Semi-global finite-time output feedback stabilization for a class of large-scale uncertain nonlinear systems

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Abstract
This paper addresses the problem of semi-global finite-time decentralized output feedback control for large-scale systems with both higher-order and lower-order terms. A new design scheme is developed by coupling the finite-time output feedback stabilization method with the homogeneous domination approach. Specifically, we first design a homogeneous observer and an output feedback control law for each nominal subsystem without the nonlinearities. Then, based on the homogeneous domination approach, we relax the linear growth condition to a polynomial one and construct decentralized controllers to render the nonlinear system semi-globally finite-time stable.

1. Introduction
In this paper, we consider the problem of semi-global finite-time stabilization via output feedback for a class of large-scale uncertain nonlinear systems

\[ \dot{x}_{ij} = x_{ij+1}^{p_j} + \varphi_{ij}(x, d(t)), \quad j = 1, \ldots, n - 1, \]
\[ \dot{x}_m = u_i + \varphi_{in}(x, d(t)), \]
\[ y_i = x_{i1}, \quad (1) \]

where \( x_i = (x_{i1}, \ldots, x_{in})^T \in \mathbb{R}^n, i = 1, \ldots, m, x = (x_1, \ldots, x_m)^T \in \mathbb{R}^{m\times n}, u_i \in \mathbb{R}, y_i \in \mathbb{R} \) are system states, control input and output, respectively. The powers \( p_j, j = 1, \ldots, n - 1 \) and \( p_m = 1 \) are odd integers. \( \varphi_{ij}(\cdot) \)'s are uncertain nonlinear functions of all the states with bounded disturbance \( d(t) \).

The problem of output feedback stabilization of (1) has attracted a great deal of attention from the nonlinear control community. The existing output feedback stabilizer does not hold for all types of nonlinearities owing to the lack of nonlinear version of separation principle [1]. Therefore certain conditions are required for most of the existing global results. Among the different kinds of assumptions, one common condition is that the unmeasurable states cannot be associated with the uncertainties. To deal with the case when the unmeasurable states are associated with the uncertainties, a feedback domination design method was proposed in [2]. Later, the works [3–5] have solved the problem of global output feedback stabilization under a higher-order growth condition by employing the homogeneous domination approach. In [6] we considered the problem of global finite-time stabilization for upper-triangular systems with unknown output gain.

To further relax the aforementioned conditions imposed on global output feedback stabilization, in this paper we pursue a less ambitious control objective, i.e. semi-global output feedback stabilization. It has been shown that the restrictive...
conditions assumed in the global output feedback stabilization problem can be relaxed for semi-global output stabilization. For example, the works [7–10] achieved the semi-global output feedback stabilization of feedback linearizable (at least partially) systems. In [11], it was shown that semi-global output feedback stabilization was achievable for uniformly completely observable and state feedback stabilizable systems. The work [12] explicitly constructed a linear output feedback controller to semi-globally exponentially stabilize a class of nonlinear systems under less restrictive conditions. One novelty of the design method is that the observer and controller are linear and independent of the higher-order nonlinearities and the mismatched uncertainties. The paper [13] considered a class of systems in which the subsystem for each output has a triangular dependence on the states of that subsystem itself, and the overall system has a block triangular form for each subsystem. In [14], semi-global robust stabilization was achieved for a class of MIMO nonlinear systems which do not necessarily have a well-defined relative degree nor need to be affine in the control inputs, but exhibit certain triangular structure. In [15], we considered the problem of semi-global stabilization by a series of linear output feedback stabilizers. The work [16] presented the semi-global output feedback stabilization for a class of nonlinear systems with unknown output gains. The problem of output tracking control was addressed for a class of nonlinear strict-feedback form systems in the presence of nonlinear uncertainties, external disturbances, unmodeled dynamics and unknown control coefficients in [17].

Most of the existing works only consider the asymptotic output feedback stabilizer that makes the trajectories of the system converge asymptotically to the equilibrium as the time goes to infinity. Our recent work [18] considered the problem of semi-global finite-time stabilisation by output feedback for a class of uncertain nonlinear systems with both higher-order and lower-order terms. In this paper, we focus on the problem of using output feedback to semi-globally stabilize a class of large-scale nonlinear systems in a finite time, namely, we are interested in designing the decentralized output feedback controllers that will render the trajectories of the closed-loop systems convergent to the origin in a finite time. The large-scale systems have more states and the unmeasurable states interconnected, which make the problem more complicated. A new output feedback design scheme will be developed by coupling the finite-time output feedback stabilization method with the homogeneous domination approach. Specifically, we will first design a homogeneous observer and an output feedback controller and appropriately adjust the gain to dominate the nonlinearities. An example is included in Section 3 to illustrate the effectiveness of the proposed decentralized output feedback controllers. Our conclusion is in Section 4.

