Diagnosability of star graphs under the comparison diagnosis model

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Abstract
In this paper, the diagnosability of \( n \)-dimensional star graph \( S_n \) under the comparison diagnosis model has been studied. It is proved that \( S_n \) is \((n-1)\)-diagnosable under the comparison diagnosis model when \( n \geq 4 \).

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1. Introduction
Fault-tolerance computing is very important for a multiprocessor system. The first step to deal with faults is to identify the faulty processors from the fault-free ones. The identification process is called the diagnosis of the system. The diagnosability of the system is defined as a number \( t \) such that the system is diagnosable as long as the number of the faulty processors is not greater than \( t \) [3].

Several diagnosis models were proposed to identify the faulty processors. One major approach is the PMC diagnosis model introduced by Preparata et al. [7]. The diagnosis of the system is achieved through two linked processors testing each other. Another major approach, proposed by Malek and Maeng [6], is called the comparison diagnosis model. The diagnosis is performed by comparing the responses of pairs of processors to the same input. The comparisons are done by other processors in the system and no central entity involves. Based on the comparison results, the faulty/fault-free status of the processors can be identified. In [8], the comparison diagnosis model was further studied. A set of criteria was given for determining whether the faulty processors in the system can be diagnosed, and most importantly a polynomial-time algorithm is presented to identify the faulty processors if the system is known to be diagnosable. The diagnosabilities of hypercube, enhanced hypercube and crossed hypercube under the comparison diagnosis model were studied in [3,9].

The star graph has proved to be a viable candidate for interconnecting the multiprocessor system [1]. The
features of the star graph include low degree of node, small diameter, partitionability, symmetry, and high degree of fault-tolerance. The diagnosabilities of star graph under the three strategies based on the PMC diagnostic model were studied in [5]. In this paper, the diagnosability of $S_n$ under the comparison diagnosis model has been studied. We prove that it is $n - 1$ when $n \geq 4$.

2. Preliminaries

In this section, the star graph, the comparison diagnosis model and some definitions and notations need for our discussion are introduced.

2.1. Star graph

An $n$-dimensional star graph, also referred to as an $n$-star or $S_n$, is an undirected graph consisting of $n!$ nodes (or vertices) and $(n - 1)n!/2$ edges (or links) [1]. Each node is uniquely assigned a label $a_1a_2\cdots a_m\cdots a_n$, which is a distinct permutation of the set of symbols $\{a_1, a_2, \ldots, a_n\}$. Without loss of generality, let this symbol set be the set of integers $\{1, 2, \ldots, n\}$. Two nodes are linked by an edge labeled $i$ if and only if the label of one node can be obtained from the label of another node by interchanging the first symbol with the $i$th symbol, $2 \leq i \leq n$. For example, in a 4-star containing 4! nodes, two nodes 1234 and 4231 are neighbors and joined through an edge labeled 4. In $S_n$, each node is connected to $n - 1$ neighbors by $n - 1$ edges. Each $S_n$ can be decomposed into $n(n - 1)$-dimensional star graph $S_{n-1}$. A 4-star graph $S_4$ is shown in Fig. 1.

2.2. The comparison diagnosis model

A multiprocessor system can be represented by an undirected graph $G = (V, E)$, where $V$ represents the set of nodes and $E$ represents the set of edges. In a multiprocessor system, each processor is represented by a node $u \in V$ and each communication link between a pair of processors $u$ and $v$ is represented by an edge $\{u, v\} \in E$.

