Hash Property and Coding Theorems for Sparse Matrices and Maximum-Likelihood Coding

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Abstract

The aim of this paper is to prove the achievability of several coding problems by using sparse matrices (the maximum column weight grows logarithmically in the block length) and maximal-likelihood (ML) coding. These problems are the Slepian-Wolf problem, the Gel’fand-Pinsker problem, the Wyner-Ziv problem, and the One-helps-one problem (source coding with partial side information at the decoder). To this end, the notion of a hash property for an ensemble of functions is introduced and it is proved that an ensemble of $q$-ary sparse matrices satisfies the hash property. Based on this property, it is proved that the rate of codes using sparse matrices and maximal-likelihood (ML) coding can achieve the optimal rate.

Index Terms

Shannon theory, hash functions, linear codes, sparse matrix, maximum-likelihood encoding/decoding, the Slepian-Wolf problem, the Gel’fand-Pinsker problem, the Wyner-Ziv problem, the One-helps-one problem

I. INTRODUCTION

The aim of this paper is to prove the achievability of several coding problems by using sparse matrices (the maximum column weight grows logarithmically in the block length) and maximal-likelihood (ML) coding, namely the Slepian-Wolf problem [39] (Fig. 1), the Gel’fand-Pinsker problem [13] (Fig. 2), the Wyner-Ziv problem [47] (Fig. 3), and the One-helps-one problem (source coding with partial side information at the decoder) [44][46] (Fig. 4). To prove these theorems, we first introduce the notion of a hash property for an ensemble of functions, where functions are not assumed to be linear. This notion is a sufficient condition for the achievability of coding theorems. Next, we prove that an ensemble of $q$-ary sparse matrices, which is an extension of [21], satisfies the hash property. Finally, based on the hash property, we prove that the rate of...
codes can achieve the optimal rate. This implies that the rate of codes using sparse matrices and ML coding can achieve the optimal rate. It should be noted here that there is a practical approximation method of ML coding by using sparse matrices and the linear programming technique introduced by [11].

The contributions of this paper are summarized in the following.

- The notion of a hash property is introduced. It is an extension of the notion of a universal class of hash functions introduced in [7]. The single source coding problem is studied in [22, Section 14.2][17] by using the hash function. We prove that an ensemble of \( q \)-ary sparse matrices has a hash property, while a weak version of hash property is proved in [26][2][10][33][34] implicitly. It should be noted that our definition of hash property is also an extension of the definition of random bin coding introduced in [3], where the set of all sequences is partitioned at random. On the other hand, the random codebook (a set of codewords/representations) generation is introduced for the proof of the original channel coding theorem [37] and lossy source coding theorem [38]. Here it is proved that random bin coding and partitioning determined by randomly generated matrix can be applied to the original channel coding theorem and lossy source coding theorem.

- The proof of the achievability of the Slepian-Wolf problem is demonstrated based on the hash property. It is the extension of [22, Section 14.2][17] and provides a new proof of [3][5][33]. By applying the theorem to the coding theorem of channel with additive (symmetric) noise, it also provides a new proof of [26][2][10][33].

- The optimality of a code is proved by using sparse matrices and ML coding for the Gel’fand-Pinsker problem. We prove the \( q \)-ary and asymmetric version of the theorem, while a binary and symmetric version is studied in [24]. It should be noted here that the column/row weight of matrices used in [24] is constant with respect to the block length, while it grows logarithmically in our construction. The detailed difference from [24] is stated in Section V-B. As a corollary, we have the optimality of codes using sparse matrices for the coding problem of an arbitrary (\( q \)-ary and asymmetric) channel, while a symmetric channel is assumed in many of the channel coding theorems by using sparse matrices. The construction is based on the coset code presented in [33][29], which is different from that presented in [12][2]. When our theorem is applied to the ensemble of sparse matrices, our proof is simpler than that in [29].

- The optimality of a code is proved by using sparse matrices and ML coding for the Wyner-Ziv problem. We prove the \( q \)-ary and biased source, and the non-additive side information version of the theorem, while a binary and unbiased source and additive side information are assumed in [24]. As a corollary, we have

\[
X \xrightarrow{\varphi_X} R_X > H(X|Y) \\
Y \xrightarrow{\varphi_Y} R_Y > H(Y|X) \\
XY \xrightarrow{\varphi^{-1}} R_X + R_Y > H(X,Y)
\]

Fig. 1. Slepian-Wolf Problem
the optimality of codes using sparse matrices for the lossy coding problem of an arbitrary (q-ary and
biased) source and a distortion measure. In [25][36][23][27][14], a lossy code is proposed by using sparse
matrices called low density generator matrices (LDGM) by assuming an unbiased source and Hamming
distortion. The column/row weight of matrices used in [24] is constant with respect to the block length,
while it grows logarithmically in our construction. The lower bounds on the rate-distortion function is
discussed in [8][19]. It should be noted that the construction of the codes is different from those presented
in [25][36][23][24][27][14]. The detailed difference is stated in Section V-C. Our construction is based
on the code presented in [28][30] and similar to the code presented in [42][43][49]. When our theorem is
applied to the ensemble of sparse matrices, our proof is simpler than that in [28].

• The achievability of the One-helps-one problem is proved by using sparse matrices and ML coding.

II. DEFINITIONS AND NOTATIONS

Throughout this paper, we use the following definitions and notations.

Column vectors and sequences are denoted in boldface. Let $A\, u$ denote a value taken by a function $A : \mathcal{U}^n \rightarrow \mathcal{U}$ at $u \in \mathcal{U}^n$ where $\mathcal{U}^n$ is a domain of the function. It should be noted that $A$ may be nonlinear. When $A$ is
a linear function expressed by an $l \times n$ matrix, we assume that $\mathcal{U} \equiv \mathbb{F}_q$ is a finite field and the range of
functions is defined by $\mathcal{U} \equiv \mathcal{U}^l$. It should be noted that this assumption is not essential for general (nonlinear)
functions because discussion is not changed if $l \log |\mathcal{U}|$ is replaced by $\log |\mathcal{U}|$. For a set $\mathcal{A}$ of functions, let $\text{Im} \mathcal{A}$ be defined as

$$\text{Im} \mathcal{A} = \bigcup_{A \in \mathcal{A}} \{A\, u : u \in \mathcal{U}^n \}.$$ 

The cardinality of a set $\mathcal{U}$ is denoted by $|\mathcal{U}|$, $\mathcal{U}^c$ denotes the compliment of $\mathcal{U}$, and $\mathcal{U} \setminus \mathcal{V} \equiv \mathcal{U} \cap \mathcal{V}^c$ denotes
the set difference. We define sets $\mathcal{C}(c)$ and $\mathcal{C}_{AB}(c, b)$ as

$$\mathcal{C}(c) \equiv \{u : A\, u = c\}$$

$$\mathcal{C}_{AB}(c, b) \equiv \{u : A\, u = c, B\, u = b\}.$$ 

In the context of linear codes, $\mathcal{C}(c)$ is called a coset determined by $c$. 

February 12, 2013 DRAFT
Let $p$ and $p'$ be probability distributions and let $q$ and $q'$ be conditional probability distributions. Then entropy $H(p)$, conditional entropy $H(q|p)$, divergence $D(p||p')$, and conditional divergence $D(q||q'|p)$ are defined as

$$H(p) \equiv \sum_u p(u) \log \frac{1}{p(u)}$$

$$H(q|p) \equiv \sum_{u,v} q(u|v)p(v) \log \frac{1}{q(u|v)}$$

$$D(p \parallel p') \equiv \sum_u p(u) \log \frac{p(u)}{p'(u)}$$

$$D(q \parallel q'|p) \equiv \sum_v p(v) \sum_u q(u|v) \log \frac{q(u|v)}{q'(u|v)},$$

where we assume the base 2 of the logarithm when the subscript of log is omitted.

Let $\mu_{UV}$ be the joint probability distribution of random variables $U$ and $V$. Let $\mu_U$ and $\mu_V$ be the respective marginal distributions and $\mu_{U|V}$ be the conditional probability distribution. Then the entropy $H(U)$, the conditional entropy $H(U|V)$, and the mutual information $I(U; V)$ of random variables are defined as

$$H(U) \equiv H(\mu_U)$$

$$H(U|V) \equiv H(\mu_{U|V}|\mu_V)$$

$$I(U; V) \equiv H(U) - H(U|V).$$

A set of typical sequences $T_{U,\gamma}$ and a set of conditionally typical sequences $T_{U|V,\gamma}(v)$ are defined as

$$T_{U,\gamma} \equiv \{u : D(\nu_u || \mu_U) < \gamma\}$$

$$T_{U|V,\gamma}(v) \equiv \{u : D(\nu_{u|v} || \mu_{U|V}|\nu_v) < \gamma\},$$

respectively, where $\nu_u$ and $\nu_{u|v}$ are defined as

$$\nu_u(u) \equiv \frac{|\{1 \leq i \leq n : u_i = u\}|}{n}$$

$$\nu_{u|v}(u|v) \equiv \frac{\nu_{uv}(u,v)}{\nu_v(v)}.$$

We define $\chi(\cdot)$ as

$$\chi(a = b) \equiv \begin{cases} 
1, & \text{if } a = b \\
0, & \text{if } a \neq b 
\end{cases}$$

$$\chi(a \neq b) \equiv \begin{cases} 
1, & \text{if } a \neq b \\
0, & \text{if } a = b. 
\end{cases}$$
Finally, for $\gamma, \gamma' > 0$, we define

$$\lambda_U \equiv \frac{|U| \log [n+1]}{n}$$  \hspace{1cm} (1)$$

$$\zeta_U(\gamma) \equiv \gamma - \sqrt{2 \gamma \log \frac{\sqrt{2\gamma}}{|U|}}$$  \hspace{1cm} (2)$$

$$\zeta_U|V(\gamma'|\gamma) \equiv \gamma' - \sqrt{2 \gamma' \log \frac{\sqrt{2\gamma'}}{|U||V|}} + \sqrt{2 \gamma \log |U|}$$  \hspace{1cm} (3)$$

$$\eta_U(\gamma) \equiv - \sqrt{2 \gamma \log \frac{\sqrt{2\gamma}}{|U|}} + \frac{|U| \log [n+1]}{n}$$  \hspace{1cm} (4)$$

$$\eta_U|V(\gamma'|\gamma) \equiv - \sqrt{2 \gamma' \log \frac{\sqrt{2\gamma'}}{|U||V|}} + \sqrt{2 \gamma \log |U|} + \frac{|U||V| \log [n+1]}{n}.$$  \hspace{1cm} (5)$$

It should be noted here that the product set $U \times V$ is denoted by $UV$ when it appears in the subscript of these functions.

III. $(\alpha, \beta)$-Hash Property

In this section, we introduce the notion of the $(\alpha, \beta)$-hash property which is a sufficient condition for coding theorems, where the linearity of functions is not assumed. The $(\alpha, \beta)$-hash property of an ensemble of linear (sparse) matrices will be discussed in Section IV. In Section V, we provide coding theorems for the Slepian-Wolf problem, the Gel’fand-Pinsker problem, the Wyner-Ziv problem, and the One-helps-one problem.

Before stating the formal definition, we explain the random coding arguments and two implications which introduce the intuition of the hash property.

A. Two types of random coding

We review the random coding argument introduced by [37]. Most of coding theorems are proved by using the combination of the following two types of random coding.

**Random codebook generation:** In the proof of the original channel coding theorem [37] and lossy source coding theorem [38], a codebook (a set of codewords/representations) is randomly generated and shared by the encoder and the decoder. It should be noted that the randomly generated codebook represents the list of typical sequences. In the encoding step of channel coding, a message is mapped to a member of randomly generated codewords as a channel input. In the decoding step, we use the maximum-likelihood decoder, which guess the most probable channel input from the channel output. In the encoding step of lossy source coding, we find a member of randomly generated representations to satisfy the fidelity criterion compared to a message, and then we let the index of this member as a codeword. In the decoding step, the codeword (index) is mapped to the reproduction. It should be noted that the encoder and the decoder have to share the large size (exponentially in the block length) of table which indicates the correspondence between a index and a member of randomly generated codewords/representations. The time complexity of the encoding and decoding step of channel coding and the encoding step of lossy source coding is exponentially in the block length. They are obstacles for the implementation.

**Random partitioning (random bin coding):** In the proof of Slepian-Wolf problem [3], the set of all sequences is partitioned at random and shared by the encoder and the decoder. In the encoding step, a pair of messages
are mapped independently to the index of bin which contains the message. In the decoding step, we use the maximum-likelihood decoder, which guess the most probable pair of messages. Random partitioning by cosets determined by randomly generated matrix can be considered as a kind of random bin coding, where the syndrome corresponds the index of bin. This approach is introduced in [9] for the coding of symmetric channel and applied to the ensemble of sparse matrices in [26][2][10]. This argument is also applied to the coding theorem for Slepian-Wolf problem in [45][5][33]. It should be noted that the time complexity of the decoding step is exponentially in the block length, but there are practical approximation methods by using sparse matrices and the techniques introduced by [1][18][11]. By using the randomly generated matrix, the size of tables shared by the encoder and the decoder has at most square order with respect to the block length.

One of the aim of introducing hash property is to replace the random codebook generation by the random partitioning. In other words, it is a unification of these two random coding arguments. It is expected that the space and time complexity can be reduced compared to the random codebook generation.

B. Two implications of hash property

We introduce the following two implications of hash property, which connect the number of bins and messages (items) and is essential for the coding by using the random partitioning. In Section III-D, these two property are derived from the hash property by adjusting the number of bins taking account of the number of sequences. **Collision-resistant property:** The good code assigns a message to a codeword which is different from the codewords of other messages, where the loss (error probability) is as small as possible. The collision-resistant property is the nature of the hash property. Figure 5 (a) represents the ideal situation of this property, where the black dots represent messages we want to distinguish. When the number of bins is greater than the number of black dots, we can find a good function that allocates the black dots to the different bins. It is because the hash property tends to avoid the collision. It should be noted that it is enough for coding problems to satisfy this property for ‘almost all (close to probability one)’ black dots by letting the ratio [the number of black dots]/[the number of bins] close to zero. This property is used for the estimation of

Fig. 5. Properties connecting the number of bins and items (black dots, messages). (a) Collision-resistant property: every bin contains at most one item. (b) Saturating property: every bin contains at least one item. (c) Pigeonhole principle: there is at least one bin which contains two or more items.
decoding error probability of lossless source coding by using maximum-likelihood decoder. In this situation, the black dots correspond to the typical sequences.

**Saturating property:** To replace the random codebook generation by the random partitioning, we prepare the method finding a typical sequence for each bin. The saturating property is the another nature of the hash property. Figure 5 (b) represents the ideal situation of this property. When the number of bins is smaller than the number of black dots, we can find a good function so that every bins has at least one black dot. It is because the hash property tends to avoid the collision. It should be noted that this property is different from the pigeonhole principle: there is at least one bin which includes two or more black dots. Figure 5 (c) represents an unusual situation, which does not contradict by the pigeonhole principle while the hash property tends to avoid this situation. It should be noted that this property is different from the pigeonhole principle: there is at least one bin which includes two or more black dots. Figure 5 (c) represents an unusual situation, which does not contradict by the pigeonhole principle while the hash property tends to avoid this situation. It should be noted that it is enough for coding problems to satisfy this property for ‘almost all (close to probability one)’ bins by letting the ratio $[\text{the number of bins}] / [\text{the number of black dots}]$ close to zero. To find a typical sequence from each bin, we use the maximum-likelihood/minimum-divergence coding introduced in Section III-D. In this situation, the black dots correspond to the typical sequences.

### C. Formal definition of $(\alpha, \beta)$-hash property

In this section, we introduce the formal definition of the hash property.

In the proof of the fixed-rate source coding theorem given in [3][5][17], it is proved implicitly that there is a probability distribution $p_A$ on a set of functions $A : \mathcal{U}^n \rightarrow \mathcal{U}^l$ such that

$$p_A(\{ A : \exists u' \in \mathcal{G} \setminus \{ u \}, A u' = A u \}) \leq \frac{|\mathcal{G}|}{|\mathcal{U}|}$$

for any $u \in \mathcal{U}^n$, where

$$\mathcal{G} \equiv \{ u' : \mu(u') \geq \mu(u), u \neq u' \}$$

and $\mu$ is the probability distribution of a source or the probability distribution of the additive noise of a channel.