2. Main results

In this section, we will employ the homogeneous domination approach to construct decentralized output feedback controllers for system (1). Specifically, we will first construct the homogeneous output feedback controller for each nominal subsystem without considering the nonlinearities \( \phi_i \). Then, we will utilize a scaling gain in the controller and observer to dominate the nonlinearities.

Firstly, we construct an output feedback stabilizer for the following nominal system

\[
\dot{z}_i = z_{i+1}^{(n)} - k_iz_i + \beta_i y, \quad i = 1, \ldots, n - 1, \quad \dot{z}_n = \nu, \quad y = z_1,
\]

where \( z = (z_1, \ldots, z_n)^T \in \mathbb{R}^n, \nu \in \mathbb{R}, y \in \mathbb{R} \) are system states, control input and output, respectively. Using the approach in [19,20], we design a homogenous output feedback stabilizer for (2) which is described in the following lemma.

**Lemma 2.1.** For any constant \( \bar{\tau} \in \left(-\frac{1}{1+\sum_{i=1}^n \rho_i}, 0\right) \), there exist constants \( k_i > 0, i = 1, \ldots, n - 1 \) and \( \beta_j > 0, j = 1, \ldots, n \) such that the homogeneous output feedback controller

\[
\dot{\bar{y}}_2 = -k_2z_{i+1}^{(n)} + \eta_2y, \quad \dot{\bar{y}}_i = -k_i\bar{y}_{i+1} + \eta_iy, \quad i = 2, \ldots, n - 1, \\
\dot{\bar{y}}_n = -k_n\bar{y}_{n-1} + \eta_ny + \beta_1 z_2^{(n)} + \beta_2 z_3^{(n)} + \cdots + \beta_n y, \\
\nu = -\beta_n \left( \bar{y}_n^{(n)} + \beta_1 z_2^{(n)} + \cdots + \beta_n z_n^{(n)} \right),
\]

with the constants \( r_i, r_i \)'s defined as

\[
r_1 = 1, \quad r_i + \bar{\tau} = r_{i+1}, \quad i = 1, \ldots, n
\]

renders system (2) globally finite-time stable.

**Proof.** The proof is very similar to the one [20, Theorem 2.1] with some modifications. For the sake of space, the detailed proof is omitted here. \( \Box \)
Remark 2.1. It should be pointed out that the output feedback controller (3) is only continuous due to the presence of the powers \( r_ip_i \), \( i = 2, \ldots, n \), which are less than one by (4) since \( \tau < 0 \). As a result, the closed-loop system (2) and (3) is not locally Lipschitz. So the uniqueness of the solution of system (2) and (3) is not guaranteed. Fortunately, as shown in the work [21], the existence of the solution can still be guaranteed for a continuous system without Lipschitz condition.

Denoting \( X := [z_1, \ldots, z_n, \eta_2, \ldots, \eta_n]_\tau \), it is straightforward to verify that the closed-loop system (2) and (3), which can be rewritten as the following compact form

\[
X = F(X) = (x_1^{p_1}, \ldots, x_n^{p_n}, u, f_{n+1}, \ldots, f_{2n-1})^T,
\]

where \( f_{n+1} = \bar{\eta}_2, f_{n+2} = \bar{\eta}_1, f_{2n-1} = \bar{\eta}_n \). Moreover, by choosing the dilation weight (refer to [22] for details)

\[
\Lambda = (r_1, \ldots, r_n; r_1, \ldots, r_n) = \left( \begin{array}{cccc}
1, & 1 + (1 + p_1 + \cdots + p_1 \cdots p_n) \tau & \cdots & 1 + (1 + p_1 + \cdots + p_1 \cdots p_n) \tau \\
\vdots & \ddots & \ddots & \vdots \\
1 + (1 + p_1 + \cdots + p_1 \cdots p_n) \tau & \cdots & 1 + (1 + p_1 + \cdots + p_1 \cdots p_n) \tau & 1 \\
\end{array} \right),
\]

it is clear that system (5) is homogeneous of degree \( \tau \) with respect to \( \Lambda \) according to the Definition A.2.

Since system (5) is globally finite-time stable, by Theorem 2 in [23], there exists a Lyapunov function \( V(X) \) with homogeneous of degree 2, such that

\[
\dot{V}(X) = \frac{\partial V}{\partial X}(X) F(X) \leq -d_1 \|X\|_\Lambda^{2+\tau},
\]

where \( d_1 > 0 \) and \( \|X\|_\Lambda = \sqrt{\sum_{i=1}^{n-1} |x_i|^{2+\tau}} \). In addition, by Lemma A.3

\[
|\partial V/\partial X| \leq d_2 \|X\|_\Lambda^{2-\tau}, \quad d_2 > 0.
\]

Together with the homogeneous controller and observer established in Lemma 2.1, we are ready to use the homogeneous domination approach to semi-globally stabilize (1) via output feedback under the following assumption.