The comparison diagnosis strategy can be modeled as a multigraph $M = (V, C)$ where $V$ is the same node set defined in $G$. $C$ is the labeled edge set. A labeled edge $\{u, v\}_w$ is said to belong to $C$ if $\{u, v\}$ is an edge labeled by $w$, which implies that the processors $u$ and $v$ are compared by processor $w$. Since different comparators can compare the same pair of processors, $M$ is a multigraph. Denote the comparison result as $r(\{u, v\}_w)$ such that $r(\{u, v\}_w) = 0$ if the outputs of $u$ and $v$ agree and $r(\{u, v\}_w) = 1$ if the outputs disagree. If the comparator $w$ is fault-free and $r(\{u, v\}_w) = 0$, the processors $u$ and $v$ are fault-free. If $r(\{u, v\}_w) = 1$, at least one of the three processors $u, v, w$ are faulty. If the comparator $w$ is faulty, the comparison result is unreliable. The collection of the comparison results, defined as a function $s : C \rightarrow \{0, 1\}$, is called the syndrome of the diagnosis. A system is diagnosable if for every syndrome $s$ there is a unique subset $F \subseteq V$ that is consistent with it. A $t$-diagnosable system is the system diagnosable if the number of the faulty processors is not greater than $t$ where $t$ is the diagnosability of the system. In [8], a polynomial algorithm is proposed to diagnose the system provided that the diagnosability of the system is known. Thus, the diagnosability of a system must be determined before using such kind of algorithms. However, this is not an easy problem as indicated in [3, 9].

Different comparison schemes can be applied to a fixed system. In this paper, we want to gain as much knowledge as possible about the faulty status of the
2.3. Definitions and notations

We denote \( N(v) = \{u \mid \{v, u\} \in E\} \) as the set of all nodes that are linked to node \( v \). For a subset \( X \subset V \), the set of nodes in \( V - X \) that are linked to nodes of \( X \), \( N(X) = \bigcup_{v \in X} N(v) - X \).

A cycle \( C \) of the graph \( G \) is defined as a closed trail with at least one edge and with no repeated vertices except that the initial vertex is the terminal vertex. The length of a cycle is the number of edges in the cycle. The number of nodes in a cycle \( C \) is denoted as \( |C| \).

Given a system \( G \) and the comparison scheme \( M \), for a node \( u \in V \), let \( X_u \) be the set of nodes such that \( X_u = \{v \mid \{u, v\} \in E \text{ or } \{u, v\} \in C \text{ for some } w\} \).

That is, a node in \( X_u \) is either linked to \( u \) or compared with \( u \) by some other node. Let \( Y_u \) be the set of edges among nodes of \( X_u \), such that

\[
Y_u = \{\{v, w\} \mid v, w \in X_u \text{ and } \{u, v\}_w \in C\}.
\]

Define \( G_u = (X_u, Y_u) \).

For a graph \( G \), a subset \( K \) of \( G \) is called a vertex cover of \( G \) if every edge of \( E \) has at least one end in \( K \). A vertex cover of minimum cardinality in \( G \) is called minimum vertex cover. For a node \( u \in V \), the cardinality of a minimum vertex cover of \( G_u \) is called the order of node \( u \).

Denote \( T(X) \) to be the set of nodes that are outside of \( X \) and are compared to some nodes of \( X \) by some nodes of \( X \) (Fig. 2). Given \( G \) and \( M \), for a subset of nodes \( X \subset V \),

\[
T(X) = \{u \mid \{u, v\}_w \in C \text{ and } v, w \in X \text{ and } u \notin X\}.
\]

![Fig. 2. An example of \( X \) and its \( T(X) \).](image)

3. Diagnosability of star graphs under the comparison model

There are several ways to characterize a \( t \)-diagnosable system under the comparison approach [8]. In this study, we use one particular characterization given in [8] which gives the three sufficient conditions for a system to be \( t \)-diagnosable.

**Theorem 1** [8]. A system with \( N \) nodes is \( t \)-diagnosable if (1) \( N \geq 2t + 1 \), (2) each node has order at least \( t \), (3) for each \( X \subset V \) such that \( |X| = N - 2t + p \) and \( 0 \leq p \leq t - 1 \), \( |T(X)| > p \).

