1}{4N^4K^2} \sum_{p=1}^{K} \sum_{q=1}^{K} \sum_{r=1}^{N} \sum_{s=1}^{N} \sum_{k=1}^{K} \sum_{l=1}^{K} \sum_{a=1}^{L} \sum_{b=1}^{L} \sum_{c=1}^{L} \sum_{d=1}^{L} \alpha^H \beta a \beta^H \beta^* c/1$$

$$\times N_{pa}(i)Y_{qb}(i)Y_{pb}(j)Y_{rc}(l)Y_{se}(k)N_{rd}(k).$$

(25)

For the case of the minimum-norm method, remember that the above simplification is not possible.

We define now $\zeta = \mathbf{J} \alpha$ and $\eta = \mathbf{J} \beta$ and evaluate terms $t_2$–$t_4$ separately. For the minimum norm method $t_1$ is the same as that of ESPRIT or MUSIC (given by Eq. (22)) and $t_2$–$t_4$ are given by

$$t_2 = \frac{1}{4}(\alpha^H \tilde{\mathbf{R}}_{\text{msf}} \beta)((\beta^H \tilde{\mathbf{J}} \tilde{\mathbf{R}}_{\text{msf}} \mathbf{J}^T)\delta)^* = \frac{1}{4N^4K^2} \sum_{p=1}^{K} \sum_{q=1}^{K} \sum_{r=1}^{N} \sum_{s=1}^{N} \sum_{k=1}^{K} \sum_{l=1}^{K} \sum_{a=1}^{L} \sum_{b=1}^{L} \sum_{c=1}^{L} \sum_{d=1}^{L} \zeta_a \beta a \zeta^H \beta^* d/1$$

$$\times Y_{pa}(i)Y_{qb}(i)Y_{pb}(j)Y_{rc}(l)Y_{se}(k)N_{rd}(k).$$

(26)
6.4.2. Case 2

Here, $I_{\beta\beta} = I_{\beta\beta}^*$. From Eq. (20) we can write the term $t_1$ as

$$t_1 = \frac{1}{4} (\overline{z^H R_{mf} \beta})(\beta R_{mf} \overline{z})^*$$

$$= \frac{1}{4N^4K^2} \sum_{p=1}^{K} \sum_{q=1}^{K} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{r=1}^{K} \sum_{s=1}^{K} \sum_{l=1}^{L} \sum_{a=1}^{L} \sum_{b=1}^{L} \sum_{c=1}^{L} \sum_{u=1}^{L} \sum_{e=1}^{L} \sum_{f=1}^{L} \overline{z}_{u}^* \beta_i \beta_s \overline{z}_{f}^*$$

$$\times N_{pd}(i) Y_{q(i)}(j) Y_{p}(j) Y_{n}^*(l) Y_{se}(l) Y_{se}(k) Y_{ra}(k).$$

(27)

Following the same steps as in Case 1, we have for ESPRIT and MUSIC

$$I_{\beta\beta} = t_1 + t_2 + t_3 + t_4$$

$$= 2(\text{Re}(t_1) + \text{Re}(t_2)),$$

(28)

where $t_2$ is given by

$$t_2 = \frac{1}{4} (\overline{z^H R_{mf} \beta})(\beta R_{mf} \overline{z})^*$$

$$= \frac{1}{4N^4K^2} \sum_{p=1}^{K} \sum_{q=1}^{K} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{r=1}^{K} \sum_{s=1}^{K} \sum_{l=1}^{L} \sum_{a=1}^{L} \sum_{b=1}^{L} \sum_{c=1}^{L} \sum_{u=1}^{L} \sum_{e=1}^{L} \sum_{f=1}^{L} \overline{z}_{u}^* \beta_i \beta_s \overline{z}_{f}^*$$

$$\times N_{pd}(i) Y_{q(i)}(j) Y_{p}(j) Y_{n}^*(l) Y_{se}(l) Y_{se}(k) Y_{ra}(k).$$