Assumption 2.1. There exists any constant \( \tau \in \left( -\frac{1}{1+\max_{1 \leq i \leq n} \left( \sum_{j=1}^{i-1} (r_{ij} - p_j) \right)} \right) \) satisfying \( r_{ij} + \tau \leq m_1 \) and \( \sum_{j=1}^{i-1} (p_j - \sum_{k=0}^{i-2} (p_j \cdots p_k)) > 0 \), for all \( 1 \leq l, i \leq m, 2 \leq k \leq j \leq n \) such that

\[
|\varphi_{ij}| \leq q_{ij} \left( |x_{i+1}|^{r_{ij}} + \cdots + |x_{ij}|^{r_{ij}} \right) + \cdots + \left( |x_{i+1}|^{r_{ij}} + \cdots + |x_{ij}|^{r_{ij}} \right) + b_{ij} (y_i) \sum_{l=1}^{m} \sum_{k=1}^{i} |x_k|^{m_{ij}}, \quad |\varphi_{ij}| \leq q_{ij} (y_i) |x_i|^{1+\tau}.
\]

where \( r_{ij} = 1, \tau = \tau_{ij} - (r_{ij} + \tau), p_{ij} = 2, \ldots, n \) and \( q_{ij} = 0, b_{ij} = 1/p_{ij}, j = 2, \ldots, n \). \( a_{ij}(y_i) \geq 0, b_{ij}(y_i) \geq 0 \) are smooth functions.

Remark 2.2. Assumption 2.1 is more general than the assumptions imposed in [18,24]. It can be easily concluded from (8) that assumption in [24] is a special case of Assumption 2.1 with \( p_i = 1 \) and \( b_i(\cdot) = 0 \). In addition, compared to [18], Assumption 2.1 is also more general. It can be seen that assumption in [18] is a special case of Assumption 2.1 with \( p_i = 1 \) and \( m = 1 \).

For simplicity, in this paper we assume \( \tau = -q_i/p_i \), with \( q_i \) an even integer and \( p_i \), an odd integer. Based on this, \( r_{ij} \) will be odd in both denominator and numerator. Note that a similar result can be achieved for a real number \( \tau \) as shown in Remark 2.3.

Theorem 2.1. Under Assumption 2.1, the problem of semi-global output feedback stabilization for uncertain nonlinear system (1) can be solved by a series of homogeneous output feedback stabilizers.

Proof. The output feedback stabilizers are constructed by introducing a scaling gain into the output feedback controller obtained in Lemma 2.1. Then we shall show that for any given domain of the initial condition, all the trajectories starting from the domain will converge to the origin in a finite time.

First, we introduce the change of coordinates

\[
z_i = \frac{X_i}{L^{r_{ij}}}, \quad i = 1, \ldots, m, j = 1, \ldots, n \quad \text{and} \quad \eta_i = \frac{u_i}{L^{r_{ii}+\tau}},
\]

where \( L \geq 1 \) is a constant to be determined later.

By construction \( r_{ij} = \tau q_{ij} + 1/(p_{ij}p_{ij-1}) \), the system (1) can be rewritten as

\[
\dot{z}_i = L_{ij-1}^p + \frac{\varphi_{ij}(\cdot)}{L_{ij}}, \quad j = 1, \ldots, n - 1, \quad \dot{z}_n = L_{iv} + \frac{\varphi_{i}(\cdot)}{L_{iv}}, \quad \eta_i = \dot{z}_i
\]

(10)
Next, we construct the observer with the scaling gain \( L \)
\[
\begin{align*}
\dot{\eta}_{i2} &= -Lk_{i2}\dot{z}_{i2} + \dot{z}_{i2} = (\eta_{i2} + k_{i1}y_{i})^{\frac{\tau_{2i}^q}{r_{i2}}}, \\
\dot{\eta}_{ij} &= -Lk_{ij}z_{ij-1}\dot{z}_{ij-1}, \quad (\eta_{ij} + k_{ij}z_{ij-1})^{\frac{q_{ij}^{r_{ij}}}{r_{ij}}}, \\
& \quad j = 3, \ldots, n,
\end{align*}
\]
where \( k_{ij} > 0, \ i = 1, \ldots, m, \ j = 1, \ldots, n - 1 \) are the gains selected by Lemma 2.1. In addition, we design \( u_i \) using the same construction of (3), specifically,
\[
\begin{align*}
u_i &= L^{\frac{n-1}{2}}v_i, \\
v_i &= -\beta_{im}^{\frac{1}{2\tau_{m}}} + \beta_{m-1}^{\frac{1}{2\tau_{m}}} \left( \beta_{m-1}^{\frac{2}{2\tau_{m}}} + \cdots + \beta_{1}^{\frac{2}{2\tau_{m}}} \right), \quad (\beta_{m-1}^{\frac{2}{2\tau_{m}}} + \cdots + \beta_{1}^{\frac{2}{2\tau_{m}}})^{\frac{n-1}{2}}.
\end{align*}
\]

