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Asymptotic analysis to blow-up points for the porous medium equation with a weighted non-local source

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This article deals with the porous medium equation with a more complicated source term,

\[ u_t = \Delta u^m + a(x)u^p(x, t) \int_{B_R} u^q(x, t) \, dx, \quad x \in B_R, \quad t > 0, \]

subject to the homogeneous Dirichlet condition, where \( B_R \subset \mathbb{R}^N \) is a ball with radius \( R \), \( m > 1 \) and the non-negative constants \( p, q \) satisfying \( p + q > m \). We investigate how the three factors (the non-local source \( \int_{B_R} u^q(x, t) \, dx \), the local source \( u^p(x, t) \) and the weight function \( a(x) \)) influence the asymptotic behaviour of the solutions. It is proved that (i) when \( p < 1 \), the non-local source plays a dominating role, i.e. the blow-up set of the system is the whole domain \( B_{R,a} \), where \( B_{R,a} = \{ x \in B_R : a(x) > 0 \} \). (ii) When \( p > m \), this system presents single blow-up patterns. In other words, the local term dominates the non-local term in the blow-up profile. Moreover, the blow-up rate estimate is established with more precise coefficients determined.

Keywords: asymptotic analysis; porous medium equation; non-linear non-local source; blow-up set; global blow-up; single blow-up; weight source

AMS Subject Classifications: 35B40; 35K55

1. Introduction

In this article, we consider the following porous medium equation

\[
\begin{cases}
    u_t = \Delta u^m + a(x)u^p(x, t) \int_{B_R} u^q(x, t) \, dx, & x \in B_R, \quad t > 0, \\
    u(x, t) = 0, & x \in \partial B_R, \quad t > 0, \\
    u(x, 0) = u_0(x), & x \in B_R,
\end{cases}
\]

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where \( B_R \subset \mathbb{R}^N \) is a ball with radius \( R \), \( m > 1 \) and the non-negative constants \( p, q \) satisfying \( p + q > m \). The source term is the product of three factors: the non-local source \( \int_{B_R} u^p(x, t)dx \), the local source \( u^q(x, t) \) and the weight function \( a(x) \). Throughout this article we always assume:

(H1) 0 is the centre of \( B_R; a(x) \in C^{2+a}(B_R) \cap C(\tilde{B}_R) \) for some \( \alpha \in (0, 1); a(x) \geq 0 \) in \( B_R; a(x) \) is radial, \( a'(r) \leq 0 \) with \( r = |x| \),

(H2) \( u_0(x) \in C^{2+a}(B_R) \cap C(\tilde{B}_R) \) for some \( \alpha \in (0, 1) \), \( u_0(x) \geq 0 \) in \( B_R \), \( \partial u_0 / \partial v < 0 \) on \( \partial B_R \),

(H3) \( u_0(x) = 0 \), \( \Delta u^m_0 + a(x)u^p_0(x) \int_{B_R} u^q_0(x)dx = 0 \) on \( \partial B_R \),

(H4) \( \Delta u^m_0(x) \leq 0 \), \( \Delta u^m_0 + a(x)u^p_0(x) \int_{B_R} u^q_0(x)dx \geq 0 \) in \( B_R \),

where \( v \) is the unit outer normal vector in \( \partial B_R \).

The porous medium equations with or without a non-local source have been studied by many authors (see [1–9] for forced reaction source and [10,11] for the non-local sources). The typical models are the following classical type: (i) Local source: \( u_t - \Delta u^m = au^p; \) (ii) Non-local source: \( u_t - \Delta u^m = a \int_{\Omega} u^q dx \), with the same initial-boundary conditions as problem (1.1). It is known that both the problems share the same blow-up criteria and the blow-up rate. However, there do exist some essential differences from the two problems. For example, the blow-up set for (i) consists of a single point under some assumptions for initial datum (such as symmetry and monotonicity of \( u_0(|x|) \)). While using the similar method in [12], we can easily see that the problem (ii) has global blow-up.