**Lemma 1** [4]. There are no odd cycles in a star graph.

**Lemma 2** [4]. There are even cycles with length \( l \) in \( S_n \) if \( l \geq 6 \) and \( l \leq n \).

From Lemmas 1 and 2, we can deduce the following corollary:

**Corollary 1**. If two nodes are adjacent, there is no common neighbor node of these two nodes, i.e., \( |N(u) \cap N(v)| = 0 \). If two nodes are not adjacent, there is at most one common neighbor node of these two nodes, i.e., \( |N(u) \cap N(v)| \leq 1 \).

**Proof**. The two cases can be proved by contradiction. For case (1), if two nodes are adjacent and they have a common neighbor node, these 3 nodes will form a cycle of length 3. It is a contradiction to Lemma 1 that there are no odd cycles in a star graph. For case (2), if two nodes are adjacent and they have two common neighbor nodes, these 4 nodes will form a cycle of length 4. It is a contradiction to Lemma 2 that there is no even cycle with length less than 6 in a star graph. \( \Box \)

**Lemma 3** [2]. If \( u \) and \( v \) are two nodes of an \( n \)-star labeled with two permutations \( P_1 \) and \( P_2 \) having different first symbols, then there are \( n - 1 \) parallel paths between \( u \) and \( v \). These paths are grouped as follows (\( c \) is the number of cycles\(^1\) of length at least two, \( m \) is the number of symbols in these cycles): (1) \( c \) paths are

\(^1\)The cycles here indicate that cyclically ordered sets of symbols with the property that each symbol’s desired position is that occu-
Proof. By contradiction. Assume for two 6-cycles paths are of length 
paths are of length \( c + m - 1 \) 
paths are of length \( c + m + 2 \).

From Lemma 3, we can deduce the following corollary.

Corollary 2. In \( S_n \), for two 6-cycles \( C_1 \) and \( C_2 \), \( |C_1 \cap C_2| \neq 3 \).

Proof. By contradiction. Assume for two 6-cycles \( C_1, C_2, |C_1 \cap C_2| = 3 \) and the three nodes in both cycles are \( u, v \) and \( w \). They can only be connected as shown in Fig. 3 if Lemmas 1 and 2 are not contradicted. Then, from node \( u \) to node \( w \), there should be one path of length 2, two paths of length 4. The three paths are parallel or node-disjoint paths. Since \( c = 1 \), \( m = 3 \) in this case, from Lemma 3, we can find that there is only one parallel path of length 4 (Lemma 3, Case 2, \( m - c - 1 = 1, c + m = 4 \)), which is a contradiction to the assumption.

Lemma 4 [5]. For any two nodes \( u, v \) in \( S_n \), \(|N((u, v))| \geq 2n - 4\).

Proof. \( S_n \) is a regular graph and the degree of each node is \( n - 1 \); this implies that, for any two arbitrary nodes \( u \) and \( v \) in \( S_n \), \(|N(u)| = |N(v)| = n - 1\).

Case 1: If \( u \) and \( v \) are adjacent, from Corollary 1, \(|N(u) \cap N(v)| = 0\), and \(|N((u, v))| = |N(u)| + |N(v)| - |[u, v]| = 2n - 4\).

Case 2: If \( u \) and \( v \) are not adjacent, from Corollary 1, \(|N(u) \cap N(v)| \leq 1\), and \(|N((u, v))| = |N(u)| + |N(v)| - |N(u) \cap N(v)| \geq 2n - 3 > 2n - 4\).

Thus, any two nodes \( u \) and \( v \) in \( S_n \) are connected to at least \( 2n - 4 \) other nodes.

Lemma 5. For any four nodes \( u, v, w, x \) in \( S_n \), 
\(|N((u, v, w, x))| \geq 4n - 10\).

Proof. We consider five cases. Denote \( N_c \) is the set of common neighbor nodes among the four nodes, i.e., \( N_c = |N(u) \cap N(v)| + |N(u) \cap N(w)| + |N(u) \cap N(x)| + |N(v) \cap N(w)| + |N(v) \cap N(x)| + |N(w) \cap N(x)| \). From Corollary 1, \( N_c \leq 6 \).