For each subsystem, similar to (5), we can obtain
\[
\begin{align*}
Z_i &= (z_{i1}, \ldots, z_{im}, \eta_{i2}, \ldots, \eta_{in})^T, \\
F_i(Z_i) &= \left( z_{i1}^{\frac{2}{2\tau_{i1}}}, \ldots, z_{im}^{\frac{2}{2\tau_{im}}}, v_i, f_{i,n+1}, \ldots, f_{i,2n-1} \right)^T,
\end{align*}
\]
where \( f_{i,n+1} = \eta_{i2}, \ldots, f_{i,2n-1} = \eta_{in} \). Now the closed-loop system (10)–(12) can be written as
\[
\begin{align*}
\dot{Z}_i &= LF_i(Z_i) + \left( \varphi_{i1}, \varphi_{i2}, \ldots, \varphi_{in}, \frac{\varphi_{in}}{L^{\frac{n}{2}}}, \tilde{0} \right)^T,
\end{align*}
\]
where \( \tilde{0} \) is a vector of zeros with suitable dimension. By Lemma 2.1, we can design the gains \( k_{ij} \) and \( \beta_{ij} \) such that
\[
\dot{Z}_i = F_i(Z_i),
\]
and globally asymptotically stable. Moreover, according to Lemma A.3, there is a Lyapunov function \( V_i(Z_i) \) for each subsystem with homogeneous degree 2, such that
\[
\begin{align*}
\frac{\partial V_i}{\partial Z_i} F_i(Z_i) &\leq c_i ||Z_i||^{2+\frac{\tau}{2}},
\end{align*}
\]
where \( c_i > 0 \) is a constant and the dilation weight \( \Delta_i = (r_{i1}, \ldots, r_{in}, r_{i1}, \ldots, r_{i,n-1}) \),

It can be seen that for each subsystem
\[
\dot{V}_i(Z_i) = L \frac{\partial V_i(Z_i)}{\partial Z_i} F_i(Z_i) + \frac{\partial V_i(Z_i)}{\partial Z_i} \left( \varphi_{i1}, \varphi_{i2}, \ldots, \varphi_{in}, \frac{\varphi_{in}}{L^{\frac{n}{2}}}, \tilde{0} \right)^T \leq -c_i ||Z_i||^{2+\frac{\tau}{2}} + \frac{\partial V_i(Z_i)}{\partial Z_i} \left( \varphi_{i1}, \varphi_{i2}, \ldots, \varphi_{in}, \frac{\varphi_{in}}{L^{\frac{n}{2}}}, \tilde{0} \right)^T.
\]

Next, we decide the gain \( L \) to guarantee the attractivity for a given domain. To this end, for an arbitrarily large number \( M > 1 \), define compact sets \( N = \{ \eta \ | \ ||\eta|| \leq M, \eta \in [0,1] \} \) and
\[
\Omega = \left\{ (Z_1, Z_2, \ldots, Z_m) \big| V(Z_1, Z_2, \ldots, Z_m) \leq S, S = \max_{Z_i \in N}V(Z_1, Z_2, \ldots, Z_m) \right\},
\]
where \( V(Z_1, Z_2, \ldots, Z_m) = \sum_{i=1}^{m} V_i(Z_i) \) is the Lyapunov function for the whole system. Clearly, \( \Omega \) is a nonempty compact set and in this set \( (Z_1, Z_2, \ldots, Z_m) \) are bounded.

According to Assumption 2.1, we have
\[
\begin{align*}
\frac{\varphi_{in}}{L^{\frac{n}{2}}} \leq a_i(y_i) \sum_{k=1}^{m} \frac{\sum_{l=1}^{m} |z_{lk}|^{q_{ik}+\frac{\tau}{2}}}{L^{q_{ik}}} + b_i(y_i) \sum_{k=1}^{m} \frac{\sum_{l=1}^{m} |z_{lk}|^{q_{ik}+\frac{\tau}{2}}}{L^{q_{ik}}}.
\end{align*}
\]

where \( b_i(y_i) \) is the power of the scaling gain \( L \) at the first part in the right-hand side of (17) is
\[
\begin{align*}
r_{ik}^q q_{ik}^r - q_{ij}^r &= q_{ik}^r (1 + \tau_{ik}) - q_{ij}^r \left( 1 + \tau_{ij} \right) \leq 1 - \frac{q_{ij}^r (1 + \tau_{ij})}{r_{ij}^r} = 1 - \frac{q_{ij}^r q_{ik}^r q_{ij}^r}{r_{ij}^r} = 1 - \frac{r_{ij}^r q_{ij}^r q_{ik}^r}{r_{ij}^r} = 1 - \frac{q_{ij}^r}{r_{ij}^r} (q_{ij}^r q_{ik}^r - q_{ij}^r q_{ik}^r).
\end{align*}
\]