For the problem with weight source, Kong et al. [13] studied the following problem with a weighted localized source

\[
\begin{align*}
\begin{cases}
   u_t = \Delta u + a(x)u^p(x, t)u^q(0, t), & x \in B, \quad t > 0, \\
   u(x, t) = 0, & x \in \partial B, \quad t > 0, \\
   u(x, 0) = u_0(x), & x \in B,
\end{cases}
\end{align*}
\]

where \( B \) is an open ball in \( \mathbb{R}^N \), non-negative constants \( p + q > 1 \). Under the radially symmetric assumptions on \( a(x) \) and \( u_0(x) \), they showed that the blow-up set consists of single point \( \{ x = 0 \} \) if \( p > 1 \). When \( 0 \leq p \leq 1 \), the blow-up takes place everywhere in \( B \). Moreover, the blow-up rate estimate is established with more precise coefficients determined.

Denote \( B_{R,a} = \{ x \in B_R; a(x) > 0 \} \), obviously, \( 0 \in B_{R,a} \). The main results of this article are stated as follows.

**Theorem 1.1** Assume that \( p \leq 1 \) and (H1)–(H4) hold. Suppose \( u(x, t) \) is a classical solution of (1.1), which blows up in finite time \( T^* \), then the blow-up set is the whole domain \( B_{R,a} \) and the following limits converge uniformly on any compact subset of \( B_{R,a}(1) \) if \( p < 1 \), then

\[
\lim_{t \to T^*} (T^* - t)^{\frac{1}{p+q-1}} u(x, t) = \lim_{t \to T^*} (T^* - t)^{\frac{1}{p+q-1}} |u(\cdot, t)|_{\infty}
= (a(x))^{\frac{1}{p+q-1}} \left( p + q - 1 \right) \int_{B_R} a^{\frac{q}{p+q-1}} dx \right)^{\frac{1}{p+q-1}} .
\]
(2) if \( p = 1 \), then the following inequalities hold as \( t \to T^* \)

\[
- \frac{a(x)}{q \int_{B_R} a(x)dx} \ln \left( q \int_{B_R} a(x)dx (T^* - t) \right) \geq \ln u(x, t) \geq - \frac{a(x)}{qa(0)} \ln (qa(0)|B_R|(T^* - t)).
\] (1.4)

**Remark 1.2** For the case \( p = 1 \), we conjure that the blow-up profile is

\[
\lim_{t \to T^*} \left| \ln \left( q \int_{B_R} a(x)dx (T^* - t) \right) \right|^{-1} \ln u(x, t) = \lim_{t \to T^*} \left| \ln \left( q \int_{B_R} a(x)dx (T^* - t) \right) \right|^{-1} \ln |u(\cdot, t)|_{\infty} = \frac{a(x)}{q \int_{\Omega} a(x)dx}.
\]

The reason is that it is not like the estimate of the upper bound of \( \ln u(x, t) \), we cannot find an optimal way to cope the inequality \( G'(\tau) \leq \int_{B_R} e^{a|\eta|G(\tau)}dx \) (see (2.15)).

**Theorem 1.3** Assume \( p > m \). Suppose the function \( a(x) \) satisfies (H1), the initial datum \( u_0(x) \) satisfies (H2)–(H4), \( a(x) \), \( u_0(x) \) are radially symmetric and \( a(t), u_0(r) \) with \( r = |x| = \sqrt{x_1^2 + \cdots + x_n^2} \) are non-increasing continuous functions in \([0, R]\), then \( x = 0 \) is the unique blow-up point.