In the proof of coding theorems for spare matrices given in [2][10][26][33][34], it is proved implicitly that there are a probability distribution on a set of $l \times n$ spare matrices and sequences $\alpha \equiv \{ \alpha(n) \}_{n=1}^\infty$ and $\beta \equiv \{ \beta(n) \}_{n=1}^\infty$ satisfying

$$\lim_{n \to \infty} \frac{\log \alpha(n)}{n} = 0$$

$$\lim_{n \to \infty} \beta(n) = 0$$

such that

$$p_A(\{ A : \exists u' \in \mathcal{G} \setminus \{ u \}, A u' = A u \}) \leq \frac{|\mathcal{G}| \alpha(n)}{|\mathcal{U}|} + \beta(n)$$

for any $u \in \mathcal{U}^n$, where $\alpha(n)$ measures how the ensemble of $l \times n$ sparse matrices differs from the ensemble of all $l \times n$ matrices and $\beta(n)$ measures the probability that the code determined by an $l \times n$ sparse matrix has low-weight codewords. It should be noted that the collision-resistant property can be derived from (6) and (8).

It is shown in Section III-D.

The aim of this paper is not only unifying the above results, but also providing several coding theorems under the general settings such as an asymmetric channel for channel coding and an unbiased source for lossy source coding. To this end, we define an $(\alpha, \beta)$-hash property in the following.
Definition 1: Let \( A \) be a set of functions \( A : \mathcal{U}^n \rightarrow \overline{\mathcal{U}}_A \) and we assume that
\[
\lim_{n \rightarrow \infty} \frac{\log |\mathcal{T}|}{n} = 0. \tag{H1}
\]
For a probability distribution \( p_A \) on \( A \), we call a pair \((A, p_A)\) an ensemble \(^3\). Then, an ensemble \((A, p_A)\) has an \((\alpha_A, \beta_A)\)-hash property if there are two sequences \( \alpha_A \equiv \{\alpha_A(n)\}_{n=1}^{\infty} \) and \( \beta_A \equiv \{\beta_A(n)\}_{n=1}^{\infty} \) such that
\[
\lim_{n \rightarrow \infty} \alpha_A(n) = 1 \tag{H2}
\]
\[
\lim_{n \rightarrow \infty} \beta_A(n) = 0, \tag{H3}
\]
and
\[
\sum_{u \in T, u' \in T'} p_A(\{A : Au = Au'\}) \leq |T \cap T'| + \frac{|T||T'|\alpha_A(n)}{|\text{Im}A|} + \text{min}\{|T|, |T'|\} \beta_A(n) \tag{H4}
\]
for any \( T, T' \subset \mathcal{U}^n \). Throughout this paper, we omit dependence on \( n \) of \( \alpha_A \) and \( \beta_A \) when \( n \) is fixed.

It should be noted that we have (8) from (H4) by letting \( T \equiv \{u\} \) and \( T' \equiv \emptyset \), and (6) is the case when \( \alpha_A(n) \equiv 1 \) and \( \beta_A(n) \equiv 0 \). In the right hand side of the inequality (H4), the first term corresponds to the sum of \( p_A(\{A : Au = Au'\}) = 1 \) over all \( u \in T \cap T' \), the second term bounds the sum of the probability \( p_A(\{A : Au = Au'\}) \) which are approximately \( 1/|\text{Im}A|\) for \( u \neq u' \), and the third term bounds the sum of the probability \( p_A(\{A : Au = Au'\}) \) far greater than \( 1/|\text{Im}A|\) for \( u \neq u' \). This intuition is explained in Section IV for the ensemble of matrices.

In the following, we present two examples of ensembles that have a hash property.

Example 1: Our terminology ‘hash’ is derived from a universal class of hash functions introduced in [7]. We call a set \( A \) of functions \( A : \mathcal{U}^n \rightarrow \overline{\mathcal{U}}_A \) a universal class of hash functions if
\[
|\{A : Au = Au'\}| \leq \frac{|A|}{|\text{Im}A|}
\]
for any \( u \neq u' \). For example, the set of all functions on \( \mathcal{U}^n \) and the set of all linear functions \( A : \mathcal{U}^n \rightarrow \mathcal{U}^{l_A} \) are classes of universal hash functions (see [7]). Furthermore, for \( \mathcal{U}^n \equiv \text{GF}(2^n) \), the set
\[
A \equiv \left\{ A : Au \equiv \text{the first } l_A \text{ bits of } au \right\}
\]
is a universal class of hash functions, where \( au \) is a multiplication of two elements \( a, u \in \text{GF}(2^n) \). It should be noted that every example above satisfies \( \text{Im}A = \overline{\mathcal{U}}_A \). When \( A \) is a class of universal hash functions and \( p_A \)

\(^2\) It should be noted that \( p_A \) does not depend on a particular function \( A \). Strictly speaking, the subscript \( A \) of \( p \) represents the random variable of a function. We use this ambiguous notation when \( A \) appears in the subscript of \( p \) because random variables are always denoted in Roman letter.

\(^3\) In the standard definition, an ensemble is defined as a set of functions and a uniform distribution is assumed for this set. It should be noted that an ensemble is defined as the probability distribution on a set of functions in this paper.

\(^4\) It should be noted that \( \alpha_A \) and \( \beta_A \) do not depend on a particular function \( A \) but may depend on an ensemble \((A, p_A)\). Strictly speaking, the subscript \( A \) represents the random variable.
is the uniform distribution on $A$, we have
\[
\sum_{u \in T} p_A (\{A : Au = Au'\}) \leq |T \cap T'| + \frac{|T||T'|}{|\text{Im}(A)|}.
\]
This implies that $(A, p)$ has a $(1, 0)$-hash property, where $1(n) \equiv 1$ and $0(n) \equiv 0$ for every $n$.

**Example 2:** In this example, we consider a set of linear functions $A : U^n \rightarrow U^l$. It was discussed in the above example that the uniform distribution on the set of all linear functions has a $(1, 0)$-hash property. The hash property of an ensemble of $q$-ary sparse matrices will be discussed in Section IV. The binary version of this ensemble is introduced in [21].

### D. Basic lemmas of hash property

In the following, basic lemmas of the $(\alpha, \beta)$-hash property are introduced. All lemmas are proved in Section VI-A. Let $A$ (resp. $B$) be a set of functions $A : U^n \rightarrow U^l_A$ (resp. $B : U^n \rightarrow U^l_B$). We assume that $(A, p_A)$ (resp. $(B, p_B)$) has an $(\alpha_A, \beta_A)$-hash (resp. $(\alpha_B, \beta_B)$-hash) property. We also assume that $p_C$ is the uniform distribution on $\text{Im}(A)$, and random variables $A$, $B$, and $C$ are mutually independent, that is,

\[
p_C(c) = \begin{cases} 
\frac{1}{|\text{Im}(A)|}, & \text{if } c \in \text{Im}(A) \\
0, & \text{if } c \in \overline{U_A} \setminus \text{Im}(A)
\end{cases}
\]

\[
p_{ABC}(A, B, c) = p_A(A)p_B(B)p_C(c)
\]

for any $A$, $B$, and $c$.

First, we demonstrate that the collision-resistant property and saturating property are derived from the $(\alpha, \beta)$-hash property.

**Lemma 1:** For any $G \subset U^n$ and $u \in U^n$,

\[
p_A (\{A : [G \setminus \{u\}] \cap C_A(Au) \neq \emptyset\}) \leq \frac{|G|\alpha_A}{|\text{Im}(A)|} + \beta_A.
\]

We prove the collision-resistant property from Lemma 1. Let $\mu_U$ be the probability distribution on $G \subset U^n$. We have

\[
E_A [\mu_U (\{u : [G \setminus \{u\}] \cap C_A(Au) \neq \emptyset\})] \leq \sum_{u \in G} \mu_U(u)p_A (\{A : [G \setminus \{u\}] \cap C_A(Au) \neq \emptyset\})
\]

\[
\leq \sum_{u \in G} \mu_U(u) \left[ \frac{|G|\alpha_A}{|\text{Im}(A)|} + \beta_A \right]
\]

\[
\leq \frac{|G|\alpha_A}{|\text{Im}(A)|} + \beta_A.
\]

By assuming that $|G|/|\text{Im}(A)|$ vanishes as $n \rightarrow \infty$, we have the fact that there is a function $A$ such that

\[
\mu_U (\{u : [G \setminus \{u\}] \cap C_A(Au) \neq \emptyset\}) < \delta
\]

for any $\delta > 0$ and sufficiently large $n$. Since the relation $[G \setminus \{u\}] \cap C_A(Au) \neq \emptyset$ corresponds to the event that there is $u' \in G$, $u' \neq u$ such that $u$ and $u'$ are the members of the same bin (have the same codeword determined by $A$), we have the fact that the members of $G$ are located in the different bins (the members of $G$
can be decoded correctly) with high probability. In the proof of fixed-rate source coding, \( G \) is defined by (7) for given probability distribution \( \mu_U \) of a source \( U \), where \( \mu_U(G^c) \) is close to zero. In the linear coding of a channel with additive noise, additive noise \( u \) can be specified by the syndrome \( Au \) obtained by operating the parity check matrix \( A \) to the channel output. It should be noted that, when Lemma 1 is applied, it is sufficient to assume that
\[
\lim_{n \to \infty} \frac{\log \alpha_A(n)}{n} = 0
\]
instead of (H2) because it is usually assumed that \( |G|/|\text{Im}A| \) vanishes exponentially by letting \( n \to \infty \). It is implicitly proved in [26][2][10][33] that some ensembles of sparse linear matrices have this weak hash property.

The second lemma introduce the saturating property. We use the folloing lemma in the proof of the coding theorems of the Gel’fand-Pinsker problem, the Wyner-Ziv problem, and the One-helps-one problem.

**Lemma 2:** If \( T \neq \emptyset \), then
\[
p_{AC}(\{(A, c) : T \cap C_A(c) = \emptyset\}) \leq \alpha_A - 1 + \frac{|\text{Im}A|}{|T|} \left[ \frac{\beta_A + 1}{|T|} \right].
\]

We prove the saturating property form Lemma 2. We have
\[
E_A[p_C(\{e : T \cap C_A(c) = \emptyset\})] = p_{AC}(\{(A, c) : T \cap C_A(c) = \emptyset\}) \leq \alpha_A - 1 + \frac{|\text{Im}A|}{|T|} \left[ \frac{\beta_A + 1}{|T|} \right].
\]

By assuming that \( |\text{Im}A|/|T| \) vanishes as \( n \to \infty \), we have the fact that there is a function \( A \) such that
\[
p_C(\{e : T \cap C_A(c) = \emptyset\}) < \delta
\]
for any \( \delta > 0 \) and sufficiently large \( n \). Since the relation \( T \cap C_A(c) = \emptyset \) corresponds to the event that there is no \( u \in T \) in the bin \( C_A(c) \), we have the fact that we can find a member of \( T \) in the randomly selected bin with high probability. It should be noted that, when Lemma 2 is applied, it is sufficient to assume that
\[
\lim_{n \to \infty} \frac{\log \beta_A(n)}{n} = 0
\]
instead of (H3) because it is usually assumed that \( |\text{Im}A|/|T| \) vanishes exponentially by letting \( n \to \infty \). In fact, the condition (H3) is required for the collision-resistant property.

Next, we prepare the lemmas used in the proof of coding theorems. The following lemmas come from Lemma 1.

**Lemma 3:** If \( G \subset U^n \) and \( u \notin G \), then
\[
p_{AC}\left(\left\{(A, c) : G \cap C_A(c) \neq \emptyset, u \in C_A(c)\right\}\right) \leq \frac{|G|\alpha_A}{|\text{Im}A|^2} + \frac{\alpha_A}{|\text{Im}A|}.
\]

**Lemma 4:** Assume that \( u_{A,c} \in U^n \) depends on \( A \) and \( c \). Then
\[
p_{ABC}(\{(A, B, c) : [G \setminus \{u_{A,c}\}] \cap C_{AB}(c, Bu_{A,c}) \neq \emptyset\}) \leq \frac{|G|\alpha_B}{|\text{Im}A||\text{Im}B|} + \beta_B
\]
for any \( G \subset U^n \).

Finally, we introduce the method for finding a typical sequence in a bin. The probability of an event where the function finds a conditionally typical sequence is evaluated by the following lemmas. They are the key lemmas.
for the coding theorems for the Gel’fand-Pinsker problem, the Wyner-Ziv problem, and the One-helps-one problem. These lemmas are proved by using Lemma 2. For $\varepsilon > 0$, let

$$l_A \equiv \frac{n[H(U|V) - \varepsilon]}{\log|\mathcal{U}|}$$

and assume that $\mathcal{A}$ is a set of functions $A : \mathcal{U}^n \to \mathcal{U}^{1_A}$ and $v \in \mathcal{T}_{V,\gamma}$.

**Lemma 5:** We define a maximum-likelihood (ML) coding function $g_A$ under constraint $u \in \mathcal{C}_A(c)$ as

$$g_A(c|v) \equiv \arg \max_{u \in \mathcal{C}_A(c)} \mu_{U|V}(u|v)$$

and assume that a set $T(v) \subset \mathcal{T}_{U|V,2\varepsilon}(v)$ satisfies

- $T(v)$ is not empty, and
- if $u \in T(v)$ and $u'$ satisfies

$$\mu_{U|V}(u|v) \leq \mu_{U|V}(u'|v) \leq 2^{-n[H(U|V) - 2\varepsilon]}$$

then $u' \in T(v)$.

In fact, we can construct such $T(v)$ by taking up $|T(v)|$ elements from $\mathcal{T}_{U|V,2\varepsilon}(v)$ in the order of probability rank. If an ensemble $(\mathcal{A}, p_A)$ of a set of functions $A : \mathcal{U}^n \to \mathcal{U}^{1_A}$ has an $(\alpha_A, \beta_A)$-hash property, then

$$p_{AC}(\{(A, c) : g_A(c|v) \notin T(v)\}) \leq \alpha_A - 1 + \frac{[\text{Im}A][\beta_A + 1]}{|T(v)|} + 2^{-n\varepsilon|\mathcal{U}|}$$

for any $v$ satisfying $\mathcal{T}_{U|V,2\varepsilon}(v) \neq \emptyset$.

**Lemma 6:** We define a minimum-divergence (MD) coding function $\hat{g}_A$ under constraint $u \in \mathcal{C}_A(c)$ as

$$\hat{g}_A(c|v) \equiv \arg \min_{u \in \mathcal{C}_A(c)} D(\nu_{u|v}\|\mu_{U|V}|v)$$

and assume that, for $\gamma > 0$, a set $T \subset \mathcal{T}_{U|V,\gamma}(v)$ satisfies that if $u \in T$ and $u'$ satisfies

$$D(\nu_{u'|v}\|\mu_{U|V}|v) \leq D(\nu_{u|v}\|\mu_{U|V}|v)$$

then $u' \in T$. In fact, we can construct such $T$ by picking up $|T|$ elements from $\mathcal{T}_{U|V,\gamma}(v)$ in the descending order of conditional divergence. Then

$$p_{AC}(\{(A, c) : \hat{g}_A(c|v) \notin T(v)\}) \leq \alpha_A - 1 + \frac{[\text{Im}A][\beta_A + 1]}{|T(v)|}$$

for any $v$ satisfying $\mathcal{T}_{U|V,\gamma}(v) \neq \emptyset$.

In Section V, we construct codes by using the maximum-likelihood coding function. It should be noted that we can replace the maximum-likelihood coding function by the minimum-divergence coding function and prove theorems simpler than that presented in this paper.
IV. HASH PROPERTY FOR ENSEMBLE OF MATRICES

In this section, we discuss the hash property of an ensemble of (sparse) matrices.

First, we introduce the average spectrum of an ensemble of matrices given in [2]. Let $\mathcal{U}$ be a finite field and $p_A$ be a probability distribution on a set of $l_A \times n$ matrices. It should be noted that $A$ represents a corresponding linear function $A: \mathcal{U}^n \rightarrow \mathcal{U}^{l_A}$, where we define $\mathcal{U}_A \equiv \mathcal{U}^{l_A}$.