Case 1: Four nodes are all adjacent. From Lemmas 1 and 2, they can only be connected as the two cases shown in Fig. 4. For case (a),

\[ |N((u, v, w, x))| \]
\[ = (|N(u)| - |v|) + (|N(v)| - |u, w|) \]
\[ + (|N(w)| - |v, x|) + (|N(x)| - |w|) \]
\[ = (n - 2) + (n - 4) + (n - 2) + (n - 2) \]
\[ = 4n - 10. \]

For case (b),

\[ |N((u, v, w, x))| \]
\[ = (|N(u)| - |v|) + (|N(v)| - |u, w, x|) \]
\[ + (|N(w)| - |v|) + (|N(x)| - |v|) \]
\[ = (n - 2) + (n - 4) + (n - 2) + (n - 2) \]
\[ = 4n - 10. \]

Case 2: Three nodes are adjacent and another is not adjacent to these three nodes. Without loss of generality, we consider \( u, v, w \) are adjacent and \( v \) is the middle node. From Lemmas 1 and 2, \(|N((u, v, w)) \cap N(x)| \leq 2\).

\[ |N((u, v, w, x))| \]
\[ = (|N(u)| - |v|) + (|N(v)| - |u, w|) \]
\[ + (|N(w)| - |v|) + |N(x)| - |N((u, v, w)) \cap N(x)| \]
\[ > (n - 2) + (n - 3) + (n - 2) + (n - 1) - 2 \]
\[ = 4n - 10. \]

Case 3: Two nodes are adjacent to each other and other two are adjacent to each other. Without loss of generality, we consider \( u, v \) are adjacent to each other and \( w, x \) are adjacent to each other. From Lemmas 1 and 2, \( N_c = |N((u, v)) \cap N((w, x))| \leq 2\).
When \( N_c \) is not equal to \( 6 \) or \( 5 \), as shown in Figs. 5(c) and (d), respectively. Thus, any four nodes in \( S_n \) are connected to at least \( 4n - 10 \) other nodes. □

**Lemma 6.** A node of \( S_n \) has order \( n - 1 \).

**Proof.** Since all nodes of \( S_n \) link to others in exactly the same way, it is sufficient to check the order for an arbitrary node \( u = a_1 a_2 \ldots a_{n-1} a_n \). By the definition of \( G_u = (X_u, Y_u) \), \( X_u \) consists of those nodes that are either linked to \( u(X_1) \) or being compared to \( u(X_2) \). So, \( X_u \) is the union of two sets \( X_1 \) and \( X_2 \):

\[
X_u = X_1 \cup X_2
\]

\[
= \{a_2 a_1 \ldots a_{n-1} a_n, \ldots, a_{n-1} a_2 \ldots a_1 a_n, \ldots, a_3 a_1 a_2 \ldots a_{n-1} a_n, \ldots, a_1 a_2 \ldots a_{n-1} a_n a_1, \ldots, a_n \}
\]

The total number of nodes in \( X_1 \) is \( n - 1 \), and the total number of nodes in \( X_2 \) is \( (n - 1)(n - 2) \).

\( Y_u \) consists of all edges \([v, w]\) such that \( w \) is a comparator of \( u \) and \( v \), i.e., \( w \) is linked to \( u \) and \( v \) is linked to \( w \). That is,

\[
Y_u = \{[v, w] \mid w \subset X_1, v \subset X_2\}
\]

It can be seen that \( G_u \) is a symmetric-structured, so-called bipartite graph (a graph whose vertices can...
be partitioned into two sets $V_1$ and $V_2$ such that every edge has one end in $V_1$ and another in $V_2$ (Fig. 6). To find the order of $u$, we need to find the size of the minimum vertex cover. From the Konig-Egervary theorem, in a bipartite graph, the size of the minimum vertex cover is equal to the size of the maximum matching. A matching is a set of edges of the graph such that no two edges in the set share a common vertex. The matching is maximum if it has the maximum number of edges over all matchings in the graph. From Fig. 6, it can be seen that the edges in $Y_u$ can be divided into $(n - 1)$ groups. The maximum matching can be then constructed by taking one edge from each group. Thus the size of the maximum matching for $G_u$ is $(n - 1)$, which is also the order of $u$. □

**Theorem 2.** An $n$-star-structured system, represented by graph $G = (V, E)$, with $V$ being the node set and $E$ being the link set, is $(n - 1)$-diagnosable under the comparison model when $n \geq 4$ and the comparison scheme is $M^* = (V, C^*)$.