**Remark 1.4** Under the conditions (H1)–(H4) and \( p + q > m \), we can easily get that the solution of system (1.1) blows up in finite time. Suppose \( a(x) \geq \delta > 0 \) on some ball \( B_h \) with \( 0 < h < R \). Then the solution \( u(x, t) \) of (1.1) satisfies

\[
\begin{align*}
& u_t \geq \Delta u^m + \delta u^p(x, t) \int_{B_h} u^q(x, t)dx, \quad x \in B_h, \ t > 0, \\
& u(x, t) \geq 0, \quad x \in \partial B_h, \ t > 0, \\
& u(x, 0) = u_0(x), \quad x \in B_h,
\end{align*}
\] (1.5)

then follows the paper [11], where we can get \( u \) blows up in finite time.

**Remark 1.5** Our methods can be used to study the following problem (more general than (1.2)):

\[
\begin{align*}
& u_t = \Delta u^m + a(x)u^p(x, t)u^q(0, t), \quad x \in B_R, \ t > 0, \\
& u(x, t) = 0, \quad x \in \partial B_R, \ t > 0, \\
& u(x, 0) = u_0(x), \quad x \in B_R.
\end{align*}
\] (1.6)

With a little change in our proofs, we can get similar results as Theorems 1.1 and 1.3. This extends a recent work of Kong et al. [13] in the following aspect: Extend \( m = 1 \) to \( m > 1 \).

In the following, we will discuss the global blow-up and prove Theorem 1.1 in Section 2 and then discuss the single blow-up and prove Theorem 1.3 in Section 3.

### 2. Global blow-up

In this section, we consider the case of \( p \leq 1 \). To investigate the blow-up profile, we begin with introducing the following transformation. Let \( u^m(x, t) = v(x, \tau), \ \tau = mt \), then system
(1.1) becomes

\[
\begin{cases}
 v_\tau = v^q \left( \Delta v + a(x)v^{p_1}(x, \tau) \int_{B_R} v^{q_1}(x, \tau) dx \right), & x \in B_R, \ t > 0, \\
v(x, \tau) = 0, & x \in \partial B_R, \ t > 0, \\
v(x, 0) = v_0(x) = u_0^{q_1}(x), & x \in B_R,
\end{cases}
\]  

(2.1)

where \(0 < \gamma = (m - 1)/m < 1\), \(p_1 = p/m\), \(q_1 = q/m\). Under this transformation, assumptions (H2)–(H4) become

(A2) \(v_0(x) \in C^{2+\alpha}(B_R) \cap C(\bar{B}_R)\) for some \(\alpha \in (0, 1)\), \(v_0(x) \geq 0\) in \(B_R\), \(\partial v_0/\partial v < 0\) on \(\partial B_R\),

(A3) \(v_0(x) = 0\), \(\Delta v_0 + a(x)v_0^{p_1}(x) \int_{B_R} v_0^{q_1}(x) dx = 0\) on \(\partial B_R\),

(A4) \(\Delta v_0(x) \leq 0\), \(\Delta v_0 + a(x)v_0^{p_1}(x) \int_{B_R} v_0^{q_1}(x) dx \geq 0\) in \(B_R\).

From now on, we will focus on the auxiliary problem (2.1) and assume that \(p_1 + q_1 > 1\) and \(v(x, \tau)\) blows up in finite time \(T_* = mT^*\). Set

\[
g(\tau) = \int_{B_R} v^{q_1}(x, \tau) dx \quad \text{and} \quad G(\tau) = \int_0^\tau g(s)ds.
\]

Lemma 2.1 Assume that \(p \equiv p_1m \leq 1\) and (H1), (A2)–(A4) hold. Suppose that \(v(x, \tau)\) is a classical solution of problem (2.1), which blows up in finite time \(T_*\), then \(\lim_{\tau \to T_*} g(\tau) = \lim_{\tau \to T_*} G(\tau) = \infty\).

Proof Suppose on the contrary that \(\lim_{\tau \to T_*} g(\tau) < \infty\). From the system (2.1), we know that \(v(x, \tau)\) exists global in time for any \(v_0(x) [14,15]\) since \(0 < p_1 < 1\). This leads to a contradiction. Therefore \(\lim_{\tau \to T_*} g(\tau) = \infty\).