Note that \( q_{ij}^r = 0, \ q_{ij}^r = q_{ij}^r q_{ij}^r \), \( q_{ij}^r \) can be recursively determined as
\[
\begin{align*}
q_{ik}^r &= \frac{1}{p_{ik}^r-1} p_{ik}^{r-1} + \cdots + \frac{1}{p_{ik}^{r-1}} p_{ik}^{r-1} - 1, \quad k \geq 2.
\end{align*}
\]

Owing to \( r_{ij}^r = 1, \ r_{ij}^r + \tau = r_{ij}^r + 1 \), one obtains
\[ r_{ij} = \frac{1 + \tau}{p_{ij}}, \]
\[ r_{il} = \frac{1 + \tau}{p_{il} p_{l}} + \frac{\tau}{p_{il} p_{l}}, \]
and recursively
\[ r_{ik} = \frac{1 + \tau}{p_{i1} \cdots p_{ik-1}} + \frac{\tau}{p_{i1} \cdots p_{ik-1}} \cdots + \frac{\tau}{p_{i1} p_{i2} \cdots p_{i,k-1}}, \quad k \geq 2. \]

In what follows, two cases will be considered.

Case 1: when \( k \geq 2 \), we can obtain
\[ q_{ij+1} r_{lk} - q_{ik} r_{lj+1} = \left( \frac{1}{p_{i1} \cdots p_{ij}} + \frac{1}{p_{ij+1}} \right) \left( \frac{1 + \tau}{p_{i1} \cdots p_{ij} \cdots p_{ij+1}} + \frac{\tau}{p_{i1} \cdots p_{ij} \cdots p_{ij+1}} \right) \]
\[ \cdots \left( \frac{1}{p_{i1} \cdots p_{ik-1}} + \frac{1}{p_{ik}} \right) \left( \frac{1 + \tau}{p_{i1} \cdots p_{ik-1} \cdots p_{ik}} + \frac{\tau}{p_{i1} \cdots p_{ik-1} \cdots p_{ik}} \right) \]
\[ = \frac{1}{(p_{i1} \cdots p_{ik}) (p_{il} \cdots p_{lj})} \left( (1 + p_{i1} + p_{i2} + \cdots + p_{ij} + p_{ij+1} + \cdots + p_{lj} + p_{lj+1}) - (1 + p_{il} + p_{il} p_{ij} + \cdots + p_{il} p_{i1} \cdots p_{ik-2}) \right). \]

By condition \( \sum_{i=1}^{j-1} (p_{i1} \cdots p_{ji}) - \sum_{i=1}^{j-2} (p_{i1} \cdots p_{ij}) > 0 \), one has
\[ q_{ij+1} r_{lk} - q_{ik} r_{lj+1} > 0, \quad \text{for all } 2 \leq k \leq j, \quad 1 \leq l, i \leq m. \]

Case 2: when \( k = 1 \), one obtains
\[ q_{ij+1} r_{lk} - q_{ik} r_{lj+1} = \frac{1}{p_{i1} \cdots p_{ij}} + \frac{1}{p_{il}} > 0. \]

In summary, it can be concluded from (21) and (23) that
\[ q_{ij+1} r_{lk} - q_{ik} r_{lj+1} > 0, \quad \text{for all } 1 \leq k \leq j \leq n, \quad 1 \leq l, i \leq m, \]
which implies that the power of the scaling gain \( L \) at the first part in the right-hand side of (17) is less than one.

Next, we consider the second part in the right-hand side of (17). The power of the scaling gain \( L \) is \( mp_j^k q_{ik} - q_{ij} \). Owing to
\[ \frac{r_{2-}^{\tau}}{\tau^2} \leq m_{ij}^k < \frac{q_{ik} - 1}{q_{ij}} \]
and \( 1 \leq l, i \leq m, 2 \leq k \leq j \leq n \), one obtains
\[ m_{ij}^k q_{ik} - q_{ij} < 1, \]
which in turn implies that there exists a constant \( x_{ij} < 1 \) such that for \( L \geq 1 \),
\[ \left| \frac{\phi_{ij}^k(y_i)}{L^{n_i}} \right| \leq a_{ij}(y_i) L^{n_i} \sum_{l=1}^{m} \sum_{k=1}^{j} \lambda_k |z_k|^\alpha_{ij} + b_{ij}(y_i) L^{n_i} \sum_{l=1}^{m} \sum_{k=1}^{j} \lambda_k |z_k|^\gamma_{ij} \]
\[ = a_{ij}(y_i) L^{n_i} \sum_{l=1}^{m} \sum_{k=1}^{j} \lambda_k |z_k|^\alpha_{ij} + b_{ij}(y_i) L^{n_i} \sum_{l=1}^{m} \sum_{k=1}^{j} \lambda_k |z_k|^\gamma_{ij} |z_k|^\alpha_{ij}. \]