Let $t(u)$ be the type$^5$ of $u \in \mathcal{U}^n$, where a type is characterized by the number $n_{u}n_{u}$ of occurrences of each symbol in the sequence $u$. Let $\mathcal{H}$ be a set of all types of length $n$ except $t(0)$, where $0$ is the zero vector. For the probability distribution $p_A$ on a set of $l_A \times n$ matrices, let $S(p_A, t)$ be defined as

$$S(p_A, t) \equiv \sum_{A} p_A(A) |\{u \in \mathcal{U}^n : Au = 0, t(u) = t\}|.$$

For $\mathcal{H} \subset \mathcal{H}$, we define $\alpha_A(n)$ and $\beta_A(n)$ as

$$\alpha_A(n) \equiv \frac{|\text{Im} A|}{|\mathcal{U}|^{l_A}} \cdot \max_{t \in \mathcal{H}} S(p_A, t)$$  
$$\beta_A(n) \equiv \sum_{t \in \mathcal{H} \setminus \mathcal{H}} S(p_A, t),$$

where $u_A$ denotes the uniform distribution on the set of all $l_A \times n$ matrices. When $\mathcal{U} \equiv \text{GF}(2)$ and $\mathcal{H}$ is a set of high-weight types, $\alpha_A$ measures how the ensemble $(A, p_A)$ differs from the ensemble of all $l_A \times n$ matrices with respect to high-weight part of average spectrum and $\beta_A$ provides the upper bound of the probability that the code $\{u \in \mathcal{U}^n : Au = 0\}$ has low-weight codewords. It should be noted that

$$\bar{\alpha}_A(n) \equiv \max_{t \in \mathcal{H}} \frac{S(p_A, t)}{S(u_A, t)}$$

is introduced in [26][2][10][33][34] instead of $\alpha_A(n)$. We multiply the coefficient $|\text{Im} A|/|\mathcal{U}|^{l_A}$ so that $\alpha_A$ satisfies (H2).

We have the following theorem.

**Theorem 1:** Let $(A, p_A)$ be an ensemble of matrices and assume that $p_A (\{A : Au = 0\})$ depends on $u$ only through the type $t(u)$. If $|\mathcal{U}_A|/|\text{Im} A|$ satisfies (H1) and $(\alpha_A(n), \beta_A(n))$, defined by (9) and (10), satisfies (H2) and (H3), then $(A, p_A)$ has an $(\alpha_A, \beta_A)$-hash property.

The proof is given in Section VI-B.

Next, we consider the independent combination of two ensembles $(A, p_A)$ and $(B, p_B)$, of $l_A \times n$ and $l_B \times n$ matrices, respectively. We assume that $(A, p_A)$ has an $(\alpha_A, \beta_A)$-hash property, where $(\alpha_A(n), \beta_A(n))$ is defined as (9) and (10). Similarly we define $(\alpha_B(n), \beta_B(n))$ for an ensemble $(B, p_B)$ and assume that $(B, p_B)$ has an $(\alpha_B, \beta_B)$-hash property. Let $p_{AB}$ be the joint distribution defined as

$$p_{AB}(A, B) \equiv p_A(A)p_B(B).$$

We have the following two lemmas. The proof is given in Section VI-B.

---

$^5$ As in [6], the type is defined in terms of the empirical probability distribution $\nu_u$. In our definition, the type is the number $n_{u}n_{u}$ of occurrences which is different from the empirical probability distribution.
Lemma 7: Let \( (\alpha_{AB}(n), \beta_{AB}(n)) \) defined as

\[
\alpha_{AB}(n) \equiv \alpha_A(n)\alpha_B(n) \\
\beta_{AB}(n) \equiv \min\{\beta_A(n), \beta_B(n)\}.
\]

Then the ensemble \((A \oplus B, p_{AB})\) of functions \(A \oplus B : \mathcal{U}^n \rightarrow \mathcal{U}^{l_A + l_B}\) defined as

\[
A \oplus B(u) \equiv (Au, Bu)
\]

has an \((\alpha_{AB}, \beta_{AB})\)-hash property.

Lemma 8: Let \((\alpha_{AB}(n), \beta'_{AB}(n))\) be defined as

\[
\alpha_{AB}(n) \equiv \alpha_A(n)\alpha_B(n) \\
\beta'_{AB}(n) \equiv \frac{\alpha_A(n)\beta_B(n)}{|\text{Im}A|} + \frac{\alpha_B(n)\beta_A(n)}{|\text{Im}B|} + \beta_A(n)\beta_B(n).
\]

Then the ensemble \((A \otimes B, p_{AB})\) of functions \(A \otimes B : \mathcal{U}^n \times \mathcal{V}^n \rightarrow \mathcal{U}^{l_A} \times \mathcal{V}^{l_B}\) defined as

\[
A \otimes B(u, v) \equiv (Au, Bv)
\]

has an \((\alpha_{AB}, \beta'_{AB})\)-hash property.

Finally, we introduce an ensemble of \(q\)-ary sparse matrices, where the binary version of this ensemble is proposed in [21]. In the following, let \(\mathcal{U} \equiv GF(q)\) and \(l_A \equiv nR\). We generate an \(l_A \times n\) matrix \(A\) with the following procedure, where at most \(\tau\) random nonzero elements are introduced in every row.

1) Start from an all-zero matrix.

2) For each \(i \in \{1, \ldots, n\}\), repeat the following procedure \(\tau\) times:
   a) Choose \((j, a) \in \{1, \ldots, l_A\} \times [GF(q) \setminus \{0\}]\) uniformly at random.
   b) Add \(a\) to the \((j, i)\) component of \(A\).

Let \((A, p_A)\) be an ensemble corresponding to the above procedure. It is proved in Section VI-C that \(p_A(\{A : Au = 0\})\) depends on \(u\) only through the type \(t(u)\). Let \((\alpha_A(n), \beta_A(n))\) be defined by (9) and (10) for this ensemble.

We assume that column weight \(\tau = O(\log n)\) is even. Let \(w(t)\) be the weight of type \(t = (t(0), \ldots, t(q - 1))\) defined as

\[
w(t) \equiv \sum_{i=1}^{q-1} t(i)
\]

and \(w(u)\) be defined as

\[
w(u) \equiv w(t(u)).
\]

We define

\[
\mathcal{H} \equiv \{t : w(t) > \xi l_A\}.
\]

(12)

Then it is also proved in Section VI-C that

\[
\text{Im}_A = \begin{cases} 
\{u \in \mathcal{U}^{l_A} : w(u) \text{ is even}\}, & \text{if } q = 2 \\
\mathcal{U}^l, & \text{if } q > 2
\end{cases}
\]
and there is $\xi > 0$ such that $(\alpha_A, \beta_A)$ satisfies (H2) and (H3). From Theorem 1, we have the following theorem.

**Theorem 2:** For the above ensemble $(A, p_A)$ of sparse matrices, let $(\alpha_A(n), \beta_A(n))$ be defined by (9), (10), (12), and suitable $\{\tau(n)\}_{n=1}^{\infty}$ and $\xi > 0$. Then $(A, p_A)$ has an $(\alpha_A, \beta_A)$-hash property.

We prove the theorem in Section VI-C. It should be noted here that, as we can see in the proof of the theorem, the asymptotic behavior (convergence speed) of $(\alpha_A, \beta_A)$ depends on the weight $\tau$.

**Remark 1:** It is proved in [26],[33] that $(\tilde{\alpha}_A(n), \beta_A(n))$, defined by (11) and (10), satisfies weaker properties

$$
\lim_{n \to \infty} \frac{\log \tilde{\alpha}_A(n)}{n} = 0 \quad (13)
$$

and (H3) when $q = 2$. It is proved in [2, Section III, Eq. (23),(82)] that $(\tilde{\alpha}_A(n), \beta_A(n))$ of another ensemble of Modulo-$q$ LDPC matrices satisfies weaker properties (13) and (H3).

### V. Coding Theorems

In this section, we present several coding theorems. We prove these theorems in Section VI based on the hash property.

Throughout this section, the encoder and decoder are denoted by $\varphi$ and $\varphi^{-1}$, respectively. We assume that the dimension of vectors $x$, $y$, $z$, and $w$ is $n$.

#### A. Slepian-Wolf Problem

In this section, we consider the Slepian-Wolf problem illustrated in Fig. 1. The achievable rate region for this problem is given by the set of encoding rate pair $(R_X, R_Y)$ satisfying

$$\begin{align*}
R_X &\geq H(X|Y) \\
R_Y &\geq H(Y|X) \\
R_X + R_Y &\geq H(X,Y).
\end{align*}$$

The achievability of the Slepian-Wolf problem is proved in [3] and [5] for the ensemble of bin-coding and all $q$-ary linear matrices, respectively. The constructions of encoders using sparse matrices is studied in [35][40][34] and the achievability is proved in [33] by using ML decoding. The aim of this section is to demonstrate the proof of the coding theorem based on the hash property. The proof is given in Section VI-D.

We fix functions

$$A : \mathcal{X}^n \to \mathcal{X}^{l_A}$$
$$B : \mathcal{Y}^n \to \mathcal{Y}^{l_B}$$

which are available to construct encoders and a decoder. We define the encoders and the decoder (illustrated in Fig. 6)

$$\varphi_X : \mathcal{X}^n \to \mathcal{X}^{l_A}$$
Fig. 6. Construction of Slepian-Wolf Source Code

\[ \varphi_Y: Y^n \rightarrow Y^B \]
\[ \varphi^{-1}: X^A \times Y^B \rightarrow X^n \times Y^n \]

as

\[ \varphi_X(x) \equiv Ax \]
\[ \varphi_Y(y) \equiv By \]
\[ \varphi^{-1}(b_X, b_Y) \equiv g_{AB}(b_X, b_Y), \]

where

\[ g_{AB}(b_X, b_Y) \equiv \arg \max_{(x', y') \in C_A(b_X) \times C_B(b_Y)} \mu_{XY}(x', y'). \]

The encoding rate pair \((R_X, R_Y)\) is given by

\[ R_X \equiv \frac{l_A \log |X|}{n} \]
\[ R_Y \equiv \frac{l_B \log |Y|}{n} \]

and the error probability \( \text{Error}_{XY}(A, B) \) is given by

\[ \text{Error}_{XY}(A, B) \equiv \mu_{XY} \left( \{(x, y) : \varphi^{-1}(\varphi_X(x), \varphi_Y(y)) \neq (x, y)\} \right). \]

We have the following theorem. It should be noted that and alphabets \(X\) and \(Y\) may not be binary and the correlation of the two sources may not be symmetric.

**Theorem 3:** Assume that \((A, p_A), (B, p_B),\) and \((A \times B, p_A \times p_B)\) have hash property. Let \((X, Y)\) be a pair of stationary memoryless sources. If \((R_X, R_Y)\) satisfies

\[ R_X > H(X|Y) \]
\[ R_Y > H(Y|X) \]
\[ R_X + R_Y > H(X, Y), \]

then for any \(\delta > 0\) and all sufficiently large \(n\) there are functions (sparse matrices) \(A \in A\) and \(B \in B\) such that

\[ \text{Error}_{XY}(A, B) \leq \delta. \]
Fig. 7. Channel Coding

Remark 2: In [3][5], random (linear) bin-coding is used to prove the achievability of the above theorem. In fact, random bin-coding is equivalent to a uniform ensemble on a set of all (linear) functions and it has a $(1, 0)$-hash property.

Remark 3: The above theorem includes the fixed-rate coding of a single source $X$ as a special case of the Slepian-Wolf problem with $|Y| \equiv 1$. This implies that the encoding rate can achieve the entropy of a source. It should be noted that source coding using a class of hash functions is studied in [22, Section 14.2][17].

Remark 4: Assuming $|Y| \equiv 1$, we can prove the coding theorem for a channel with additive noise $X$ by letting $A$ and $\{x : Ax = 0\}$ be a parity check matrix and a set of codewords (channel inputs), respectively. This implies that the encoding rate of the channel can achieve the channel capacity. The coding theorem for a channel with additive noise is proved by using a low density parity check (LDPC) matrix in [26][2][10].

B. Gel’fand-Pinsker Problem

In this section we consider the Gel’fand-Pinsker problem illustrated in Fig. 2.

First, we construct a code for the standard channel coding problem illustrated in Fig. 7, which is a special case of Gel’fand-Pinsker problem.

A channel is given by the conditional probability distribution $\mu_{Y|X}$, where $X$ and $Y$ are random variables corresponding to the channel input and channel output, respectively. The capacity of a channel is given by

$$\text{Capacity} \equiv \max_{\mu_X} I(X; Y),$$

where the maximum is taken over all probability distributions $\mu_X$ and the joint distribution of random variable $(X, Y)$ is given by

$$\mu_{XY}(x, y) \equiv \mu_{Y|X}(y|x)\mu_X(x).$$

The code for this problem is given below (illustrated in Fig. 7). We fix functions

$$A : \mathcal{X}^n \to \mathcal{X}^{l_A}$$

$$B : \mathcal{X}^n \to \mathcal{X}^{l_B}$$

and a vector $c \in \mathcal{X}^{l_A}$ available to construct an encoder and a decoder, where

$$l_A \equiv \frac{n[H(X|Y) + \varepsilon_A]}{\log |\mathcal{X}|}$$

$$l_B \equiv \frac{n[I(X; Y) - \varepsilon_B]}{\log |\mathcal{X}|}.$$ 

We define the encoder and the decoder

$$\varphi : \mathcal{X}^{l_B} \to \mathcal{X}^n.$$
Fig. 8. Construction of Channel Code

\[ \varphi^{-1} : \mathcal{Y}^n \to \mathcal{X}^n \]

as

\[ \varphi(m) \equiv g_{AB}(c, m) \]
\[ \varphi^{-1}(m) \equiv B g_A(c | y), \]

where

\[ g_{AB}(c, m) \equiv \arg \max_{x' \in \mathcal{C}_{AB}(c, m)} \mu_X(x') \]
\[ g_A(c | y) \equiv \arg \max_{x' \in \mathcal{C}_A(c)} \mu_{XY}(x' | y). \]

Let \( M \) be the random variable corresponding to the message \( m \), where the probability \( p_M(m) \) is given by

\[ p_M(m) \equiv \begin{cases} 1, & \text{if } m \in \text{Im} \mathcal{B} \\ 0, & \text{if } m \notin \text{Im} \mathcal{B}. \end{cases} \]

The rate \( R(B) \) of this code is given by

\[ R(B) \equiv \frac{\log |\text{Im} \mathcal{B}|}{n} = \frac{l_B \log |\mathcal{W}|}{n} - \frac{\log |\mathcal{W}|}{n} \]

and the decoding error probability \( \text{Error}_{Y|X}(A, B, c) \) is given by

\[ \text{Error}_{Y|X}(A, B, c) \equiv \sum_{m, y} p_M(m) \mu_{Y|X}(y | \varphi(m)) \chi(\varphi^{-1}(y) \neq m). \]

In the following, we provide an intuitive interpretation of the construction of the code, which is illustrated in Fig. 8. Assume that \( c \) is shared by the encoder and the decoder. For \( c \) and a message \( m \), the function \( g_{AB} \) generates a typical sequence \( x \in \mathcal{T}_{X, \gamma} \) as a channel input. The decoder reproduces the channel input \( x \) by using \( g_A \) from \( c \) and a channel output \( y \). Since \( (x, y) \) is jointly typical and \( Bx = m \), the decoding succeed if
the amount of information of \( c \) is greater than \( H(X|Y) \) to satisfy the collision-resistant property. On the other hand, the total rate of \( c \) and \( m \) should be less than \( H(X) \) to satisfy the saturating property. Then we can set the encoding rate of \( m \) close to \( H(X) - H(X|Y) = I(X;Y) \).

We have the following theorem: It should be noted that alphabets \( X \) and \( Y \) are allowed to be non-binary, and the channel is allowed to be asymmetric.

**Theorem 4:** For given \( \varepsilon_A, \varepsilon_B > 0 \) satisfying
\[
\varepsilon_B - \varepsilon_A \leq \sqrt{6[\varepsilon_B - \varepsilon_A] \log |X|} < \varepsilon_A,
\]
assume that \((A, p_A)\) and \((A \times B, p_A \times p_B)\) have hash property. Let \( \mu_{Y|X} \) be the conditional probability distribution of a stationary memoryless channel. Then, for all \( \delta > 0 \) and sufficiently large \( n \) there are functions (sparse matrices) \( A \in \mathcal{A}, B \in \mathcal{B} \), and a vector \( e \in \text{Im}A \) such that
\[
R(B) \geq I(X;Y) - \varepsilon_B - \delta
\]
\[
\text{Error}_{Y|X}(A, B, e) < \delta.
\]
By assuming that \( \mu_X \) attains the channel capacity and \( \delta \to 0, \varepsilon_B \to 0 \), the rate of the proposed code is close to the capacity.