Next, we infer that \(\lim_{\tau \to T_*} G(\tau) = \infty\). Denote \(V(\tau) = \max_{x \in B_R} v(x, \tau)\), which is Lipschitz continuous [16] and

\[
V'(\tau) \leq a(x)V^{q_1 + \gamma}(\tau)g(\tau) \quad \text{a.e. in } [0, T_*).
\]  

(2.2)

Integrating (2.2) over \((0, \tau)\), we obtain

(i) If \(p_1 + \gamma < 1\), then

\[
\frac{1}{1 - \gamma - p_1} V^{1 - \gamma - p_1}(\tau) \leq a(x) \int_0^\tau g(s)ds + \frac{1}{1 - \gamma - p_1} V^{1 - \gamma - p_1}(0)
\]

(2.3)

(ii) If \(p_1 + \gamma = 1\), then

\[
\ln V(\tau) \leq a(x) \int_0^\tau g(s)ds + \ln V(0) = a(x)G(\tau) + \ln V(0) \leq a(0)G(\tau) + \ln V(0).
\]  

(2.4)

Noticing that \(\lim_{\tau \to T_*} V(\tau) = \infty\), it follows that \(\lim_{\tau \to T_*} G(\tau) = \infty\). The proof of Lemma 2.1 is completed.

To prove Theorem 1.1, we try to show the relationships among \(v(x, \tau)\) and \(G(\tau)\). Sometimes we use the notation \(f(\tau) \sim g(\tau)\) as \(\tau \to T_*\) for \(\lim_{\tau \to T_*} f(\tau)/g(\tau) = 1\).
Lemma 2.2 Under the conditions of Lemma 2.1, the following statements hold uniformly on any compact subset of $B_{R,a}$.

1. If $p < 1$, then $v^{1-p_1}(x, \tau) \sim a(x)(1 - \gamma - p_1)G(\tau) \quad \text{as } \tau \to T_*$. (2.5)

2. If $p = 1$, then $\ln v(x, \tau) \sim a(x)G(\tau) \quad \text{as } \tau \to T_*$. (2.6)

Proof

1. If $p < 1$, we have $1 - \gamma - p_1 > 0$, and a direct computation yields

$$v_\tau = v^{p_1} \left( \frac{\Delta v^{1-p_1}}{1 + p_1} + p_1 v^{-1-p_1} |\nabla v|^2 + a(x)g(\tau) \right) \geq v^{p_1} \left( \frac{\Delta v^{1-p_1}}{1 + p_1} + a(x)g(\tau) \right).$$

Consequently, $v^{1-p_1}(x, \tau)$ is a supersolution of the following problem

$$\left\{
\begin{array}{ll}
\omega_\tau = \omega^{p_1} (\Delta \omega + a(x)(1 - p_1)g(\tau)), & x \in B_R, \ 0 < \tau < T_*, \\
\omega(x, \tau) = 0, & x \in \partial B_R, \ 0 < \tau < T_*, \\
\omega(x, 0) = v^{1-p_1}(x), & x \in B_R.
\end{array}
\right.$$ 

Noticing that $0 < \gamma/(1 - p_1) < 1$ and using the similar method in [12], we have

$$\lim_{\tau \to T_*} \frac{\omega^{(1-\gamma-p_1)/(1-p_1)}(x, \tau)}{a(x)(1 - \gamma - p_1)G(\tau)} = \lim_{\tau \to T_*} \frac{|\omega(\cdot, \tau)|^{(1-\gamma-p_1)/(1-p_1)}}{a(x)(1 - \gamma - p_1)G(\tau)} = 1,$$

uniformly on any compact subset of $B_{R,a}$. By comparison method, we have

$$v^{1-p_1}(x, \tau) \geq \omega(x, \tau), \quad (x, \tau) \in [0, T_*).$$

Furthermore, the following inequality holds uniformly on any compact subset of $B_{R,a}$

$$\liminf_{\tau \to T_*} \frac{v^{1-p_1}}{a(x)(1 - \gamma - p_1)G(\tau)} \geq 1. \quad (2.7)$$