**Proof.** We prove that $S_n$ satisfies all three sufficient conditions of Theorem 1 for $n \geq 4$.

As we know, the number of nodes $N$ in $S_n$ is $n!$. It is easy to show that $n! \geq 2(n - 1) + 1$ is true when $n \geq 3$. Thus the first condition holds. According to Lemma 6, the second condition holds. It remains to show that the third condition holds for $S_n$, i.e., for an arbitrary $X \subseteq V$ such that $|X| = n! - 2(n - 1) + p$, $0 \leq p \leq n - 2$, $|T(X)| \geq p$ is true.

We prove it in two steps. First, we show that this is true for $p = n - 2$. Then we prove that it also holds for $p = 0, 1, \ldots, n - 3$.

If $p = n - 2$, then $|X| = n! - 2(n - 1) + n - 2 = n! - n$ and

$$|V - X| = n. \tag{1}$$

We prove by contradiction and assume $|T(X)| \leq n - 2$, which means that $V - X$ contains at least two nodes not in $T(X)$. Let $u, v \in V - X$ and $u, v \notin T(X)$. From the definition of $T(X)$, if $u \notin T(X)$, then $u$ must belong to the following two cases: (a) $N(u) \cap X = \emptyset$, (b) if $u' \in N(u) \cap X$, then $N(u') \cap X = \emptyset$ (Fig. 7). If two nodes $u, v \notin T(X)$, they must belong to one of the three cases shown in Fig. 8.

**Case 1:** Both nodes belong to case (a). From Lemma 4, $|N([u, v])] \geq 2n - 4$. Then $|V - X| \geq |N([u, v])] + |[u, v]| \geq 2n - 4 + 2 = 2n - 2$

**Case 2:** One node belongs to case (a), one node belongs to case (b). Without loss of generality, we consider $u$ belongs to case (b), $v$ belongs to case (a). $u'$ and $v$ are not adjacent because $v$ is not connected to any node in $X$. From Case 2 of Lemma 4, $|N((u', v))] \geq 2n - 3$. Then $|V - X| \geq |N((u', v))] + |[v]| \geq 2n - 3 + 1 = 2n - 2$

**Case 3:** Both nodes belong to case (b). In this case, $u'$ and $v'$ are not adjacent. From Case 2 of Lemma 4, $|N((u', v'))] \geq 2n - 3$. Then $|V - X| \geq |N((u', v'))] \geq 2n - 3$.

![Fig. 6. The illustration of $u, X_1, X_2, Y_u$ and $G_u$.](image)

![Fig. 7. Two cases that a node does not belong to $T(X)$.](image)

![Fig. 8. Three cases that two nodes $u, v \notin T(X)$.](image)
For all three cases, \(|V - X| \geq 2n - 3\). But \(2n - 3 > n = |V - X|\) when \(n \geq 4\), a contradiction to (1).

For \(p = 0, \ldots, n - 3\), let \(|X| = n! - 2(n - 1) + p\), then
\[|V - X| = 2(n - 1) - p. \quad (2)\]

Again, we prove by contradiction and assume \(|T(X)| \leq p\), which means \(V - X\) contains at least \(2(n - 1) - p = 2(n - p - 1)\) nodes that are not in \(T(X)\). Since \(p \leq n - 3\), we have \(2(n - p - 1) \geq 4\). Let \(u, v, w, x \in V - X\) be four nodes such that \(u, v, w, x \notin T(X)\). For those four nodes, they must belong to one of the five cases shown in Fig. 9.