(29)

For the minimum norm method $t_1$ is the same as that of ESPRIT or MUSIC (given by Eq. (27)) and $t_2, t_3$ and $t_4$ are given by

$$t_2 = \frac{1}{4} (\overline{z^H R_{mf} \beta})(\eta R_{mf} \overline{z})^*$$

$$= \frac{1}{4N^4K^2} \sum_{p=1}^{K} \sum_{q=1}^{K} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{r=1}^{K} \sum_{s=1}^{K} \sum_{l=1}^{L} \sum_{a=1}^{L} \sum_{b=1}^{L} \sum_{c=1}^{L} \sum_{u=1}^{L} \sum_{e=1}^{L} \sum_{f=1}^{L} \overline{z}_{u}^* \eta_i \overline{z}_{f}^*$$

$$\times N_{pd}(i) Y_{q(i)}(j) Y_{p}(j) Y_{n}^*(l) Y_{se}(l) Y_{se}(k) Y_{ra}(k),$$

$$t_3 = \frac{1}{4} (\overline{z^H R_{mf} \beta})(\beta R_{mf} \overline{z})^*$$

$$= \frac{1}{4N^4K^2} \sum_{p=1}^{K} \sum_{q=1}^{K} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{r=1}^{K} \sum_{s=1}^{K} \sum_{l=1}^{L} \sum_{a=1}^{L} \sum_{b=1}^{L} \sum_{c=1}^{L} \sum_{u=1}^{L} \sum_{e=1}^{L} \sum_{f=1}^{L} \overline{z}_{u}^* \eta_i \beta_s \overline{z}_{f}^*$$

$$\times Y_{p}(j) Y_{q(i)}(j) Y_{p}(j) Y_{n}^*(l) Y_{se}(l) Y_{se}(k) Y_{ra}(k).$$

(30)

Remark. Thus, we can see that some simplification is possible for ESPRIT and Root-MUSIC in the calculation of $I_{\beta\beta}$ and $I_{\beta\beta}^*$ using the modified forward–backward smoothed covariance matrix. For the minimum-norm method no simplification is possible.

The statistical analysis has not yielded tractable closed-form expressions and it is difficult to simplify even for the one- and two-source cases. In order to validate the analytical results and also to study the performance of the methods, extensive simulations have been carried out and a few examples are presented next.

7. Simulation studies

In order to compare the performance of the subspace methods with the MSS and the conventional spatial smoothing schemes, a numerical study is undertaken. A few typical examples are considered for studying the comparative performance. For all the cases considered,

- The data vector is generated using Eq. (2) for the case of the ULA.
- The parameter $d_1/\lambda_0$ is chosen to be $\frac{1}{2}$. 

The noise vector is generated as a Gaussian random vector of unit variance ($\sigma^2_n = 1$).

The signal-to-noise ratio (SNR) is defined as $10 \log P_1/\sigma^2_n$ where $P_1$ is the power in each source. The signal amplitude vector is taken to be a gaussian random vector which is independent of the noise vector.

The $M \times M$ covariance matrix was generated in a single trial which had 100 snapshots. The different covariance estimates RF, RMF, RFB and RMFB were all formed from the same $M \times M$ covariance matrix. ESPRIT, Root-MUSIC and Root-Minimum Norm Method were implemented for all the four covariance matrix estimates.

For each example, 500 independent trials were carried out to obtain the estimates of the signal zeros and hence the DOA for each method.

To compute the statistics of the DOA, the actual DOA are arranged first in the order of increasing magnitude. The estimates obtained are also sorted in the same manner and assigned to the corresponding DOA. The mean-squared error (MSE) in DOA is computed in degrees.

To compare the performance of the subspace methods using various covariance matrix estimates, a plot of $-10 \log($MSE$)$ versus the parameter of interest is presented. Thus, the higher the value, the better the performance of the method. In the tables presented the notation (th) implies that the entries in that column are theoretical results. In all the tables, the MSE (dB) in the DOA of $18^\circ$ is considered.