On the other hand, it follows from (2.3) that

$$\limsup_{\tau \to T_*} \frac{v^{1-p_1}}{a(x)(1 - \gamma - p_1)G(\tau)} \leq 1. \quad (2.8)$$

Combining (2.7) and (2.8), we conclude that

$$\lim_{\tau \to T_*} \frac{v^{1-p_1}}{a(x)(1 - \gamma - p_1)G(\tau)} = \lim_{\tau \to T_*} \frac{|\omega(\cdot, \tau)|^{1-\gamma-p_1}}{a(x)(1 - \gamma - p_1)G(\tau)} = 1,$$

uniformly on any compact subset of $B_{R,a}$.

2. If $p = 1$, i.e. $\gamma + p_1 = 1$, using the similar computation as (1), we obtain

$$v_\tau \geq v \left( \frac{\Delta v^{1-p_1}}{1 + p_1} + a(x)g(\tau) \right).$$

Hence $v^{1-p_1}$ is a supersolution of the following problem

$$\left\{
\begin{array}{ll}
\omega_\tau = \omega (\Delta \omega + a(x)(1 - p_1)g(\tau)), & x \in B_R, \ 0 < \tau < T_*, \\
\omega(x, \tau) = 0, & x \in \partial B_R, \ 0 < \tau < T_*, \\
\omega(x, 0) = v^{1-p_1}(x), & x \in B_R.
\end{array}
\right.$$
Set $\xi(x, \tau) = a(1 - p_1)G(\tau) - \ln \omega$ and $\eta(\tau) = \int_{B_R} \xi(y, \tau)\varphi(y)dy$, where $\varphi(y) > 0$ is the eigenfunction corresponding to the first eigenvalue $\lambda_1$ of $-\Delta$ in $B_R$ with $\int_{B_R} \varphi(y)dy = 1$. Using the similar method in [12], we have the following limit that converges uniformly on any compact subset of $B_{R,a}$

$$\lim_{\tau \to T, a(x)(1 - p_1)G(\tau)} = \lim_{\tau \to T, a(x)(1 - p_1)G(\tau)} = 1.$$ Proceeding as (1), we arrive at the corresponding conclusion. The proof of Lemma 2.2 is complete.

**Lemma 2.3** Assume that the conditions of Lemma 2.1 hold, then the following limits converge uniformly on any compact subset of $B_{R,a}$:

1. If $p < 1$, then
   $$\lim_{\tau \to T, a(x)(1 - p_1)G(\tau)} = \lim_{\tau \to T, a(x)(1 - p_1)G(\tau)} = 1.$$

   \begin{equation}
   (2.9)
   \end{equation}

2. If $p = 1$, then
   $$\ln v(x, \tau) \leq \frac{a(x)}{q_1} \ln(\int_{B_R} a(x)dx),$$

   \begin{equation}
   (2.10)
   \end{equation}

**Proof** (1) From (2.5), we have

$$G'(\tau) = g(\tau) = \int_{B_R} v^{q_1}dx \sim \int_{B_R} ((1 - \gamma - p_1)a(x)G(\tau))^{\frac{q_1}{\gamma - p_1}}dx, \quad \text{as } \tau \to T_*.$$ Integrating (2.11) over $(\tau, T_*)$, we get

$$G(\tau) \sim \frac{1}{1 - \gamma - p_1} \left(\int_{B_R} a^{\frac{q_1}{\gamma - p_1}}dx(p_1 + q_1 + \gamma - 1)(T_* - \tau)^{\frac{1}{1 - \gamma - p_1}}\right).$$

since $\lim_{\tau \to T_*} G(\tau) = \infty$. Combining (2.12) with (2.5), we obtain that (2.9) converges uniformly on any compact subset of $B_{R,a}$.