Next, we consider the Gel’fand-Pinsker problem illustrated in Fig. 2. A channel with side information is given by the conditional probability distribution \( \mu_{Y|XZ} \), where \( X, Y \) and \( Z \) are random variables corresponding to the channel input, channel output, and channel side information, respectively. The capacity of a channel with side information is given by
\[
\text{Capacity} = \max_{\mu_{XW|Z}} [I(W;Y) - I(W;Z)],
\]
where the maximum is taken over all conditional probability distributions \( \mu_{XW|Z} \) and the joint distribution of random variable \((X,Y,Z,W)\) is given by
\[
\mu_{XZW}(x, y, z, w) \equiv \mu_{XW|Z}(x, w|z)\mu_{Y|XZ}(y|x, z)\mu_{Z}(z).
\]

In the following, we assume that \( \mu_{XW|Z} \) is fixed. We fix functions
\[
A : \mathcal{W}^n \to \mathcal{W}^{l_A}
\]
\[
B : \mathcal{W}^n \to \mathcal{W}^{l_B}
\]
\[
\hat{A} : \mathcal{X}^n \to \mathcal{X}^{l_b_A}
\]
and vectors \( e \in \mathcal{W}^{l_A} \) and \( \hat{e} \in \mathcal{X}^{l_b_A} \) available to construct an encoder and a decoder, where
\[
l_A \equiv \frac{n[H(W|Y) + \varepsilon_A]}{\log |\mathcal{W}|}
\]
\[
l_B \equiv \frac{n[H(W|Z) - H(W|Y) - \varepsilon_B]}{\log |\mathcal{W}|}
\]
\[
= \frac{n[I(W;Y) - I(W;Z) - \varepsilon_B]}{\log |\mathcal{W}|}
\]
\[
l_{\hat{A}} \equiv \frac{n[H(X|Z,W) - \varepsilon_{\hat{A}}]}{\log |\mathcal{X}|}.
\]
We define the encoder and the decoder
\[
\varphi : \mathcal{W}^{nB} \times Z^n \to X^n
\]
\[
\varphi^{-1} : Y^n \to \mathcal{W}^{nB}
\]
as
\[
\varphi(m | z) \equiv g_{\tilde{A}}(\tilde{c} | z, g_{AB}(c, m | z))
\]
\[
\varphi^{-1}(m) \equiv Bg_A(c | y),
\]
where
\[
g_{AB}(c, m | z) \equiv \max_{w' \in \mathcal{C}_{AB}(c, m)} \mu_{W | Z}(w' | z)
\]
\[
g_{\tilde{A}}(\tilde{c} | z, w) \equiv \max_{w' \in \mathcal{C}_{\tilde{A}}(\tilde{c})} \mu_{X | ZW}(x | z, w)
\]
\[
g_A(c | y) \equiv \max_{w' \in \mathcal{C}_A(c)} \mu_{W | Y}(w' | y).
\]

Let $M$ be the random variable corresponding to the message $m$, where the probability $p_M(m)$ and $p_{MZ}(m, z)$ are given by
\[
p_M(m) \equiv \begin{cases} 1 & \text{if } m \in \text{Im}\mathcal{B} \\ 0 & \text{if } m \notin \text{Im}\mathcal{B} \end{cases}
\]
\[
p_{MZ}(m, z) \equiv p_M(m) \mu_Z(z).
\]
The rate $R(B)$ of this code is given by
\[
R(B) \equiv \frac{\log |\text{Im}\mathcal{B}|}{n}
\]
and the decoding error probability $\text{Error}_{Y|XZ}(A, B, \hat{A}, c, \hat{c})$ is given by

$$\text{Error}_{Y|XZ}(A, B, \hat{A}, c, \hat{c}) = \sum_{m, z, \varphi} p_M(m) \mu_Z(z) \mu_{Y|XZ}(y|\varphi(m, z), z) \chi(\varphi^{-1}(y) \neq m).$$

In the following, we provide an intuitive interpretation of the construction of the code, which is illustrated in Fig. 9. Assume that $c$ is shared by the encoder and the decoder. For $c$, a message $m$, and a side information $z$, the function $g_{AB}$ generates a typical sequence $w \in T_{Z, \gamma}(z)$ and the function $g_{\hat{A}}$ generates a typical sequence $x \in T_{X|WZ, \gamma}(w, z)$ as a channel input. The decoder reproduces the channel input $w$ by using $g_A$ from $c$ and a channel output $y$. Since $(w, y)$ is jointly typical and $Bw = m$, the decoding succeed if the rate of $c$ is greater than $H(W|Y)$ to satisfy the collision-resistant property. On the other hand, the rate of $\hat{c}$ should be less than $H(X|Z, W)$ and the total rate of $c$ and $m$ should be less than $H(W|Z)$ to satisfy the saturating property. Then we can set the encoding rate of $m$ close to $H(W|Z) - H(W|Y) = I(W; Y) - I(W; Z)$.

We have the following theorem. It should be noted that alphabets $X$, $Y$, $W$, and $Z$ are allowed to be non-binary, and the channel is allowed to be asymmetric.

**Theorem 5:** For given $\varepsilon_A, \varepsilon_B, \varepsilon_\hat{A} > 0$ satisfying

$$\varepsilon_B - \varepsilon_\hat{A} \leq \sqrt{6|\varepsilon_B - \varepsilon_A| \log |Z| |W| < \varepsilon_A}$$

$$2\varepsilon_{Y|W}(6\varepsilon_\hat{A}) < \varepsilon_A,$$  \hspace{1cm} (18)

assume that $(A, p_A)$, $(\hat{A} \times B, p_A \times p_B)$ and $(\hat{A}, p_{\hat{A}})$ have hash property. Let $\mu_{Y|XZ}$ be the conditional probability distribution of a stationary memoryless channel. Then, for all $\delta > 0$ and sufficiently large $n$ there are functions (sparse matrices) $A \in A$, $B \in B$, $\hat{A} \in \hat{A}$, and vectors $c \in \text{Im}A$, $\hat{c} \in \text{Im}\hat{A}$ such that

$$R(B) \geq I(W; Y) - I(W; Z) - \varepsilon_B - \delta$$

$$\text{Error}_{Y|XZ}(A, B, \hat{A}, c, \hat{c}) < \delta.$$  \hspace{1cm} (19)

By assuming that $\mu_{X|W|Z}$ attains the Gel’fand-Pinsker bound, and $\delta \to 0$, $\varepsilon_B \to 0$, the rate of the proposed code is close to this bound.

The proof is given in Section VI-E. It should be noted that Theorem 4 is a special case of Gel’fand-Pinsker problem with $|Z| \equiv 1$ and $W \equiv X$.

**Remark 5:** In [24], the code for the Gel’fand-Pinsker problem is proposed by using a combination of two sparse matrices when all the alphabets are binary and the channel side information and noise are additive. In their constructed encoder, they obtain a vector called the ‘middle layer’ by using one of two matrices and obtain a channel input by operating another matrix on the middle layer and adding the side information. In our construction, we obtain $w$ by using two matrices $A$, $B$, and $g_{AB}$, where the dimension of $w$ differs from that of the middle layer. We obtain channel input $x$ by using $\hat{A}$ and $g_{\hat{A}}$ instead of adding the side information. It should be noted that our approach is based on the construction of the channel code presented in [33][29], which is also different from the construction presented in [12][2].
C. Wyner-Ziv Problem

In this section we consider the Wyner-Ziv problem introduced in [47] (illustrated in Fig. 3).

First, we construct a code for the standard lossy source coding problem illustrated in Fig. 10, which is a special case of the Wyner-Ziv problem.

Let $\rho : \mathcal{X} \times \mathcal{Y} \to [0, \infty)$ be the distortion measure satisfying

$$\rho_{\text{max}} \equiv \max_{x, y} \rho(x, y) < \infty.$$ 

We define $\rho_n(x, y)$ as

$$\rho_n(x, y) \equiv \sum_{i=1}^{n} \rho(x_i, y_i)$$

for each $x \equiv (x_1, \ldots, x_n)$ and $y \equiv (y_1, \ldots, y_n)$. For a probability distribution $\mu_x$, the rate-distortion function $R_X(D)$ is given by

$$R_X(D) = \min_{E_{XY} \in \mathbf{E}_{XY \mid X} \mid D} I(X; Y),$$

where the minimum is taken over all conditional probability distributions $\mu_{Y \mid X}$ and the joint distribution $\mu_{XY}$ of $(X, Y)$ is given by

$$\mu_{XY}(x, y) \equiv \mu_X(x) \mu_{Y \mid X}(y \mid x).$$

The code for this problem is given in the following (illustrated in Fig. 11). We fix functions

$$A : \mathcal{Y}^n \to \mathcal{Y}^A$$
and a vector $c \in \mathcal{Y}^l_A$ available to construct an encoder and a decoder, where

$$l_A \equiv \frac{n[H(Y|X) - \varepsilon_A]}{\log |\mathcal{Y}|}$$

$$l_B \equiv \frac{n[I(Y;X) + \varepsilon_B]}{\log |\mathcal{Y}|}.$$

We define the encoder and the decoder

$$\varphi : \mathcal{X}^n \to \mathcal{Y}^l_B$$

$$\varphi^{-1} : \mathcal{Y}^l_B \to \mathcal{Y}^n$$

as

$$\varphi(x) \equiv Bg_A(c|x)$$

$$\varphi^{-1}(b) \equiv g_{AB}(c,b),$$

where

$$g_A(c|x) \equiv \arg \max_{y' \in \mathcal{C}_A(c)} \mu_{Y|X}(y'|x)$$

$$g_{AB}(c,b) \equiv \arg \max_{y' \in \mathcal{C}_{AB}(c,b)} \mu_{Y}(y').$$

In the following, we provide an intuitive interpretation of the construction of the code, which is illustrated in Fig. 11. Assume that $c$ is shared by the encoder and the decoder. For $c$ and $x$, the function $g_A$ generates $y$ such that $Ay = c$ and $(x, y)$ is a jointly typical sequence. The rate of $c$ should be less than $H(Y|X)$ to satisfy the saturation property. Then the encoder obtains the codeword $By$. The decoder obtains the reproduction $y$ by using $g_{AB}$ from $c$ and the codeword $By$ if the rate of $c$ and $By$ is greater than $H(Y)$ to satisfy the collision-resistant property. Then we can set the encoding rate close to $H(Y) - H(Y|X) = I(X;Y)$. Since $(x, y)$ is jointly typical, $\rho_n(x, y)$ is close to the distortion criterion.

We have the following theorem. It should be noted that a source is allowed to be non-binary and unbiased and the distortion measure $\rho$ is arbitrary.

**Theorem 6:** For given $\varepsilon_A, \varepsilon_B > 0$ satisfying

$$\varepsilon_A + 2\zeta_Y(3\varepsilon_A) < \varepsilon_B,$$

assume that $(A, p_A)$ and $(B, p_B)$ have hash property. Let $X$ be a stationary memoryless source. Then for all sufficiently large $n$ there are functions (sparse matrices) $A \in \mathcal{A}$, $B \in \mathcal{B}$, and a vector $c \in \text{Im} A$ such that

$$R(B) = I(X;Y) + \varepsilon_B$$

$$\frac{E_X [\rho_n(X^n, \varphi^{-1}(\varphi(X^n)))]}{n} \leq E_{XY} [\rho(X,Y)] + 3|\mathcal{X}||\mathcal{Y}| \mu_{\text{max}} \sqrt{\varepsilon_A}.$$
Next, we consider the Wyner-Ziv problem introduced in [47] (illustrated in Fig. 3). Let \( \rho : \mathcal{X} \times \mathcal{W} \rightarrow [0, \infty) \) be the distortion measure satisfying

\[
\rho_{\text{max}} \equiv \max_{x, w} \rho(x, w) < \infty.
\]

We define \( \rho_n(x, w) \) as

\[
\rho_n(x, w) \equiv \sum_{i=1}^{n} \rho(x_i, w_i)
\]

for each \( x \equiv (x_1, \ldots, x_n) \) and \( w \equiv (w_1, \ldots, w_n) \). For a probability distribution \( \mu_{XZ} \), the rate-distortion function \( R_{X|Z}(D) \) is given by

\[
R_{X|Z}(D) = \min_{\mu_{Y|X}, f} \left[ I(X; Y) - I(Y; Z) \right],
\]

where the minimum is taken over all conditional probability distributions \( \mu_{Y|X} \) and functions \( f : \mathcal{Y} \times \mathcal{Z} \rightarrow \mathcal{W} \) and the joint distribution \( \mu_{XYZ} \) of \( (X, Y, Z) \) is given by

\[
\mu_{XYZ}(x, y, z) \equiv \mu_{XZ}(x, z) \mu_{Y|X}(y|x).
\]

In the following, we assume that \( \mu_{Y|X} \) is fixed. We fix functions

\[
A : \mathcal{Y}^n \rightarrow \mathcal{Y}^{l_A}
\]

\[
B : \mathcal{Y}^n \rightarrow \mathcal{Y}^{l_B}
\]

and a vector \( c \in \mathcal{Y}^{l_A} \) available to construct an encoder and a decoder, where

\[
l_A \equiv \frac{n[H(Y|X) - \varepsilon_A]}{\log |\mathcal{Y}|}
\]

\[
l_B \equiv \frac{n[H(Y|Z) - H(Y|X) + \varepsilon_B]}{\log |\mathcal{Y}|}
\]

\[
= \frac{n[I(X; Y) - I(Y; Z) + \varepsilon_B]}{\log |\mathcal{Y}|}.
\]
We define the encoders and the decoder (illustrated in Fig. 12)

\[ \varphi : X^n \rightarrow Y^n \]

\[ \varphi^{-1} : Y^n \times Z^n \rightarrow W^n \]

as

\[ \varphi(x) \equiv Bg_A(c|x) \]

\[ \varphi^{-1}(b|z) \equiv f_n(g_{AB}(c, b|z), z) \]

where

\[ g_A(c|x) \equiv \arg \max_{y' \in C^A(c)} \mu_{Y|X}(y'|x) \]

\[ g_{AB}(c, b|z) \equiv \arg \max_{y' \in C^{AB}(c, b)} \mu_{Y|Z}(y'|z) \]

and we define \( f_n(y, z) \equiv (w_1, \ldots, w_n) \) by

\[ w_i \equiv f(y_i, z_i) \]

for each \( y \equiv (y_1, \ldots, y_n) \) and \( z \equiv (z_1, \ldots, z_n) \).

The rate \( R(B) \) of this code is given by

\[ R(B) \equiv \frac{l_B \log |Y|}{n}. \]

In the following, we provide an intuitive interpretation of the construction of the code, which is illustrated in Fig. 12. Assume that \( c \) is shared by the encoder and the decoder. For \( c \) and \( x \), the function \( g_A \) generates \( y \) such that \( Ay = c \) and \( (x, y) \) is a jointly typical sequence. The rate of \( c \) should be less than \( H(Y|X) \) to satisfy the saturation property. Then the encoder obtains the codeword \( By \). The decoder obtains the reproduction \( y \) by using \( g_{AB} \) from \( c \), the codeword \( By \), and the side information \( z \) if the rate of \( c \) and \( By \) is greater than \( H(Y|Z) \) to satisfy the collision-resistant property. Then we can set the encoding rate close to \( H(Y|Z) - H(Y|X) = I(X; Y) - I(Y; Z) \). Since \( (x, y, z) \) is jointly typical, \( \rho_n(x, f(y, z)) \) is close to the distortion criterion.

We have the following theorem. It should be noted that a source is allowed to be non-binary and unbiased, side information is allowed to be asymmetric, and the distortion measure \( \rho \) is arbitrary.

**Theorem 7:** For given \( \varepsilon_A, \varepsilon_B > 0 \) satisfying

\[ \varepsilon_A + 2\varepsilon_A 3\varepsilon_A < \varepsilon_B, \]  

assume that \((A, p_A)\) and \((B, p_B)\) have hash property. Let \((X, Z)\) be a pair of stationary memoryless sources. Then for all sufficiently large \( n \) there are functions (sparse matrices) \( A \in A, B \in B \), and a vector \( c \in \text{Im}A \) such that

\[ R(B) = I(X; Y) - I(Y; Z) + \varepsilon_B \]

\[ \frac{E_{XZ} [\rho_n(X^n, \varphi^{-1}(\varphi(X^n), Z^n))]}{n} \leq E_{XYZ} [\rho(X, f(Y, Z))] + 3|X||Y||Z|\rho_{\text{max}} \sqrt{\varepsilon_A}. \]

By assuming that \( \mu_{Y|X} \) and \( f \) attain the Wyner-Ziv bound and letting \( \varepsilon_A, \varepsilon_B \rightarrow 0 \), the rate-distortion pair of the proposed code is close to this bound.
The proof is given in Section VI-F. It should be noted that Theorem 6 is a special case of the Wyner-Ziv problem with \( |Z| = 1, W \equiv Y \), and \( f(y, z) \equiv y \).