Owing to \( \frac{r_{2-}^{\tau}}{\tau^2} \geq 0 \). From (25), we know that in the compact set \( \Omega \) there is a constant \( d_{ij} \), such that
\[ \left| \frac{\phi_{ij}^k(y_i)}{L^{n_i}} \right| \leq d_{ij} L^{n_i} \sum_{l=1}^{m} |z_k|^\alpha_{ij}. \]

Recall that for \( j = 1, \ldots, n \), \( \partial V_i / \partial Z_{ij} \) is homogeneous of degree \( 2 - r_{ij} \), then
\[ \left| \frac{\partial V_i}{\partial Z_{ij}} \right| \sum_{l=1}^{m} \sum_{k=1}^{j} |z_k|^\alpha_{ij}, \]

is homogeneous of degree \( 2 + \tau \) by Lemma A.1. With (26) and (27) in mind, by Lemma A.3, we can find a constant \( \rho_{ij} > 0 \) such that
\[ \left| \frac{\partial V_i}{\partial Z_{ij}} \right| \frac{\phi_{ij}^k(y_i)}{L^{n_i}} \leq \rho_{ij} L^{n_i} |Z_i|^{2+\tau}, \]
where \( Z = (Z_1, Z_2, \ldots, Z_m)^T \) and \( \Delta = (\Delta_1, \Delta_2, \ldots, \Delta_m) \). Substituting (28) into (16) yields
It can be seen that the time derivative for the whole system is
\[
\dot{V}(Z) = \sum_{i=1}^{m} \dot{V}_i \leq -Lc_i ||Z||_{\Delta}^{2+\tau} + \sum_{j=1}^{n} \rho_j Lx_j ||Z||_{\Delta}^{2+\tau},
\]
(30)
where \(c_{\text{max}} = \max_{1 \leq i \leq m, 1 \leq j \leq n} (c_i) < 1, c > 0 \) and \(\rho > 0\) that do not depend on \(L\). Apparently, by choosing a large enough \(L\), the right-hand side of (30) is negative definite.
In addition, \(V\) is homogeneous of degree 2 with respect to \(\Delta\). By Lemma A.3, there is a constant \(c_1 > 0\), such that
\[
\dot{V}(Z) \leq c_1 ||Z||_{\Delta}^{2}.
\]
(31)
Since the right-hand side of (30) is homogenus of degree 2 + \(\tau\), by Lemma A.3 there is a constant \(c_2 > 0\) such that
\[
\dot{V}(Z) \leq -c_2 ||Z||_{\Delta}^{2+\tau}.
\]
(32)
Combining (31) and (32), it can be deduced from (30) that
\[
\dot{V}(Z) + k_1 \dot{V}^{\frac{2+\tau}{2}}(Z) \leq 0,
\]
(33)
for a positive constant \(k_1 > 0\). By Lemma A.2 (\(k = k_1, x = \frac{2+\tau}{2} < 1\)), (33) leads to the conclusion that the closed-loop system (1), (11) and (12) is semi-globally finite-time stable. □

**Remark 2.3.** During the proof, it is assumed that the fraction number \(\tau\) has an even numerator and an odd denominator. This requirement can be easily removed. We are still able to design a homogeneous controller globally stabilizing the system with necessary modification to preserve the sign of function \(\bar{\psi}_{i/\tau+1}\). Specifically, for any real number \(r_i/r_{i-1} > 0\), we define \(\bar{\psi}_{i/\tau+1} = \text{sign}(\cdot) \cdot |\cdot|^{1/\tau+1}\). Moreover, it can be verified that the function \(\bar{\psi}_{i/\tau+1}\) is \(C^1\).

### 3. Numerical example

In this section, we provide an example to show the effectiveness of the proposed method.