(2) Choose two sequences $\{\xi_i\}_{i=1}^\infty$ and $\{\eta_i\}_{i=1}^\infty$ $(0 < \xi_i < 1, \eta_i > 1)$ with $\xi_i, \eta_i \to 1$ as $i \to \infty$. It follows from (2.6) that for such sequence $\{\xi_i\}_{i=1}^\infty, \{\eta_i\}_{i=1}^\infty$, there exists $\{\tau_i\}_{i=1}^\infty$ $(\tau_i < T_*)$ with $\tau_i \to T_*$ as $i \to \infty$ such that

$$a(x)\xi_i G(\tau) \leq \ln v(x, \tau) \leq a(x)\eta_i G(\tau), \quad \tau_i \leq \tau < T_*.$$ Namely

$$\int_{B_R} e^{a(x)\xi_i G(\tau)}dx \leq \int_{B_R} v^{q_1}(x, \tau)dx = g(\tau) = G'(\tau) \leq \int_{B_R} e^{a(x)\eta_i G(\tau)}dx, \quad \tau_j \leq \tau < T_*.$$ \begin{equation}
(2.13)
\end{equation}
Since \( e^{q_1 G(T)} \int_{B_R} e^{a(x)dx} \leq \int_{B_R} e^{a(x)G(T)} dx \), in view of the left-hand side of (2.13), we get
\[
e^{-q_1 \xi_i \int_{B_R} a(x)dx} dG \geq d\tau, \quad \tau_i \leq \tau < T_*.
\]
Integrating the above inequality from \( \tau \) to \( T_* \) yields that
\[
\left( q_1 \xi_i \int_{B_R} a(x)dx \right)^{-1} e^{-q_1 \xi_i \int_{B_R} a(x)dx G(T_*)} \geq (T_* - \tau), \quad \tau_i \leq \tau < T_*,
\]
because of \( \lim_{\tau \to T_*} G(\tau) = \infty \). Furthermore, for \( \tau_i \leq \tau < T_* \), we have
\[
- \int_{B_R} a(x)dx G(\tau) \geq \frac{1}{q_1 \xi_i} \ln \left( q_1 \xi_i \int_{B_R} a(x)dx (T_* - \tau) \right).
\]  
(2.14)

For the right-hand side of (2.13)
\[
G'(\tau) \leq \int_{B_R} e^{a(x)q_1 \eta_i G(\tau)} dx \leq |B_R| e^{a(0)q_1 \eta_i G(\tau)}.
\]  
(2.15)

Integrating (2.15) from \( \tau \) to \( T_* \) yields that
\[
\frac{1}{a(0)q_1 \eta_i} e^{-a(0)q_1 \eta_i G(\tau)} \leq |B_R|(T_* - \tau), \quad \tau_i \leq \tau < T_*,
\]
because of \( \lim_{\tau \to T_*} G(\tau) = \infty \). Furthermore, for \( \tau_i \leq \tau < T_* \), we have
\[
-a(0)G(\tau) \leq \frac{1}{q_1 \eta_i} \ln(a(0)|B_R|q_1 \eta_i(T_* - \tau)).
\]  
(2.16)

Consequently, (2.14) and (2.16) guarantee that for \( \tau_i \leq \tau < T_* \)
\[
- \frac{1}{q_1 \xi_i \int_{B_R} a(x)dx} \ln \left( q_1 \xi_i \int_{B_R} a(x)dx (T_* - \tau) \right)
\]
\[
\geq G(\tau) \geq - \frac{\ln(a(0)|B_R|q_1 \eta_i(T_* - \tau))}{q_1 \eta_i a(0)}.
\]  
(2.17)

We take \( i \to \infty \) in (2.17) and obtain
\[
- \frac{1}{q_1 \int_{B_R} a(x)dx} \ln \left( q_1 \int_{B_R} a(x)dx (T_* - \tau) \right) \geq G(\tau) \geq - \frac{\ln(a(0)|B_R|q_1(T_* - \tau))}{q_1 a(0)}.
\]