Remark 6: In [25][36][23][27][14], the lossy source code is proposed using sparse matrices for the binary alphabet and Hamming distance. In their constructed encoder proposed in [36][23][27][14], they obtain a codeword vector called the ‘middle layer’ (see [23]) by using a matrix. In their constructed decoder, they operate another matrix on the codeword vector. In our construction of the decoder, we obtain the reproduction \( y \) by using a sparse matrix \( A \) and \( g_A \) and compress \( y \) with another matrix \( B \). It should be noted that the dimension of \( y \) is different from that of the middle layer and we need ML decoder \( g_{AB} \) in the construction of the decoder, because \( y \) is compressed by using \( B \). In [24], the code for the Wyner-Ziv problem is proposed and there are similar differences. Our approach is based on the code presented in [28][30] and similar to the code presented in [42][43][49].

D. One-helps-one Problem

In this section, we consider the One-helps-one problem illustrated in Fig. 4. The achievable rate region for this problem is given by a set of encoding rate pair \((R_X, R_Y)\) satisfying

\[
R_X \geq H(X|Z)
\]

\[
R_Y \geq I(Y; Z),
\]

where the joint distribution \( \mu_{XYZ} \) is given by

\[
\mu_{XYZ}(x, y, z) = \mu_{XY}(x, y)\mu_{Z|Y}(z|y).
\]  

(22)

In the following, we construct a code by combining a Slepian-Wolf code and a lossy source code. We assume that \( \mu_{Z|Y} \) is fixed. We fix functions

\[
\hat{B} : \mathcal{X}^n \rightarrow \mathcal{X}^l \beta
\]
A : \mathbb{Z}^n \rightarrow \mathbb{Z}^{l_A}
B : \mathbb{Z}^n \rightarrow \mathbb{Z}^{l_B}

and a vector \( c \in \mathbb{Z}^{l_A} \) available to construct an encoder and a decoder, where

\[
l_B = \frac{n[H(X|Z) + \varepsilon_B]}{\log |X|}
\]

\[
l_A = \frac{n[H(Z|Y) - \varepsilon_A]}{\log |Z|}
\]

\[
l_B = \frac{n[I(Y;Z) + \varepsilon_B]}{\log |Z|}.
\]

We define the encoders and the decoder (illustrated in Fig. 13)

\[
\varphi_X : \mathcal{X}^n \rightarrow \mathcal{X}^{l_B}
\]

\[
\varphi_Y : \mathcal{Y}^n \rightarrow \mathcal{Z}^{l_B}
\]

\[
\varphi^{-1} : \mathcal{X}^{l_B} \times \mathcal{Z}^{l_B} \rightarrow \mathcal{X}^n
\]

as

\[
\varphi_X(x) \equiv \hat{B}x
\]

\[
\varphi_Y(y) \equiv Bg_A(c|y)
\]

\[
\varphi^{-1}(b_X, b_Y) \equiv g_B(b_X, g_{AB}(c, b_Y)),
\]

where

\[
g_A(c|y) \equiv \arg \max_{z' \in C_A(c)} \mu_{Z|Y}(z'|y)
\]

\[
g_{AB}(c, b_Y) \equiv \arg \max_{z' \in C_{AB}(c, b_Y)} \mu_Z(z')
\]

\[
g_B(b_X|z) \equiv \arg \max_{x' \in C_B(b_X)} \mu_{X|Z}(x'|z).
\]

The pair of encoding rates \((R_X, R_Y)\) is given by

\[
R_X \equiv \frac{l_B \log |X|}{n}
\]

\[
R_Y \equiv \frac{l_B \log |Z|}{n}
\]

and the decoding error probability \(\text{Error}_{XY}(A, B, \hat{B}, c)\) is given by

\[
\text{Error}_{XY}(A, B, \hat{B}, c) \equiv \mu_{XY} \left( \{(x, y) : \varphi^{-1}(\varphi_X(x), \varphi_Y(y)) \neq x \} \right).
\]

We have the following theorem.

**Theorem 8:** For given \( \varepsilon_A, \varepsilon_B, \varepsilon_B > 0 \) satisfying

\[
\varepsilon_B > \varepsilon_A + \zeta_Z(3\varepsilon_A)
\]

\[
\varepsilon_B > 2\zeta_X Z(3\varepsilon_A),
\]
assume that \((A, p_A), (B, p_B), \) and \((\hat{B}, p_{\hat{B}})\) have hash property. Let \((X, Y)\) be a pair of stationary memoryless sources. Then, for any \(\delta > 0\) and and all sufficiently large \(n\), there are functions (sparse matrices) \(A \in A, B \in B, \hat{B} \in \hat{B}\), and a vector \(c \in \text{Im}A\) such that

\[
R_X = H(X|Z) + \varepsilon_B
\]

\[
R_Y = I(X; Z) + \varepsilon_B
\]

\[
\text{Error}_{XY}(A, B, \hat{B}, c) \leq \delta.
\]

The proof is given in Section VI-G.

VI. PROOF OF LEMMAS AND THEOREMS

In the proof, we use the method of types, which is given in Appendix. Throughout this section, we assume that the probability distributions of \(p_C, p_{\hat{C}}, p_M\) are uniform and the random variables \(A, B, \hat{A}, \hat{B}, C, \hat{C}\) and \(M\) are mutually independent.

A. Proof of Lemmas 1–6

We prepare the following two lemmas, which come from the fact that \(p_C\) is the uniform distribution on \(\text{Im}A\) and random variables \(A, C\) are mutually independent.

Lemma 9: Let \(p_A\) be the distribution on the set of functions and \(p_C\) be the uniform distribution on \(\text{Im}A\). We assume that a joint distribution \(p_{AC}\) satisfies

\[
p_{AC}(A, c) = p_A(A)p_C(c)
\]

for any \(A\) and \(c \in \text{Im}A\). Then

\[
\sum_c p_C(c)\chi(Au = c) = \frac{1}{|\text{Im}A|} \tag{25}
\]

for any \(A\) and \(u \in \mathcal{U}^n\),

\[
\sum_{A, c} p_{AC}(A, c)\chi(Au = c) = \frac{1}{|\text{Im}A|} \tag{26}
\]

for any \(u \in \mathcal{U}^n\), and

\[
\sum_{A, c} p_{AC}(A, c)|G \cap C_A(c)| = \sum_{u \in G} \sum_{A, c} p_{AC}(A, c)\chi(Au = c) = \frac{|G|}{|\text{Im}A|} \tag{27}
\]

\[
p_{AC}((A, c) : G \cap C_A(c) \neq \emptyset) \leq \frac{|G|}{|\text{Im}A|} \tag{28}
\]

for any \(G \subset \mathcal{U}^n\).

Proof: First, we prove (25). Since \(Au\) is determined uniquely, we have

\[
\sum_c \chi(Au = c) = 1.
\]

Then we have

\[
\sum_c p_C(c)\chi(Au = c) = \sum_c \frac{\chi(Au = c)}{|\text{Im}A|}
\]
Next, we prove (26). From (25), we have
\[
\sum_{A, c} p_{AC}(A, c) \chi(Au = c) = \sum_{A} p_{A}(A) \sum_{c} p_{c}(c) \chi(Au = c) = \sum_{A} \frac{p_{A}(A)}{|\text{Im} A|} = \frac{1}{|\text{Im} A|}.
\]

Next, we prove (27). From (26), we have
\[
\sum_{A, c} p_{AC}(A, c) |G \cap C_A(c)| = \sum_{A, c} p_{AC}(A, c) \sum_{u \in G} \chi(Au = c) = \sum_{u \in G} \sum_{A, c} p_{AC}(A, c) \chi(Au = c) = \sum_{u \in G} \frac{1}{|\text{Im} A|} = \frac{|G|}{|\text{Im} A|}.
\]

Finally, we prove (28). From (27), we have
\[
p_{AC}((A, c) : G \cap C_A(c) \neq \emptyset) = p_{AC}([(A, c) : \exists u \in G \cap C_A(c)]) \leq \sum_{u \in G} p_{AC}([(A, c) : u \in C_A(c)]) = \sum_{u \in G} \sum_{A, c} p_{AC}(A, c) \chi(Au = c) = \frac{|G|}{|\text{Im} A|}.
\]

**Proof of Lemma 1:** Since $(A, p_A)$ has an $(\alpha_A, \beta_A)$-hash property, we have
\[
p_A([(A : [G \setminus \{u\}] \cap C_A(Au) \neq \emptyset) \leq \sum_{u' \in G \setminus \{u\}} p_A([(A : Au = Au') \leq |\{u\} \cap [G \setminus \{u\}]| + \frac{|G \setminus \{u\}| \alpha_A}{|\text{Im} A|} + \min\{|\{u\}|, |G \setminus \{u\}|\} \beta_A \leq \frac{|G| \alpha_A}{|\text{Im} A|} + \beta_A.
\]

**Proof of Lemma 2:** First, since $(A, p_A)$ has an $(\alpha_A, \beta_A)$-hash property, we have
\[
\sum_{u, u' \in T} \sum_{A, c} p_{AC}(A, c) \chi(Au = c) \chi(Au' = c) = \sum_{u, u' \in T} \sum_{A, c} p_{AC}(A, c) \chi(Au = c) \chi(Au = Au') = \sum_{u, u' \in T} p_A(A) \chi(Au = Au') \sum_{c} p_{c}(c) \chi(Au = c) = \frac{1}{|\text{Im} A|} \sum_{u, u' \in T} \sum_{A} p_A(A) \chi(Au = Au').
\]
where the third equality comes from (25).

Next, we have

\[
\sum_{A,c} p_{AC}(A,c) \left[ \sum_{u \in T} \chi(Au = c) - \frac{|T|}{|\text{Im}A|} \right]^2
\]

= \sum_{A,c} p_{AC}(A,c) \left[ \sum_{u \in T} \chi(Au = c) \right]^2 - 2 \frac{|T|}{|\text{Im}A|} \sum_{A,c} p_{AC}(A,c) \sum_{u \in T} \chi(Au = c) + \frac{|T|^2}{|\text{Im}A|^2}

= \sum_{u,u' \in T} \sum_{A,c} p_{AC}(A,c) \chi(Au = c) \chi(Au' = c) - 2 \frac{|T|}{|\text{Im}A|} \sum_{u \in T} \sum_{A,c} p_{AC}(A,c) \chi(Au = c) + \frac{|T|^2}{|\text{Im}A|^2}

\leq \frac{|T|^2 [\alpha_A - 1]}{|\text{Im}A|^2} + \frac{|T| |\beta_A + 1|}{|\text{Im}A|},

(30)

where the last inequality comes from (27) and (29).

Finally, from the fact that \( T \neq \emptyset \), we have

\[
p_{AC} \left( \{(A,c) : \forall u \in T, Au \neq c \} \right) = p_{AC} \left( \{(A,c) : T \cap C_A(c) = \emptyset \} \right)
\]

= \sum_{A,c} p_{AC}(A,c) \left[ \sum_{u \in T} \chi(Au = c) - \frac{|T|}{|\text{Im}A|} \right] \left[ \sum_{u \in T} \chi(Au = c) - \frac{|T|}{|\text{Im}A|} \right]

\leq \sum_{A,c} p_{AC}(A,c) \left[ \sum_{u \in T} \chi(Au = c) - \frac{|T|}{|\text{Im}A|} \right]^2

\leq \frac{|T|^2 [\alpha_A - 1]}{|\text{Im}A|^2} + \frac{|T| |\beta_A + 1|}{|\text{Im}A|}

= \alpha_A - 1 + \frac{|\text{Im}A| |\beta_A + 1|}{|T|},

where the second inequality comes from the Markov inequality and the third inequality comes from (30). \(\blacksquare\)

**Proof of Lemma 3:** Since \((A,p_A)\) has an \((\alpha_A, \beta_A)\)-hash property, we have

\[
p_{AC} \left( \left\{(A,c) : G \cap C_A(c) \neq \emptyset \right\} \right) = p_{AC} \left( \left\{(A,c) : \forall u \in C_A(c) \right\} \right)
\]

= \sum_A p_A(A) \chi(G \cap C_A(Au) \neq \emptyset) \sum_c p_C(c) \chi(Au = c)

\leq \frac{|G| \alpha_A}{|\text{Im}A|^2} + \frac{\beta_A}{|\text{Im}A|},

where the second equality comes from the fact that random variables A and C are independent, the third equality comes from (25), and the inequality comes from Lemma 1. \(\blacksquare\)
Proof of Lemma 4: By applying Lemma 1 to the set \([G \setminus \{u_{A,c}\}] \cap C_A(c)\), we have

\[
p_{ABC} \left(\{(A, B, c) \mid [G \setminus \{u_{A,c}\}] \cap C_{AB}(e, Bu_{A,c}) \neq \emptyset\}\right) \leq \sum_{A, c} p_{AC}(A, c) \left[\frac{|[G \setminus \{u_{A,c}\}] \cap C_A(c)| \alpha_B + \beta_B}{|\text{Im} B|}\right]
\]

\[
\leq \sum_{A, c} p_{AC}(A, c) \left[\frac{|G \cap C_A(c)| \alpha_B + \beta_B}{|\text{Im} B|}\right]
\]

\[
= \frac{|G| \alpha_B}{|\text{Im} A| |\text{Im} B|} + \beta_B,
\]

where the last equality comes from (27).

Proof of Lemma 5: Let

\[
G(v) \equiv \{u : \mu_{U|V}(u|v) > 2^{-n[H(U|V) - 2\varepsilon]}\}
\]

If \(G(v) \cap C_A(c) = \emptyset\) and \(T(v) \cap C_A(c) \neq \emptyset\), then there is \(u \in T(v) \cap C_A(c)\) and \(g_A(c|v)\) satisfies

\[
\mu_{U|V}(g_A(c|v)|v) \leq 2^{-n[H(U|V) - 2\varepsilon]}.
\]

Since \(u \in C_A(c)\), we have

\[
\mu_{U|V}(g_A(c|v)|v) \geq \mu_{U|V}(u|v).
\]

This implies that \(g_A(c|v) \in T(v)\) from the assumption of \(T(v)\). From Lemma 2 and (28), we have

\[
p_{AC} \left(\{(A, c) : g_A(c|v) \notin T(v)\}\right) \leq 1 - p_{AC} \left(\{(A, c) : g_A(c|v) \in T(v)\}\right)
\]

\[
\leq 1 - p_{AC} \left(\left\{(A, c) : \frac{T(v) \cap C_A(c) = \emptyset}{G(v) \cap C_A(c) = \emptyset}\right\}\right)
\]

\[
\leq p_{AC} \left(\{(A, c) : T(v) \cap C_A(c) = \emptyset\}\right) + p_{AC} \left(\{(A, c) : G(v) \cap C_A(c) \neq \emptyset\}\right)
\]

\[
\leq \alpha_A - 1 + \frac{|\text{Im} A| [\beta_A + 1]}{|T(v)|} + \frac{|G(v)|}{|\text{Im} A|}
\]

\[
\leq \alpha_A - 1 + \frac{|\text{Im} A| [\beta_A + 1]}{|T(v)|} + \frac{2^{-n \varepsilon} |\text{Im} A|}{|\text{Im} A|},
\]

where the last inequality comes from the fact that

\[
|G(v)| \leq 2^n |H(U|V) - 2\varepsilon|.
\]

Proof of Lemma 6: When \(T(v) \cap C_A(c) \neq \emptyset\), we can always find the member of \(T(v)\) by using \(\hat{g}_A\). From Lemma 2, we have

\[
p_{AC} \left(\{(A, c) : \hat{g}_A(c|v) \notin T(v)\}\right) \leq p_{AC} \left(\{(A, c) : T(v) \cap C_A(c) = \emptyset\}\right)
\]

\[
\leq \alpha_A - 1 + \frac{|\text{Im} A| [\beta_A + 1]}{|T(v)|}.
\]
B. Proof of Theorem 1 and Lemmas 7 and 8

For a type \( t \), let \( C_t \) be defined as
\[
C_t \equiv \{ u \in U^n : t(u) = t \}.
\]

We assume that \( p_A(\{ A : Au = 0 \}) \) depends on \( u \) only through the type \( t(u) \). For a given \( u \in C_t \), we define
\[
u_{A,t} \equiv u_A(\{ A : Au = 0 \}),
\]
\[
p_{A,t} \equiv p_A(\{ A : Au = 0 \}),
\]
where \( u_A \) denotes the uniform distribution on the set of all \( l_A \times n \) matrices and we omit \( u \) from the left hand side because the probabilities \( u_A(\{ A : Au = 0 \}) \) and \( p_A(\{ A : Au = 0 \}) \) depend on \( u \in C_t \) only through the type \( t \).

We use the following lemma in the proof.