**Example 3.1.** Consider the uncertain nonlinear system
\[
\Sigma_1: \begin{cases}
\dot{x}_{11} = x_{12} + d(t)x_{11}^{4/5} \\
\dot{x}_{12} = u_1 + d(t)x_{12} \ln (1 + x_{12}^3) + x_{11}^{15/11}x_{12}^{6/11} \\
y_1 = x_{11}
\end{cases}
\]
\[
\Sigma_2: \begin{cases}
\dot{x}_{21} = x_{22} \\
\dot{x}_{22} = u_2 + d(t)x_{22}^{1/3} + (1 + x_{21})x_{22}^{3/5} \\
y_2 = x_{21}
\end{cases}
\]
(34)
where \(|d(t)| \leq 1\). Clearly, \(\varphi_{21}\) is trivial. By choosing \(\tau = -2/5, r_{11} = 1, r_{12} = 3/5, r_{21} = 1, r_{22} = 3/5\), it can be verified that
\[
|\varphi_{11}| = |d(t)x_{11}^{4/5}| \leq (1 + |x_{11}|)|x_{11}|^{3/5},
\]
(35)
satisfies Assumption 2.1. In addition, by mean value theory and Lemma A.4, we have
\[
\ln (1 + x_{22}^3) \leq \frac{3s^2}{1 + s^3} |x_{22}|^{2/3} \leq \frac{2s^3 + 1}{1 + s^3} |x_{22}|^{2/3} \leq 2|x_{22}|^{2/3},
\]
for a \(s\) between \([0, |x_{22}|^{2/3}]\). It can be verified that
\[
|d(t)x_{22} \ln (1 + x_{22}^3)| \leq 2|x_{22}|^{5/3}.
\]
(36)
By Lemma A.4, one obtains
\[
|x_{11}^{2/15}x_{12}^{1/9}| \leq |x_{11}|^{1/5} + |x_{12}|^{1/3}.
\]
(37)
Define \(m_{22}^z = 5/3\), which satisfies \(\frac{1}{r_{22}} \leq m_{22}^z < \frac{2s_{0}^{4/5}}{30}\). Combining (36) and (37), it can be seen that \(\varphi_{12}\) satisfies Assumption 2.1. Moreover, by defining \(m_{22}^x = 3/5\), we can verify that
\[
|\varphi_{22}| \leq |x_{12}|^{m_{22}^x \tau_{22}^{1/\tau_{22}}} + (1 + x_{21}^3)x_{22}^{m_{22}^x},
\]
which implies \(\varphi_{22}\) also satisfies Assumption 2.1. Therefore, according to Theorem 2.1, there are homogeneous output feedback controllers rendering the whole nonlinear system semi-globally finite-time stable. Specifically, the output feedback controllers can be constructed as follows:
\[ \dot{\eta}_{12} = -Lk_{11}\xi_{12}^{p_{11}}, \quad \dot{\xi}_{12} = (\eta_{12} + k_{11}y_1)^{3/5}, \]
\[ u_1 = -L^2\beta_{12}(\xi_{12}^{p_{11}} + \beta_{11}^{1/2}y_1)^{1/5}, \]
\[ \dot{\eta}_{22} = -Lk_{21}\xi_{22}^{p_{21}}, \quad \dot{\xi}_{22} = (\eta_{22} + k_{21}y_2)^{3/5}, \]
\[ u_2 = -L^2\beta_{22}(\xi_{22}^{p_{21}} + \beta_{21}^{1/2}y_2)^{1/5}, \]  
where \( k_{11} = 2, \beta_{11} = 1, \beta_{12} = 4, k_{21} = 15, \beta_{21} = 2, \beta_{22} = 6 \) and \( L = 8 \). In the simulation, it is assumed that \( d(t) = \sin(t) \). The simulation result shown in Fig. 1 demonstrates the semi-global finite-time stability property of the closed-loop system (34) and (38) for the initial condition \( \dot{x}_{11}(0), x_{12}(0), \eta_{12}(0), x_{21}(0), x_{22}(0), \eta_{22}(0) = [-1, 3, 0, 2, 1, 0] \).

4. Conclusions

This paper discusses the problem of semi-global finite-time output feedback control for large-scale nonlinear systems with both higher-order and lower-order terms. The subsystems interconnected by nonlinearities which are functions of the measurable and unmeasurable states, and their linearization cannot guarantee to be either controllable or observable. A new design scheme is developed by coupling the finite-time output feedback stabilization method with the homogeneous domination approach. The output feedback controllers with the appropriate gain will guarantee that the trajectories starting from the given initial region will be bounded all the time and converge to the origin in a finite time.

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Appendix A

The appendix collects some basic definitions and useful lemmas which serve as the basis for main results.