Due to \( \ln v(x, t) \sim a(x)G(\tau) \), we obtain
\[
- \frac{a(x)}{q_1 \int_{B_R} a(x)dx} \ln \left( q_1 \int_{B_R} a(x)dx (T_* - \tau) \right) \geq \ln v(x, \tau) \geq - \frac{a(x)}{q_1 a(0)} \ln(a(0)|B_R|q_1(T_* - \tau)).
\]

We conclude the statement (2) in Lemma 2.3. The proof of Lemma 2.3 is complete. \( \blacksquare \)

Theorem 1.1 follows Lemma 2.3 easily.
3. Single blow-up

Throughout this section, we assume that $p > m$ and (H1)–(H4) hold. The solution $u(x, t)$ of problem (1.1) blows up in finite time $T^*$. Then we can consider the blow-up set of the solution under the assumption: the initial data $u_0(x)$, $a(x)$ are radially symmetric and $a(r)$, $u_0(r)$ with $r = |x| = \sqrt{x_1^2 + \cdots + x_N^2}$ are non-increasing continuous functions in $[0, R]$.

By using the similar argument of Lemma 4.1 in [12] to the auxiliary problem (2.1), we can easily get $v_t(x, \tau) \geq 0$ and $\Delta v(x, \tau) \leq 0$ in $B_R \times (0, T_\ast)$. Moreover, applying the standard methods in [16, 17], we get $v_t(x, \tau) = v(r, \tau) > 0$ and $v_t(r, \tau) \leq 0$ in $(0, R) \times (0, T_\ast)$, where $r = |x| = \sqrt{x_1^2 + \cdots + x_N^2}$.

**Lemma 3.1** Assume $p > m$ and the function $a(x)$ satisfies (H1), the initial datum $v_0(x)$ satisfies (A2)–(A4). Suppose that $a(x)$, $v_0(x)$ are radially symmetric and $a(r)$, $v_0(r)$ with $r = |x| = \sqrt{x_1^2 + \cdots + x_N^2}$ are non-increasing continuous functions in $[0, R]$, then $x = 0$ is the unique blow-up point.

**Proof** Suppose on the contrary that $v(x, \tau)$ blows up at another point $x' \neq 0$ as $\tau \to T_\ast$. Set $r' = |x'|$, then $\limsup_{\tau \to T_\ast} v(r, \tau) = \infty$ for $r \in [0, r']$ with $r = |x|$ since $v(r, \tau)$ is non-increasing in $r$. Let $B_{r'} = B_{r'} \cap \{x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N; x_1 > \delta\}$, where $\delta = r'/3$. Define $J(x, \tau) = v(x_1) - (x_1 - \delta)^2$ with $\delta$ is a constant to be determined later. Choosing any $T (0 < T < T_\ast)$, and for $0 < \tau < T$, a series of computation yields

$$J_t - v^\gamma \Delta J$$

$$= (v_t - v^\gamma \Delta v)x_1 + \gamma v^{\gamma-1} v x_1 \Delta v + p_0 \phi(x_1) v^{p_0-1}(v_t - v^\gamma \Delta v)$$

$$- 2\epsilon v^{p_0+\gamma} - 4p_0 \epsilon v^{p_0+\gamma -1}(x_1 - \delta) v x_1 - p_0(p_0 - 1) \phi(x_1) v^{p_0+\gamma -2} |\nabla v|^2$$

$$\leq d' \frac{p_1}{r} v^{p_1+\gamma} \int_{B_R} v^{p_1+\gamma} dx + \left( a(x)p_1 v^{p_1+\gamma -1} \int_{B_R} v^{p_1+\gamma -1} dx - 4p_0 \epsilon v^{p_1+\gamma -1} \right) J$$