Lemma 10:
\[
\alpha_A(n) = |\text{Im} A| \max_{t \in H} p_{A,t},
\]
\[
\beta_A(n) = \sum_{t \in H \setminus \{ 0 \}} |C_t| p_{A,t},
\]
where \( H \) is a set of all types of length \( n \) except the type of the zero vector.

Proof: Since we can find \( |U|^{|n-1|l_A} \) matrices \( A \) to satisfy \( Au = 0 \) for \( u \in C_t \), we have
\[
u_{A,t} = \frac{|U|^{|n-1|l_A}}{|U|^{|n|l_A}} = |U|^{-l_A}.
\]

We have
\[
S(p_A, t) = \sum_A p_A(A) \sum_{\substack{u \in C_t \\ A u = 0}} 1
\]
\[
= \sum_{u \in C_t} \sum_{\substack{A : A u = 0}} p_A(A)
\]
\[
= |C_t| p_{A,t}.
\]

Similarly, we have
\[
S(u_{A,t}) = |C_t| u_{A,t}.
\]

The lemma can be shown immediately from (9), (10), and the above equalities. \hfill \Box

Proof of Theorem 1: Without loss of generality, we can assume that \( |T| \leq |T'| \). We have
\[
\sum_{u \in T, \; u' \in T'} p_A(\{ A : Au = Au' \}) = \sum_{u \in T} \sum_{u' \in T'} p_A(\{ A : A[u - u'] = 0 \})
\]
\[
\leq \sum_{u \in T \cap T'} p_A(\{ A : A0 = 0 \}) + \sum_{t \in H} \sum_{\substack{u \in T \\ t(u - u') = t}} p_{A,t}
\]
to the proof of Theorem 1, we have

\[ \sum_{u \in T \cap T'} 1 + \sum_{u \in T} \sum_{u' \in T'} p_{A,t} p_{u',t} + \sum_{t \in \tilde{H}} \sum_{u \in T} |C_t| p_{A,t} \]

\[ \leq |T \cap T'| + \sum_{t \in \tilde{H}} \sum_{u \in T} \frac{\alpha_A(n)}{|\text{Im}A|} + |T| \sum_{t \in \tilde{H}} S(p_{A,t}) \]

\[ \leq |T \cap T'| + \frac{|T||T'|\alpha_A(n)}{|\text{Im}A||\text{Im}B|} + |T| \beta_A(n) \]

\[ = |T \cap T'| + \frac{|T||T'|\alpha_A(n)}{|\text{Im}A||\text{Im}B|} + \min\{|T|,|T'|\} \beta_A(n), \]

where the third inequality comes from (31) and the last equality comes from the assumption $|T| \leq |T'|$. Since $(\alpha_A, \beta_A)$ satisfies (H2) and (H3), we have the fact that $(A, p_A)$ has an $(\alpha_A, \beta_A)$-hash property. 

**Proof of Lemma 7:** Without loss of generality, we can assume that $|T| \leq |T'|$ and $\beta_A(n) \leq \beta_B(n)$. Similar to the proof of Theorem 1, we have

\[ \sum_{u \in T \cap T'} p_{AB} \left( \{ (A, B) : (Au, Bu) = (Au', Bu') \} \right) \]

\[ = \sum_{u \in T \cap T'} p_A \left( \{ A : A0 = 0 \} p_B \left( \{ B : B0 = 0 \} \right) \right) + \sum_{t \in \tilde{H}} \sum_{u \in T} \sum_{u' \in T'} p_{A,t} p_{B,t} \]

\[ \leq |T \cap T'| + \sum_{t \in \tilde{H}} \sum_{u \in T} \frac{\alpha_A(n)\alpha_B(n)}{|\text{Im}A||\text{Im}B|} + |T| \sum_{t \in \tilde{H}} S(p_{A,t}) \]

\[ \leq |T \cap T'| + \frac{|T||T'|\alpha_A(n)\alpha_B(n)}{|\text{Im}A||\text{Im}B|} + |T| \beta_A(n) \]

\[ = |T \cap T'| + \frac{|T||T'|\alpha_A(n)\alpha_B(n)}{|\text{Im}A||\text{Im}B|} + \min\{|T|,|T'|\} \beta_{AB}(n), \]

where the first inequality comes from the fact that $p_{B,t} \leq 1$. Since $(\alpha_{AB}(n), \beta_{AB}(n))$ satisfies (H2) and (H3), $(A \times B, p_{AB})$ has an $(\alpha_{AB}, \beta_{AB})$-hash property. 

**Proof of Lemma 8:** Without loss of generality, we can assume that $|T| \leq |T'|$. Let $\mathcal{H}_U$ and $\mathcal{H}_V$ be defined similarly to the definition of $\mathcal{H}$, and $\tilde{\mathcal{H}}_U$ and $\tilde{\mathcal{H}}_V$ be defined similarly to the definition of $\tilde{\mathcal{H}}$. Similar to the proof of Theorem 1, we have

\[ \sum_{(u,v) \in T \cap T'} p_{AB} \left( \{ (A, B) : (Au, Bu) = (Au', Bu') \} \right) \]

\[ = \sum_{(u,v) \in T \cap T'} p_A \left( \{ A : A0 = 0 \} \right) p_B \left( \{ B : B0 = 0 \} \right) + \sum_{t_U \in \tilde{\mathcal{H}}_U} \sum_{t_V \in \tilde{\mathcal{H}}_V} \sum_{(u,v) \in T} \sum_{(u',v') \in T'} p_{A,t_U} p_{B,t_V} \]
where the first inequality comes from the fact that $E|T \cap T'| \leq |T \cap T'| = |T \cap T'|$.

We consider a random-walk on $\mathbb{C} \times \mathbb{B}$, $p_A$ has an $(\alpha, \beta, \beta')$-hash property.

\section{Proof of Theorem 2}

Throughout this section, let $\mathcal{U} \equiv \text{GF}(q)$, $l \equiv nR$, and $p_A$ be an ensemble of $l \times n$ sparse matrices as specified in Section IV, where we omit dependence on the ensemble of $l$. It should be noted that $l \rightarrow \infty$ by letting $n \rightarrow \infty$.

First, we prepare lemmas that provide the analytic expression of $p_{AT}$. Let $c_n \in \text{GF}(q^l)$ be the position after $n$ steps. At each unit step, the position is renewed in the following rule.

1) Choose $(u, v) \in \{1, \ldots, l\} \times \text{GF}(q)$ uniformly at random.

2) Add $u$ to the $i$-th element of $c_n$.

Then, the probability $P_n(c)$ of the position $c$ after $n$ steps starting from the zero vector is described by

$$P_n(c) = \frac{1}{q^l} \sum_{t=0}^{n} \binom{n}{t} \left(1 - \frac{aq}{q - 1}\right)^t \binom{w(c)}{k'} \binom{l - w(c)}{k - k'} (-1)^{k'} (q - 1)^{k-k'}.$$  \hspace{1cm} (33)

**Proof:** Let $\hat{C} \subset \text{GF}(q^l)$ be defined as

$$\hat{C} \equiv \left\{ \overset{\text{j \in \{1, \ldots, l\}}{0, \ldots, 0, c, 0, \ldots, 0} : \overset{\text{c \in \text{GF}(q)}}{[0]} \right\}.$$  

Then the transition rule of this random walk is equivalent to the following.

1) Choose $\hat{c} \in \hat{C}$ uniformly at random.

2) Add $c$ to $c_n$, that is,

$$c_{n+1} \equiv c_n + \hat{c}.$$  

February 12, 2013
We have the following recursion formula for $P_n(c)$.

\[
P_1(c) = \begin{cases} 
\frac{1}{q-1}, & \text{if } c \in \mathbb{C}, \\
0, & \text{otherwise}.
\end{cases}
\]

\[
P_{n+1}(c) = \sum_{c' \in \text{GF}(q')} P_n(c')P_1(c-c') = [P_n * P_1](c),
\]

where $P_n * P_1$ denotes the convolution. We have (33) by using the following formulas

\[
\mathfrak{F}P_n = [\mathfrak{F}P_{n-1}][\mathfrak{F}P_1] = \cdots = [\mathfrak{F}P_1]^n
\]

\[
P_n = \mathfrak{F}^{-1}P_n = \mathfrak{F}^{-1}[[\mathfrak{F}P_1]^n],
\]

where $\mathfrak{F}$ is the discrete Fourier transform and $\mathfrak{F}^{-1}$ is its inverse.

\[\blacksquare\]

**Lemma 12:** The probability $p_A (\{A : Au = 0\})$ depends on $u$ only through the type $t(u)$, that is, if $w(t) = w(t')$ then $p_{A,t} = p_{A,t'}$. Furthermore,

\[
p_{A,t} = \frac{1}{q'} \sum_{k=0}^{l} \left( 1 - \frac{qk}{q-1} \right)^{w(t)\tau} \binom{\frac{l}{k}}{q-1}^k.
\]

**Proof:** For $u \equiv (u_1, \ldots, u_n)$, we define $u^* \equiv (u_1^*, \ldots, u_n^*)$ as

\[
u_i^* \equiv \begin{cases} 
1, & \text{if } u_i \neq 0 \\
0, & \text{if } u_i = 0.
\end{cases}
\]

Similarly as in the proof of [10, Lemma 1], we can prove that two sets $\{A : Au = 0\}$ and $\{A : Au^* = 0\}$ are in one-to-one correspondence. Then we have

\[
p_A (\{A : Au = 0\}) = p_A (\{A : Au^* = 0\}),
\]

that is, $p_A (\{A : Au = 0\})$ depends on $u$ only through $w(t)$.

Since $p_A (\{A : Au^* = 0\})$ is equal to the probability that the position of the random walk defined in Lemma 11 starts from the zero vector and returns to the zero vector after $w(u^*)\tau$ steps, we have

\[
p_{A,t} = P_{w(t)\tau}(0)
\]

\[
= \frac{1}{q'} \sum_{k=0}^{l} \left( 1 - \frac{qk}{q-1} \right)^{w(t)\tau} \binom{\frac{l}{k}}{q-1}^k.
\]

\[\blacksquare\]

Next, we prove the following lemma.

**Lemma 13:** If the column weight $\tau$ is even, then

\[
\text{Im} A = \begin{cases} 
\{u \in \mathcal{U}^l : w(u) \text{ is even}\}, & \text{if } q = 2 \\
\mathcal{U}^l, & \text{if } q > 2,
\end{cases}
\]

which implies

\[
\frac{|\text{Im} A|}{|\mathcal{U}|} = \begin{cases} 
2, & \text{if } q = 2 \\
1, & \text{if } q > 2.
\end{cases}
\]
Proof: Let $a_{i,j}$ be the $(i,j)$ element of $A$.

First, we assume that $q = 2$. Then it is sufficient to prove that $w(Au)$ is even for any possible $A$ and $u \in U^l$ because

$$
\sum_{w(e) \text{ is even}} 1 - \sum_{w(e) \text{ is odd}} 1 = \sum_{w=0}^{l} \binom{n}{w} (-1)^w
$$

which implies that $|\text{Im} A| = |U|^l / 2$. Without loss of generality, we can assume that $w(u) = w$ and $u = (1, \ldots, 0, \ldots, 0)$. Let $a_i \equiv (a_{i,1}, \ldots, a_{i,w(u)})$. Since every column vector has an even weight, we have the fact that $\sum_{i=1}^{w(u)} w(a_i)$ is even. In addition, we have

$$
\sum_{w(a_i) \text{ is odd}} w(a_i) = \sum_{i=1}^{w(u)} w(a_i) - \sum_{w(a_i) \text{ is even}} w(a_i).
$$

This implies that the number of odd-weight vectors $a_i$ is even because the right hand side of the above equality is even. Since $w(Au)$ is a number of odd-weight vectors $a_i$, we have the fact that $w(Au)$ is even for any $A$ and $u \in U^l$.

Next, we assume that $q > 2$. It is sufficient to prove that, for any $c = (c_1, \ldots, c_l) \in U^l$, there is $A$ generated by the scheme and $u \in U^l$ such that $Au = c$. This fact implies that $\text{Im} A = U^l$. Let $u = (1, \ldots, 1)$. It is possible to generate $A$ satisfying

$$
a_{i,j} = \begin{cases} 
2a, & \text{if } i = j \\
0, & \text{if } i \neq j,
\end{cases}
$$

where $a \in GF(q)$ is arbitrary. Since $q > 3$, we have $Au = c$ by letting $a \equiv c_i / 2$.

Finally, we prove that $(\alpha_A, \beta_A)$ satisfies (H2) and (H3). We define the function $h$ as

$$
h(\theta) \equiv -\theta \log_e(\theta) - [1 - \theta] \log_e(1 - \theta),
$$

where $e$ is the base of the natural logarithm. We use the following lemmas to derive the asymptotic behavior of $(\alpha_A, \beta_A)$.

Lemma 14: Let $a$ be a real number. Then

$$
\max_{0 \leq \theta \leq 1} [h(\theta) + a\theta] \leq \log_e (1 + e^a).
$$

If $a \leq -\log_e(l - 1)$, then

$$
\max_{1/2 \leq \theta \leq 1} [h(\theta) + a\theta] \leq h \left( \frac{1}{7} \right) + \frac{a}{l}.
$$

Lemma 15:

$$
\ln \left( \frac{1}{7} \right) \leq 1 + \log_e l.
$$

Lemma 16:

$$
\sum_{k=1}^{l-1} \left| \frac{1 - 2k}{l} \right|^{wt} \binom{l}{k} \leq 2 \sum_{k=1}^{l} \exp \left( - \frac{2k}{l} \right) \binom{l}{k}
$$

Proof: Since

$$
\left| 1 - \frac{2k}{l} \right|^{wt} \binom{l}{k} = \left| 1 - \frac{2[l - k]}{l} \right|^{wt} \binom{l}{1 - k},
$$
then we have
\[
\sum_{k=1}^{l-1} \left| 1 - \frac{2k}{l} \right|^{w\tau} \binom{l}{k} = 2 \sum_{k=1}^{\lfloor \frac{l}{2} \rfloor} \left| 1 - \frac{2k}{l} \right|^{w\tau} \binom{l}{k} \\
\leq 2 \sum_{k=1}^{\lfloor \frac{l}{2} \rfloor} \exp \left( -\frac{2kw\tau}{l} \right) \binom{l}{k},
\]
where the inequality comes from the fact that \(2k/l \leq 1\).

**Lemma 17:**
\[
\sum_{k=1}^{l} \left| 1 - \frac{qk}{[q-1]l} \right|^{w\tau} \binom{l}{k} [q-1]^k \leq \sum_{k=1}^{\lfloor \frac{[q-1]l}{q} \rfloor} \exp \left( -\frac{qkw\tau}{[q-1]l} \right) \binom{l}{k} [q-1]^k + \sum_{k=\lfloor \frac{[q-1]l}{q} \rfloor}^{l} \binom{l}{k} [q-1]^{k-w\tau}. \tag{34}
\]

**Proof:** We can show the lemma from the fact that
\[
\frac{qk}{[q-1]l} \leq 1
\]
when \(k \leq [q-1]l/q\) and
\[
\left| 1 - \frac{qk}{[q-1]l} \right| = \frac{q(k-l) + l}{[q-1]l} \\
\leq \frac{l}{[q-1]l} \\
= \frac{1}{q-1}
\]
when \([q-1]l/q < k \leq l\).

Let \(\tau\) be the parameter given in the procedure used for generating a sparse matrix. We assume that \(\tau\) and \(\xi\) satisfy
\[
\tau \equiv 2 \left\lceil \log_e \frac{l^2}{R} \right\rceil \tag{35}
\]
\[
\frac{h(\xi R)}{R} + \xi \log_e (q-1) < \frac{1}{3}. \tag{36}
\]

Then we have
\[
\xi \tau \geq 3 \log_e l \tag{37}
\]
for all sufficiently large \(l\).

Now we are in position to prove the following two lemmas which provides the proof of Theorem 2.