**Case 1**: All four nodes belong to case (a) (Fig. 9(a)). From Lemma 5, \(|V - X| \geq |N(u, v, w, x)| + |u, v, w, x| \geq 4n - 10 + 4 = 4n - 6\).

**Case 2**: One node belongs to case (b), three nodes belong to case (a) (Fig. 9(b)). Without loss of generality, we assume \(u\) is connected to \(u'\) in \(X\). Then \(u'\) is not adjacent to \(v, w, x\) or \(x\). For \(v, w, x\), we consider three subcases.

**Subcase 2.1**: \(u, v, w, x\) are adjacent. From Case 2 of Lemma 5, \(|V - X| \geq |N((u', v, w, x))| + |u, v, w, x| \geq 4n - 10 + 3 = 4n - 7\).

**Subcase 2.2**: Two nodes are adjacent, one is not. From Case 4 of Lemma 5, \(|V - X| \geq |N((u', v, w, x))| + |u, v, w, x| \geq 4n - 8 + 3 = 4n - 5\).

**Subcase 2.3**: \(u, v, w, x\) are not adjacent to each other. From Case 5 of Lemma 5, \(|V - X| \geq |N((u', v, w, x))| + |u, v, w, x| \geq 4n - 8 + 3 = 4n - 5\).

**Case 3**: Two nodes belong to case (a), the other two belong to case (b) (Fig. 9(c)). Without loss of generality, we assume \(u\) is connected to \(u'\) in \(X\), \(v\) is connected to \(v'\) in \(X\). Then \(u'\) and \(v'\) are not adjacent to each other, also not adjacent to \(w\) or \(x\). For \(w, x\), we consider two subcases.

**Subcase 3.1**: \(w\) and \(x\) are adjacent. From Case 4 of Lemma 5, \(|V - X| \geq |N((u', v', w, x))| + |w, x| \geq 4n - 8 + 2 = 4n - 6\).

**Subcase 3.2**: \(w\) and \(x\) are not adjacent. From Case 5 of Lemma 5, \(|V - X| \geq |N((u', v', w, x))| + |w, x| \geq 4n - 8 - 2 = 4n - 6\).

**Case 4**: Three nodes belong to case (b), the other one belongs to case (a) (Fig. 9(d)). Without loss of generality, we assume \(u\) is connected to \(u'\) in \(X\), \(v\) is connected to \(v'\) in \(X\). Then \(u', v', w'\) are not adjacent to each other, also \(u', v', w'\) are not adjacent to \(x\). From Case 5 of Lemma 5, \(|V - X| \geq |N((u', v', w', x))| + |x| \geq 4n - 8 + 1 = 4n - 7\).

**Case 5**: Four nodes belong to case (b) (Fig. 9(e)). In this case, \(u', v', w', x'\) are not adjacent to each other. From the Case 5 of Lemma 5, \(|V - X| \geq |N((u', v', w', x'))| \geq 4n - 8\).

For all 5 cases, \(|V - X| \geq 4n - 8\). But, \(4n - 8 > 2(n - p - 1) = |V - X|\) when \(n \geq 4\). Again, a contradiction to (2).
Finally, we point out that $S_3$ is the least star graph satisfying the three sufficient conditions in Theorem 1. In Fig. 10, we give an example to show that $S_3$ is not 2-diagnosable.

4. Conclusion

The diagnosability of star graph under the comparison diagnosis model is studied in this paper. Under this model, the system is self-diagnosable if we know the diagnosability of the system. We prove that a system with the $n$-dimensional star graph structure is $(n−1)$-diagnosable under the comparison model if $n ≥ 4$. Based on the result, a polynomial-time algorithm proposed in [8] can be directly used to diagnose the system if there are at most $n − 1$ faulty processors. The diagnosis involves only one test phase to identify the faulty processors and one repair/replacement phase. Thus it is applicable in the environment that the components are reliable and periodic and quick testings are affordable. Furthermore, the algorithm can be used as a component of a larger diagnosis scheme to perform a given phase of fault location, as opposed to being used as a stand-alone diagnosis tool.

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