**Example 1.** A ULA with $M = 16$, $N = 100$, DOA = $18^\circ$, $28^\circ$ is considered.

Table 1 presents the MSE in DOA corresponding to $18^\circ$ obtained from simulations and compared with the MSE obtained from the analytical expressions derived above.

- The theoretical MSE in the DOA using RMF and RMFB closely match the simulation results! This was observed for all the examples considered in the simulation study.
- The use of RMFB covariance matrix yields lower MSE compared with the use of the RMF estimator.
- The MSE increases for extreme cases of $K$ and there seems to be an optimal value of $K$. Performance analysis presented in [15] indicates an optimal value of $K$ for which the MSE is minimum, which depends upon the source parameters. A similar trend is observed in the above analysis too. But an interesting result was presented in [6] where the optimal value of $K$ for two equipowered, closely spaced, coherent sources was found to be independent of the source parameters. It would be interesting if such a result holds true for the MSS technique.

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Table 1
MSE (dB) in DOA corresponding to $18^\circ$ with $M = 6$, SNR = 20 dB, DOA = $18^\circ$, $28^\circ$, $\rho = 1$
Henceforth, the MSE obtained through simulations is presented as the theoretical values are close to the simulation results.

**Example 2.** A ULA with $M = 16$, $N = 100$, DOA = $18^\circ$, $23^\circ$ is considered.

Tables 2–4 present the performance of all the subspace methods as a function of the number of subarrays ($K$), correlation coefficient $\rho$ and separation between the two sources (sep), respectively.

- For all subspace methods, the use of RMF and RMFB yields better performance, in general, than the use of RF and RFB, respectively, when $K$ is large. Also, the use of RFB and RMFB yields better performance than the use of RF and RMF, respectively.

- Considering Table 2, as $K$ increases, the MSE gets lower but for larger values of $K$, the MSE again increases. However, the range of values of $K$. $V$. S. HARI, B. V. RAMAKRISHNAN / Signal Processing 79 (1999) 73–85

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Table 4

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Henceforth, the MSE obtained through simulations is presented as the theoretical values are close to the simulation results.
\( K \) for which the MSE is lower is larger when RMF (RMFB) is used, compared with the corresponding range when RF (RFB) is used. This also indicates that there is a wider choice of \( K \) which can be used in the RMF (RMFB) case.

- Considering Table 3, for all subspace methods, the use of RF (RFB) results in a better performance than the use of RMF (RMFB) for low values of \( \rho \) and equals the performance only at \( \rho = 1 \). This is because RMF and RMFB introduce artificial correlation at \( \rho = 0 \) as explained before. It has also been observed that RMF (RMFB) and RF (RFB) have the same relative cross term magnitude for \( \rho = 1 \). But the use of RMF and RMFB results in slightly better performance than RF and RFB, respectively, because the signal eigenvalues of RMF and RMFB are larger and far separated from the noise eigenvalues when compared with those of RF and RFB.

- Considering Table 4, for all the subspace methods, the MSE in the DOA improves with DOA separation for all covariance matrices. Only at a small DOA separation of 2° to 3°, use of RMF (RMFB) yields better performance than RF (RFB).

- Comparing the performance of the subspace methods, for the examples considered, in most cases, ESPRIT yields lower MSE compared with the root-MUSIC and minimum-norm methods. Based on other simulation examples, it was observed that the MSE decreases as signal-to-noise ratio and the snapshots increase. This is expected for obvious reasons.

8. Conclusion

Theoretical expressions for the MSE in DOA estimates from the subspace methods, resulting from the use of the modified smoothed covariance matrix are derived and validated with simulations. It is observed that the modified smoothing case does improve the performance compared to the conventional smoothing, but introduces some artificial correlation for uncorrelated sources.

References