**Lemma 18:**
\[
\lim_{n \to \infty} \alpha_A(n) = 1.
\]

**Proof:** In the following, we first show that
\[
\lim_{l \to \infty} \sum_{k=1}^{\lfloor \frac{[q-1]l}{q} \rfloor} \exp \left( -\frac{qkw\tau}{[q-1]l} \right) \binom{l}{k} [q-1]^k = 0 \tag{38}
\]
for all \( q \geq 2 \) and \( w > \xi l \). By assuming \( w > \xi l \), we have

\[
\sum_{k=1}^{\lfloor \frac{|q-1|}{q-1} \rfloor} \exp \left( \frac{-qk\tau}{|q-1|l} \right) \left( \begin{array}{c} l \\ k \end{array} \right) \leq l \max_{1/\theta \leq 1} \exp \left( -w\theta \right) \exp \left( l h(\theta) \right) \left( q-1 \right)^{0} \\
\leq l \max_{1/\theta \leq 1} \exp \left( -\xi l \tau + l h(\theta) + l \log_{e}(q-1) \right) \\
\leq l \max_{1/\theta \leq 1} \exp \left( l \left[ h(\theta) + \log_{e}(q-1) - \xi \tau \right] \right) \\
\leq l \exp \left( l \left[ h \left( \frac{1}{l} \right) + \log_{e}(q-1) - \xi \tau \right] \right) \\
\leq \exp \left( 1 + \log_{e} l + \log_{e}(q-1) - \xi \tau + \log_{e} l \right) \\
\leq \exp \left( -\xi \tau + 2 \log_{e} l + \log_{e}(q-1)e \right),
\]

where the fifth inequality comes from (37) and Lemma 14, and the sixth inequality comes from Lemma 15. Hence we have (38) for all \( q \geq 2 \) and \( w > \xi l \).

Next, we show the lemma by assuming that \( q = 2 \). From Lemma 16, (38), and the fact that \( w \tau \) is even, we have

\[
\lim_{l \to \infty} \max_{w > \xi l} \sum_{k=1}^{l-1} \left[ 1 - \frac{2k}{l} \right]^{w} \left( \begin{array}{c} l \\ k \end{array} \right) = \lim_{l \to \infty} \max_{w > \xi l} \sum_{k=1}^{l-1} \left[ 1 - \frac{2k}{l} \right]^{w} \left( \begin{array}{c} l \\ k \end{array} \right) \\
\leq 2 \lim_{l \to \infty} \max_{w > \xi l} \sum_{k=1}^{\lfloor \frac{l}{2} \rfloor} \exp \left( -\frac{2kw\tau}{l} \right) \left( \begin{array}{c} l \\ k \end{array} \right) \\
= 0.
\]

From (31) and Lemma 13, we have

\[
\lim_{n \to \infty} \alpha_{A}(n) = \lim_{n \to \infty} \frac{1}{2} \max_{w > \xi l} \sum_{k=0}^{l} \left[ 1 - \frac{2k}{l} \right]^{w} \left( \begin{array}{c} l \\ k \end{array} \right) \\
= 1 + \frac{1}{2} \lim_{n \to \infty} \max_{w > \xi l} \sum_{k=1}^{l-1} \left[ 1 - \frac{2k}{l} \right]^{w} \left( \begin{array}{c} l \\ k \end{array} \right) \\
= 1.
\]

Finally, we show the lemma by assuming that \( q > 2 \). From (38), the first term on the right hand side of (34) vanishes by letting \( l \to \infty \). Since \( |q-1|/q \geq 1/2 \), then the second term on the right hand side of (34) is evaluated by

\[
\sum_{k=\lfloor \frac{|q-1|}{q} \rfloor}^{l} \left( \begin{array}{c} l \\ k \end{array} \right) \left[ q-1 \right]^{k-w\tau} \leq l \left( \begin{array}{c} l \\ \left[ (q-1)/q \right] \end{array} \right) \left[ q-1 \right]^{l-w\tau} \\
\leq l \exp \left( lh \left( \frac{q-1}{q} \right) \right) \left[ q-1 \right]^{l-w\tau} \\
\leq l \exp \left( l \log_{e} eq - w\tau \log_{e}(q-1) \right) \\
\leq \exp \left( -l \left[ \xi \tau \log_{e}(q-1) - \log_{e} eq - \log_{e} l \right] \right),
\]
where the third inequality comes from $h(\theta) \leq 1$. From $q > 2$ and (37), the second term on the right hand side of (34) vanishes by letting $l \to \infty$. From the above two observations and the fact that $w\tau$ is even, we have

$$\lim_{l \to \infty} \max_{w > \xi l} \sum_{k=1}^{l} \left[ 1 - \frac{qk}{|q-1|l} \right]^{w\tau} \binom{l}{k} [q - 1]^k = \lim_{l \to \infty} \max_{w > \xi l} \sum_{k=1}^{l} \left[ 1 - \frac{qk}{|q-1|l} \right]^{w\tau} \binom{l}{k} [q - 1]^k$$

$$= 0.$$

From (31) and Lemma 13, we have

$$\lim_{n \to \infty} \alpha_A(n) = \lim_{n \to \infty} \max_{w > \xi l} \sum_{k=0}^{l} \left[ 1 - \frac{qk}{|q-1|l} \right]^{w\tau} \binom{l}{k} [q - 1]^k$$

$$= 1 + \lim_{n \to \infty} \max_{w > \xi l} \sum_{k=1}^{l} \left[ 1 - \frac{qk}{|q-1|l} \right]^{w\tau} \binom{l}{k} [q - 1]^k$$

$$= 1.$$

**Lemma 19:**

$$\lim_{n \to \infty} \beta_A(n) = 0.$$

**Proof:** Let $C_w \equiv \{ x : w(x) = w \}$. Then we have

$$|C_w| = \binom{n}{w} [q - 1]^w \leq \exp \left( n h \left( \frac{w}{n} \right) + w \log_e(q - 1) \right). \quad (39)$$

In the following, we first show that

$$\lim_{l \to \infty} \sum_{w=1}^{\xi l} \frac{|C_w|}{q^l} \sum_{k=0}^{l} \exp \left( - \frac{qk w\tau}{|q-1|l} \right) \binom{l}{k} [q - 1]^k = 0. \quad (41)$$

We have

$$\sum_{w=1}^{\xi l} \frac{|C_w|}{q^l} \sum_{k=0}^{l} \exp \left( - \frac{qk w\tau}{|q-1|l} \right) \binom{l}{k} [q - 1]^k$$

$$= \sum_{w=1}^{\xi l} \frac{|C_w|}{q^l} \left[ 1 + [q - 1] \exp \left( - \frac{qw\tau}{|q-1|l} \right) \right]^l$$

$$= \sum_{w=1}^{\xi l} \frac{|C_w|}{q^l} \left[ 1 + [q - 1] \exp \left( - \frac{qw\tau}{|q-1|l} \right) \right]^l.$$ 

The first term on the right hand side of (42) is evaluated by

$$\sum_{w=1}^{\xi l} \frac{|C_w|}{q^l} \left[ 1 + [q - 1] \exp \left( - \frac{qw\tau}{|q-1|l} \right) \right]^l \leq \sum_{w=1}^{\xi l} \frac{|C_w|}{q^l} \left[ 1 + [q - 1] \left[ 1 - \frac{qw\tau}{2|q-1|l} \right] \right]^l$$

$$= \sum_{w=1}^{\xi l} \frac{|C_w|}{q^l} \left[ 1 + [q - 1] \left[ 1 - \frac{qw\tau}{2l} \right] \right]^l.$$
\[ \frac{1}{q} \left( \log_e \frac{l^2}{R} - \frac{\tau}{2} \right) \leq \frac{q}{4 \log_e l^2}. \]  

The first inequality comes from the fact that \( \exp(-x) \leq 1 - x/2 \) for \( 0 \leq x \leq 1/2 \). The first equality comes from (39). The second inequality comes from the fact that \( 1 + x \leq \exp(lx) \). The third inequality comes from the fact that \( n^w q^w \exp\left(-\frac{w\tau}{2}\right) \) is a non-increasing function of \( w \). The fifth inequality comes from (35).

From (43), the first term on the right hand side of (42) vanishes by letting \( l \to \infty \). The second term of (42) is evaluated by

\[
\xi l \sum_{w=1}^{\lfloor l/2 \rfloor} |C_w| \left[ 1 + \left(\frac{q-1}{q-1}\right) \exp\left(-\frac{q-1}{q-1}\right) \right]^l \leq \xi l \sum_{w=1}^{\lfloor l/2 \rfloor} |C_w| \left[ \frac{1 + [q-1] \exp\left(-\frac{q-1}{q-1}\right)}{q} \right]^l \leq \xi l \left[ \frac{h(\xi R)}{R} + \xi \log_e(q-1) - \frac{1}{3} \right],
\]

where the second inequality comes from the fact that

\[
\frac{1 + [q-1] \exp\left(-\frac{q-1}{q-1}\right)}{q} \leq e^{-\frac{q}{q-1}}
\]

and the third inequality comes from (40). From (36) and (44), the second term on the right hand side of (42) vanishes by letting \( l \to \infty \). From the above two observations, we have (41).

Next, we show the lemma by assuming that \( q = 2 \). From (32), the fact that \( w\tau \) is even, and Lemma 16, we have

\[
\beta_A(n) = \xi l \sum_{w=1}^{\lfloor l/2 \rfloor} |C_w| \sum_{k=0}^{l} \left[ 1 - \frac{2k}{l} \right]^{w\tau} \binom{l}{k} \leq 2 \sum_{w=1}^{\lfloor l/2 \rfloor} |C_w| \sum_{k=0}^{l} \exp\left(-\frac{2kw\tau}{l}\right) \binom{l}{k} \leq 2 \sum_{w=1}^{\lfloor l/2 \rfloor} |C_w| \sum_{k=0}^{l} \exp\left(-\frac{2kw\tau}{l}\right) \binom{l}{k}
\]

From (41), we have the lemma for \( q = 2 \).

Finally, we show the lemma by assuming that \( q > 2 \). From (32), and Lemmas 12 and 17, we have

\[
\beta_A(n) = \xi l \sum_{w=1}^{\lfloor l/2 \rfloor} |C_w| \sum_{k=0}^{l} \left[ 1 - \frac{qk}{[q-1]l} \right]^{w\tau} \binom{l}{k} \binom{[q-1]k}{k].
\]
Since a decoding error occurs when at least one of the conditions (SW1)–(SW4) is satisfied, the error probability is upper bounded by

\[ P_e \leq \sum_{w=1}^{q^l} \sum_{k=0}^{l} \exp \left( - \frac{q^k w \tau}{|q-1|} \right) \binom{l}{k} (q-1)^k + \sum_{w=1}^{q^l} \sum_{k=0}^{l} \exp \left( - \frac{q^k w \tau}{|q-1|} \right) \binom{l}{k} (q-1)^{k-w\tau} \]

From (41), the first term on the right hand side of (45) vanishes by letting \( l \to \infty \). From (40), the second term on the right hand side of (45) is evaluated by

\[ \sum_{w=1}^{q^l} |C_w| (q-1)^{-w\tau} \leq \sum_{w=1}^{q^l} \exp \left( n h \left( \frac{w}{n} \right) + w[1-\tau]\log_e(q-1) \right) \]

\[ \leq \xi l \exp \left( n \max_{1/n \leq \theta \leq 1} [h(\theta) + n[1-\tau]\log_e(q-1)\theta] \right) \]

\[ \leq \xi l \exp \left( n h \left( \frac{1}{n} \right) + [1-\tau]\log_e(q-1) \right) \]

\[ \leq \exp \left( 1 + \log_e n + [1-\tau]\log_e(q-1) + \log_e \xi l, \right) \]

where the third inequality comes from Lemma 14 and the fact that

\[ [1-\tau]\log_e(q-1) < -\log_e(n-1) \]

for all sufficiently large \( n \) and \( q > 2 \). The fourth inequality comes from Lemma 15. From (35), we have

\[ 1 + \log_e n + [1-\tau]\log_e(q-1) + \log_e \xi l \to -\infty \]

by letting \( n \to \infty \). Then the third term on the right hand side of (45) vanishes by letting \( n \to \infty \). Hence we have the lemma for \( q > 2 \).

\[ \Box \]

\section*{D. Proof of Theorem 3}

We define the set \( T \) as

\[ T \equiv \left\{ (x, y) : - \frac{1}{n} \log \mu_{X|Y}(x|y) \leq H(X|Y) + \gamma \right\} \]

It should be noted that the above definition can be replaced by that defined in [33]. This implies that the theorem is valid for general correlated sources.

Let \((x, y)\) be the output of correlated sources. We define

\begin{itemize}
  \item \((x, y) \notin T\) \quad (SW1)
  \item \(\exists x' \neq x \) s.t. \( x' \in C_A(Ax), \mu_{X,Y}(x', y) \geq \mu_{X,Y}(x, y) \) \quad (SW2)
  \item \(\exists y' \neq y \) s.t. \( y' \in C_B(By), \mu_{X,Y}(x, y') \geq \mu_{X,Y}(x, y) \) \quad (SW3)
  \item \(\exists (x', y') \neq (x, y) \) s.t. \( x' \in C_A(Ax), \ y' \in C_B(By), \mu_{X,Y}(x', y') \geq \mu_{X,Y}(x, y) \) \quad (SW4)
\end{itemize}

Since a decoding error occurs when at least one of the conditions (SW1)–(SW4) is satisfied, the error probability is upper bounded by

\[ \text{Error}_{XY}(A, B) \leq \mu_{XY}(E_1^c) + \mu_{XY}(E_1^c \cap E_2) + \mu_{XY}(E_1^c \cap E_3) + \mu_{XY}(E_1^c \cap E_4), \]

(46)
where we define

\[ \mathcal{E}_i \equiv \{(x, y) : (SW_i)\}. \]

First, we evaluate \( E_{AB}[\mu_{XY}(\mathcal{E}_1)] \). From Lemma 26, we have

\[ E_{AB}[\mu_{XY}(\mathcal{E}_1)] \leq \frac{\delta}{4} \tag{47} \]

for all sufficiently large \( n \).

Next, we evaluate \( E_{AB}[\mu_{XY}(\mathcal{E}_1 \cap \mathcal{E}_2)] \) and \( E_{AB}[\mu_{XY}(\mathcal{E}_1 \cap \mathcal{E}_3)] \). Since

\[ \mu_{X|Y}(x'|y) \geq \mu_{X|Y}(x|y) \geq 2^{-n[H(X|Y)+\gamma]} \]

When (SW1) and (SW2), we have

\[ [\mathcal{G}(y) \setminus \{x\}] \cap \mathcal{C}_A(Ax) \neq \emptyset, \]

where

\[ \mathcal{G}(y) \equiv \{(x : \mu_{X|Y}(x|y) \geq 2^{-n[H(X|Y)+\gamma]} \}. \]

From Lemma 1, we have

\[
E_{AB}[\mu_{XY}(\mathcal{E}_1 \cap \mathcal{E}_2)] = \sum_{(x,y) \in T} \mu_{XY}(x,y)p_{AB} \left( \{(A,B) : (SW2)\} \right) \\
\leq \sum_{(x,y) \in T} \mu_{XY}(x,y)p_{A} \left( \{A : [\mathcal{G}(y) \setminus \{x\}] \cap \mathcal{C}_A(Ax) \neq \emptyset\} \right) \\
\leq \sum_{(x,y) \in T} \mu_{XY}(x,y) \left[ \frac{[\mathcal{G}(y)|\alpha_A]}{|\Im A|} + \beta_A \right] \\
\leq \sum_{(x,y) \in T} \mu_{XY}(x,y) \left[ \frac{2^{n[H(X|Y)+\gamma]|\alpha_A}}{|\Im A|} + \beta_A \right] \\
\leq 2^{n[H(X|Y)+\gamma]}|X|^{-1} |\alpha_A| \frac{|Y|}{|\Im A|} + \beta_A \\
\leq \frac{\delta}{4} \tag{48} \]

for all sufficiently large \( n \) by taking an appropriate \( \gamma > 0 \), where the last inequality comes from (14) and an \((\alpha_A, \beta_A)\)-hash property of \((A, p_A)\). Similarly, we have

\[
E_{AB}[\mu_{XY}(\mathcal{E}_1 \cap \mathcal{E}_3)] \leq 2^{n[H(Y|X)+\gamma]}|Y|^{-1} |\alpha_B| \frac{|X|}{|\Im B|} + \beta_B \\
\leq \frac{\delta}{4} \tag{49} \]

for all sufficiently large \( n \) by taking an appropriate \( \gamma > 0 \), where the last inequality comes from (15) and an \((\alpha_B, \beta_B)\)-hash property of \((B, p_B)\).