Definition A.1 [25]. Consider the following autonomous system

\[ \dot{x} = f(x), \quad \text{with} \quad f(0) = 0, \quad x \in D, \quad x(0) = x_0, \]  
where \( f : D \to \mathbb{R}^n \) is continuous on an open neighborhood \( D \subseteq \mathbb{R}^n \) of the origin. The zero solution of (A1) is finite-time convergent if there are an open neighborhood \( U \subseteq D \) of the origin and a function \( T_x : U \setminus \{0\} \to (0, \infty) \), such that \( \forall x_0 \in U \), the solution trajectory \( x(t, x_0) \) of (A1) starting from the initial point \( x_0 \in U \setminus \{0\} \) is well-defined and unique in forward time for
\( t \in [0, T_s(x_0)], \) and \( \lim_{t \to T_s(x_0)} x(t, x_0) = 0. \) Then, \( T(x_0) \) is called the settling time. The zero solution of (A1) is finite-time stable if it is Lyapunov stable and finite-time convergent. When \( \mathcal{U} = \mathcal{D} = \mathbb{R}^n, \) the zero solution is said to be globally finite-time stable.

**Definition A.2.** [22] For real numbers \( r_i > 0, i = 1, \ldots, n \) and fixed coordinates \( (x_1, \ldots, x_n) \in \mathbb{R}^n, \)

- The dilation \( \Delta_i(x) \) is defined by \( \Delta_i(x) = (\varepsilon^r x_1, \ldots, \varepsilon^r x_n), \) \( \forall \varepsilon > 0, \) with \( r_i \) being called as the weights of the coordinates. For simplicity, we define dilation weight \( \Delta = (r_1, \ldots, r_n). \)
- A function \( V \in C(\mathbb{R}^n, \mathbb{R}) \) is said to be homogeneous of degree \( r \) if there is a real number \( r \in \mathbb{R} \) such that \( \forall x \in \mathbb{R}^n \setminus \{0\}, \quad V(\Delta(x)) = \varepsilon^r V(x_1, \ldots, x_n). \)
- A vector field \( f \in C(\mathbb{R}^n, \mathbb{R}^n) \) is said to be homogeneous of degree \( r \) if there is a real number \( r \in \mathbb{R} \) such that for \( i = 1, \ldots, n \)
  \[ \forall x \in \mathbb{R}^n \setminus \{0\}, \quad f_i(\Delta(x)) = \varepsilon^{r_i} f_i(x_1, \ldots, x_n). \]
- A homogeneous \( p \)-norm is defined as \( \|x\|_{\Delta, p} = \left( \sum_{i=1}^n |x_i|^{p/r_i} \right)^{1/p}, \forall x \in \mathbb{R}^n, \) for a constant \( p \geq 1. \) For simplicity, we choose \( p = 2 \) and write \( \|x\|_{\Delta, 2} \).

**Lemma A.1** [22]. Let \( f \) be a continuous vector field on \( \mathbb{R}^n \) such that the origin is a locally asymptotical stable equilibrium point. Assume that \( f \) is homogeneous of degree \( k \) for some \( r \in (0, +\infty)^n. \) Then, for any \( p \in \mathbb{N} \) and any \( m > p \cdot \max\{r_i\}, \) there exists a strict Lyapunov function \( V \) for system (A1), which is homogeneous of degree \( m \) and of class \( C^p. \) As a direct consequence, the time derivative \( V \) is homogeneous of degree \( m + k. \)

**Lemma A.2** [26]. For a continuous system
\[
\dot{x} = f(x), \quad f(0) = 0, \quad x \in \mathbb{R}^n, \tag{A2}
\]
suppose that there exists a \( C^1 \) positive definite and proper function \( V : \mathbb{R}^n \to \mathbb{R} \) and real numbers \( k > 0 \) and \( \alpha \in (0, 1) \) such that \( V + kV^\alpha \) is negative semi-definite. Then the origin is a finite-time stable equilibrium of (A2).

**Lemma A.3.** [22] Suppose \( V : \mathbb{R}^n \to \mathbb{R} \) is a homogeneous function of degree \( r \) with respect to the dilation weight \( \Delta. \) Then the following hold: (i) \( \partial V/\partial x \) is homogeneous of degree \( r - r_i \) with \( r_i \) being the homogeneous weight of \( x_i; \) (ii) there is a constant \( \bar{c} \) such that \( V(x) \leq \bar{c} \|x\|^r_\Delta. \) Moreover, if \( V(x) \) is positive definite, then there exists a constant \( \tilde{c} > 0, \) such that \( V(x) \geq \tilde{c} \|x\|^r_\Delta. \)

**Lemma A.4.** Suppose \( c \) and \( d \) are two positive real numbers. Given any real-valued function \( \gamma(x, y) > 0, \) the following inequality holds
\[
|x|^c |y|^d \leq \frac{c}{c+d} \gamma(x, y) |x|^{c-d} + \frac{d}{c+d} \gamma^{-1}(x, y) |y|^{c-d}. \tag{A3}
\]

**References**