$$- \phi(x_1) v^{p_0+\gamma} \left( a(x)(p_1 - p_0) v^{p_1} \int_{B_R} v^{p_1+\gamma -1} dx - 4p_0 \epsilon v^{p_1+\gamma -1} + 2(x_1 - \delta)^2 \right)$$

$$\leq c(x, \tau) J - \phi(x_1) v^{p_0+\gamma} \left( a(x)(p_1 - p_0) v^{p_1} \int_{B_R} v^{p_1+\gamma -1} dx - 4p_0 \epsilon v^{p_1+\gamma -1} + 2R^2 \right),$$

where $c(x, \tau) \equiv a(x)p_1 v^{p_1+\gamma -1} \int_{B_R} v^{p_1+\gamma -1} dx - 4p_0 \epsilon v^{p_1+\gamma -1}$. $c(x, \tau)$ is continuous and bounded in $B_{2R} \times (0, T)$. There exists a constant $\sigma > 0$ such that $v(0, \tau) > \sigma$ since $v_t(r, \tau) > 0$ in $[0, R] \times [0, T_\ast)$ and $v(0, \tau) = \max_{0 \leq \tau \leq R} v_t(r, \tau)$ for $\tau \in [0, T_\ast)$. Hence, there exists a small constant $\epsilon_1 > 0$ such that for $0 < \epsilon \leq \epsilon_1$, $a(x)(p_1 - p_0) v^{p_1} \int_{B_{2R}} v^{p_1+\gamma -1} dx - 4p_0 \epsilon v^{p_1+\gamma -1} + 2R^2 > 0$ in $B_{2R} \times (0, T)$. Hence, we get

$$J_t - v^\gamma \Delta J - c(x, \tau) J \leq 0, \quad (x, \tau) \in B_{2R} \times (0, T).$$

(3.1)

We claim that $v_0$ is non-positive and non-trivial. Otherwise, $v_0 \equiv 0$, which contradicts the assumption on $v_0$. Then by the standard methods, we can deduce that $v_t(r, \tau) < 0$ for $r > 0$, $\tau > 0$. Furthermore, we have $v_{x_1} < 0$ for $x, \tau) \in B_{2R} \times (0, T)$. Hence,

$$J(x, \tau) = v(x, \tau) < 0, \quad \text{on } \partial B_{2R} \times (0, T).$$

(3.2)
and there exists a small constant $\varepsilon_2$ such that
\[
J(x, 0) = v_{x_1}(x, 0) + \phi(x_1)v_{x_1}^{\rho_0} \leq v_{x_1}(x, 0) + \varepsilon R^2 \max_{B_R} v_{x_1}^{\rho_0} \leq 0,
\]
for $x \in B_R^\delta, 0 < \varepsilon \leq \varepsilon_2$. Choosing $\varepsilon = \min\{1, \varepsilon_1, \varepsilon_2\}$ and implying the comparison principle [11] to (3.1)–(3.3), we obtain
\[
J(x, \tau) \leq 0, \quad (x, \tau) \in B_R^\delta \times (0, T).
\]
By the arbitrariness of $T < T_\ast$, we get
\[
\phi(x_1) \leq -v^{\rho_0}(x, \tau)v_{x_1}(x, \tau), \quad (x, \tau) \in B_R^\delta \times (0, T_\ast).
\]
Fix $(a_2, a_3, \ldots, a_N) \in \mathbb{R}^{N-1}$, and take $y = (\delta, a_2, \ldots, a_N)$, $z = (3\delta, a_2, \ldots, a_N)$, then integrate (3.4) from $y$ to $z$ to get
\[
0 < \int_y^z \phi(x_1) dx_1 \leq \frac{1}{p_0 - 1} v^{1-\rho_0}(z, \tau), \quad 0 < \tau < T_\ast.
\]
This contradicts $v(z, \tau) \to \infty$ as $\tau \to T_\ast$ and $p_0 > 1$. The proof of Lemma 3.1 is complete.

Theorem 1.3 follows Lemma 3.1 easily.

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References