Next, we evaluate \( E_{AB}[\mu_{XY}(\mathcal{E}_1 \cap \mathcal{E}_4)] \). When (SW1) and (SW4), we have

\[ [\mathcal{G} \setminus \{(x, y)\}] \cap \mathcal{C}_A(Ax) \times \mathcal{C}_B(By) \neq \emptyset, \]

where

\[ \mathcal{G} \equiv \{(x, y) : \mu_{XY}(x, y) \geq 2^{-n[H(XY)+\gamma]} \}. \]
Applying Lemma 1 to the joint ensemble $(A \times B, p_{AB})$ of a set of functions $AB : \mathcal{X}^n \times \mathcal{Y}^n \to \mathcal{X}^{l_A} \times \mathcal{Y}^{l_B}$, we have

$$E_{AB}[\mu_{XY}(E_1 \cap E_2)] = \sum_{(x,y) \in T} \mu_{XY}(x,y)p_{AB} \left(\{(A,B) : (SW4)\}\right)$$

$$\leq \sum_{(x,y) \in T} \mu_{XY}(x,y) \left[\frac{|G|\alpha_{AB}}{|\text{Im} A||\text{Im} B|} + \beta'_{AB}\right]$$

$$\leq 2^n[H(X,Y) + \gamma] |\mathcal{X}|^{-l_A} |\mathcal{Y}|^{-l_B} \frac{|\mathcal{X}^{l_A} \mathcal{Y}^{l_B}| \alpha_{AB}}{|\text{Im} A||\text{Im} B|} + \beta'_{AB}$$

$$\leq \frac{\delta}{4}$$

(50)

for all sufficiently large $n$ by taking an appropriate $\gamma > 0$, where the last inequality comes from (16) and an $(\alpha_{AB}, \beta_{AB})$-hash property of $(A \times B, p_A \times p_B)$.

Finally, from (46)–(50), for all $\delta > 0$ and for all sufficiently large $n$ there are $A$ and $B$ such that

$$\text{Error}_{XY}(A, B) < \delta.$$

E. Proof of Theorem 5

For $\beta_A$ satisfying $\lim_{n \to \infty} \beta_A(n) = 0$, let $\kappa \equiv \{\kappa(n)\}_{n=1}^\infty$ be a sequence satisfying

$$\lim_{n \to \infty} \kappa(n) = \infty$$

(51)

$$\lim_{n \to \infty} \kappa(n)\beta_A(n) = 0$$

(52)

$$\lim_{n \to \infty} \frac{\log \kappa(n)}{n} = 0.$$  

(53)

For example, there is such $\kappa$ by letting

$$\kappa(n) \equiv \begin{cases} n^\xi & \text{if } \beta_A(n) = o\left(n^{-\xi}\right) \\ \frac{1}{\sqrt{\beta_A(n)}}, & \text{otherwise} \end{cases}$$

for every $n$. If $\beta_A(n)$ is not $o\left(n^{-\xi}\right)$, there is $\kappa' > 0$ such that $\beta_A(n)n^\xi > \kappa'$ and

$$\frac{\log \kappa(n)}{n} = \frac{\log \frac{1}{\beta_A(n)}}{2n} \leq \frac{\frac{\xi \log n - \log \kappa'}{2n}}{2n}$$

for all sufficiently large $n$. This implies that $\kappa$ satisfies (53). In the following, $\kappa$ denotes $\kappa(n)$.

Let $\varepsilon \equiv \varepsilon_B - \varepsilon_A$. Then, from (53), there are $\gamma$ and $\gamma'$ such that

$$0 < \gamma < \sqrt{2\varepsilon} \log |Z| < \varepsilon$$

(54)

February 12, 2013 DRAFT
\begin{equation}
0 < \gamma' \leq 2\varepsilon
\end{equation}

\begin{equation}
\eta_{W|Z}(\gamma'|\gamma) \leq \varepsilon - \frac{\log \kappa}{2n}
\end{equation}

\begin{equation}
\eta_{XW|Z}(\gamma + 2\varepsilon) \leq \varepsilon - \varepsilon A - \frac{\log \kappa}{2n}
\end{equation}

for all sufficiently large \( n \). It should be noted here that there is such \( \gamma \) and \( \gamma' \) by assuming \( (18) \).

Let \( z \) be the output of channel side information, and \( x \) and \( y \) be an input and an output of the channel, respectively, and \( m \) be a message. From \( (55) \), we have \( T_{W|Z,\gamma'}(z) \neq \emptyset \) for all \( z \) and sufficiently large \( n \). Then we have

\begin{equation}
|T_{W|Z,2\varepsilon}(z)| \geq |T_{W|Z,\gamma}(z)|
\end{equation}

\begin{equation}
\geq 2^{n[H(W|Z) - \eta_{W|Z}(\gamma)|]} 
\end{equation}

\begin{equation}
\geq \sqrt{2}n^{n[H(W|Z) - \varepsilon A + \varepsilon S]}
\end{equation}

\begin{equation}
= \sqrt{n} W^{\varepsilon A + \varepsilon S}
\end{equation}

\begin{equation}
\geq \sqrt{n} |\text{Im}A||\text{Im}B|
\end{equation}

for all \( z \in T_{Z,\gamma} \) and sufficiently large \( n \), where the first inequality comes from \( (55) \), the second inequality comes from Lemma 27, the third inequality comes from \( (56) \), and the fourth inequality comes from the fact that \( \text{Im}A \subset W^{\varepsilon A} \) and \( \text{Im}B \subset W^{\varepsilon S} \). This implies that for all \( z \in T_{Z,\gamma} \) there is \( T_{W|Z}(z) \subset T_{W|Z,2\varepsilon}(z) \) such that

\begin{equation}
\sqrt{n} \leq \frac{|T_{W|Z}(z)|}{|\text{Im}A||\text{Im}B|} \leq 2\sqrt{n}
\end{equation}

for all \( z \in T_{Z,\gamma} \) and sufficiently large \( n \). We assume that \( T_{W|Z}(z) \) satisfies the assumption described in Lemma 5. Similarly, from \( (57) \), we obtain \( T_{X|ZW}(z, w) \subset T_{X|ZW,2\varepsilon}(z, w) \) such that

\begin{equation}
\sqrt{n} \leq \frac{|T_{X|ZW}(z, w)|}{|\text{Im}A|} \leq 2\sqrt{n}
\end{equation}

for all \((z, w) \in T_{ZW,2\varepsilon}\) and sufficiently large \( n \).

We define

- \( z \in T_{Z,\gamma} \) (GP1)
- \( g_{AB}(c, m|z) \in T_{W|Z}(z) \) (GP2)
- \( g_{\tilde{A}|z, g_{AB}(c, m|z)} \in T_{X|ZW}(z, g_{AB}(c, m|z)) \) (GP3)
- \( y \in T_{Y|XZ,\gamma_0}(g_{\tilde{A}|z, g_{AB}(c, m|z)}, z) \) (GP4)
- \( g_A(c|y) = g_{AB}(c, m|z) \). (GP5)

Under condition (GP5), we have

\[ \varphi^{-1}(y) = Bg_A(c|y) = Bg_{AB}(c, m|z) = m, \]
which implies that the decoding succeeds. Then the error probability is upper bounded by

\[
\text{Error}_{Y|XZ}(A, B, \hat{A}, c, \hat{c}) = \sum_{m, y} p_m(m) \mu_Z(\gamma) \left| g_{A}(c, m, z) \right| \left| g_{A}(\hat{c}, y) \right| \geq g_{AB}(c, m, z)
\]

\[
\leq p_{MYZ}(S_i^c) + p_{MYZ}(S_1 \cap S_2^c) + p_{MYZ}(S_1 \cap S_2 \cap S_3^c) + p_{MYZ}(S_4^c) + p_{MYZ}(S_1 \cap S_2 \cap S_3 \cap S_4 \cap S_5^c),
\]

where

\[
S_i \equiv \{(m, y, z) : (GP_i)\}.
\]

Let \( \delta \) be an arbitrary positive number.

First we evaluate \( E_{AB\hat{A}C\hat{C}}[p_{MYZ}(S_i^c)] \) and \( E_{AB\hat{A}C\hat{C}}[p_{MYZ}(S_1^c)] \). From Lemma 26, we have

\[
E_{AB\hat{A}C\hat{C}}[p_{MYZ}(S_i^c)] \leq \frac{\delta}{5}
\]

(61)

\[
E_{AB\hat{A}C\hat{C}}[p_{MYZ}(S_1^c)] \leq \frac{\delta}{5}
\]

(62)

for sufficiently large \( n \).

Next we evaluate \( E_{ABC}[p_{MYZ}(S_1 \cap S_2^c)] \) and \( E_{AC\hat{C}}[p_{MYZ}(S_1 \cap S_2 \cap S_3^c)] \). From Lemma 5, we have

\[
E_{ABC}[p_{MYZ}(S_1 \cap S_2^c)] = \sum_{z \in T_{Z,\gamma}} p_z(z) \mu_{ABCM}(\{(A, B, c, m) : g_{AB}(c, m, z) \notin T_{W|Z}(z)\})
\]

\[
\leq \sum_{z \in T_{Z,\gamma}} p_z(z) \left[ \alpha_{AB} - 1 + \frac{|\text{Im}A||\text{Im}B| |\beta_{AB} + 1|}{|T_{W|Z}(z)|} + \frac{2^{-n\epsilon}}{|\text{Im}A||\text{Im}B|} \right]
\]

\[
\leq \alpha_{AB} - 1 + \frac{\beta_{AB} + 1}{\sqrt{\kappa}} + \frac{2^{-n\epsilon}}{|\text{Im}A||\text{Im}B|}
\]

\[
\leq \frac{\delta}{5}
\]

(63)

for all sufficiently large \( n \), where the second inequality comes from (58), and the last inequality comes from (51) and the properties of \( (\alpha_{AB}, \beta_{AB}) \) and \( |W|^\alpha/|\text{Im}A| \). Similarly, by using (59), we have

\[
E_{AC\hat{C}}[p_{MYZ}(S_1 \cap S_2 \cap S_3^c)] \leq \alpha_{\hat{A}} - 1 + \frac{\beta_{\hat{A}} + 1}{\sqrt{\kappa}} + \frac{2^{-n\epsilon}}{|\text{Im}A|}
\]

\[
\leq \frac{\delta}{5}
\]

(64)

for all sufficiently large \( n \).

Next, we evaluate \( E_{AB\hat{A}C\hat{C}}[p_{MYZ}(S_1 \cap S_2 \cap S_3 \cap S_4 \cap S_5^c)] \). In the following, we assume that

\[
\bullet z \in T_{Z,\gamma}
\]

\[
\bullet w \in T_{W|Z}(z) \subset T_{W|Z,2\epsilon}(z)
\]

\[
\bullet x \in T_{X|ZW}(z, w) \subset T_{X|ZW,2\epsilon}(z, w)
\]

\[
\bullet y \in T_{Y|XZ,\gamma}(x, z) = T_{Y|XZ,\gamma}(x, z, w)
\]

\[
\bullet g_A(c, y) \neq w,
\]
Then, we have
\[ \mu_{W|Y}(w'|y) \geq \mu_{W|Y}(w|y) \]
\[ = \frac{\mu_{Y}(y)}{\mu_{Y}(y)} \]
\[ \geq \frac{2^{-n[H(W|Y)+\zeta_{YW}(6\epsilon_A)]}}{2^{-n[H(W|Y)\Sigma_{YW}(6\epsilon_A)]}} \]
\[ \geq 2^{-n[H(W|Y)+2\zeta_{YW}(6\epsilon_A)]}, \]
where the second inequality comes from Lemma 26. This implies that
\[ [\mathcal{G}(y) \setminus \{w\}] \cap C_{A}(c) \neq \emptyset, \]
where
\[ \mathcal{G}(y) \equiv \left\{ w' : \mu_{W|Y}(w'|y) \geq 2^{-n[H(W|Y)+2\zeta_{YW}(6\epsilon_A)]} \right\}. \]

Then, we have
\[
E_{AB\tilde{AC}G}[\rho_{MYZ}(S_1 \cap S_2 \cap S_3 \cap S_4 \cap S_0)] \\
\leq E_{AB\tilde{AC}MG} \left[ \sum_{z \in T_{Z,\gamma}} \sum_{w \in T_{W|Z}(z)} \chi(g_{AB}(c, m|z) = w) \right. \\
\sum_{x \in T_{X|Z,W}(x, w)} \chi(g_{X|Z,Y}(y|x, z) = x) \sum_{y \in T_{Y|X,Z,\gamma}(x, z)} \mu_{Y|X,Z}(y|x, z) \chi(g_{A}(c|y) \neq w) \right] \\
\leq E_{AB\tilde{AC}MG} \left[ \sum_{z \in T_{Z,\gamma}} \sum_{w \in T_{W|Z}(z)} \chi(Aw = c) \chi(Bw = m) \right. \\
\left. \sum_{x \in T_{X|Z,W}(x, w)} \chi(\tilde{A}x = \tilde{c}) \sum_{y \in T_{Y|X,Z,\gamma}(x, z)} \mu_{Y|X,Z}(y|x, z) \chi(g_{A}(c|y) \neq w) \right] \\
= \sum_{z \in T_{Z,\gamma}} \sum_{w \in T_{W|Z}(z)} \sum_{x \in T_{X|Z,W}(x, w)} \sum_{y \in T_{Y|X,Z,\gamma}(x, z)} \mu_{Y|X,Z}(y|x, z) \\
\cdot E_{AC} \left[ \chi(g_{A}(c|y) \neq w) \chi(Aw = c) \right] \cdot E_{B\tilde{AC}G} \left[ \chi(Bw = m) \chi(\tilde{A}x = \tilde{c}) \right] \\
\leq \frac{1}{|\text{ImB}||\text{ImA}|} \sum_{z \in T_{Z,\gamma}} \sum_{w \in T_{W|Z}(z)} \sum_{x \in T_{X|Z,W}(x, w)} \sum_{y \in T_{Y|X,Z,\gamma}(x, z)} \mu_{Y|X,Z}(y|x, z) \\
\cdot p_{AC} \left( \left\{ (A, c) : [\mathcal{G}(y) \setminus \{w\}] \cap C_{A}(c) \neq \emptyset \right\} \right) \\
\leq \frac{1}{|\text{ImB}||\text{ImA}|} \sum_{z \in T_{Z,\gamma}} \sum_{w \in T_{W|Z}(z)} \sum_{x \in T_{X|Z,W}(x, w)} \sum_{y \in T_{Y|X,Z,\gamma}(x, z)} \mu_{Y|X,Z}(y|x, z) \\
\cdot \left[ \frac{2^{n[H(W|Y)+2\zeta_{YW}(6\epsilon_A)]}}{|\text{ImA}|^2} + \frac{\beta_{A}}{|\text{ImA}|} \right] \\
\leq \frac{2^{n[H(W|Y)+2\zeta_{YW}(6\epsilon_A)]}}{|\text{ImA}|} + \frac{\beta_{A}}{|\text{ImA}|} \sum_{z \in T_{Z,\gamma}} \sum_{w \in T_{W|Z}(z)} \mu_{Z}(z) \\
\frac{1}{|\text{ImB}||\text{ImA}|} \sum_{x \in T_{X|Z,W}(x, w)} \frac{1}{|\text{ImA}|} \sum_{y \in T_{Y|X,Z,\gamma}(x, z)} \frac{1}{|\text{ImA}|} \]
\[
\leq 4\kappa |W||A|2^{-n|\varepsilon_A - 2\zeta_{YW}(6\varepsilon_A)|}|\alpha_A| + 4\kappa |\beta_A|
\]
\[
\leq \frac{\delta}{5}.
\]  
(65)

where the third inequality comes from (9), the fourth inequality comes from Lemma 3 and the fact that

\[|\mathcal{G}(y)| \leq 2^n|H(W|Y) + 2\zeta_{YW}(6\varepsilon_A)|,\]

the sixth inequality comes from (58) and (59), and the last inequality comes from (19), (52), and the properties of \((\alpha_A, \beta_A)\) and \(|\operatorname{Im} A|\).

Finally, from (60)–(65), we have the fact that for all \(\delta > 0\) and sufficiently large \(n\) there are \(A \in A, B \in B, \hat{A} \in \hat{A}, c \in \operatorname{Im} A, \) and \(\hat{c}\) such that

\[\operatorname{Error}_{Y|XZ}(A, B, \hat{A}, c, \hat{c}) \leq \delta.\]

\[\square\]

\section*{F. Proof of Theorem 7}

We define

\begin{itemize}
  \item \((x, z) \in T_{XZ}, \gamma\) \hspace{1cm} (WZ1)
  \item \(g_A(c|x) \in T_{Y|XZ,2\varepsilon_A}(x, z)\) \hspace{1cm} (WZ2)
  \item \(g_{AB}(c, Bg_A(c|x)) = g_A(c|x)\) \hspace{1cm} (WZ3)
\end{itemize}

and assume that \(\gamma > 0\) satisfies

\[\gamma + \sqrt{2\gamma \log |X||Z|} < \varepsilon_A.\]

(66)

We prove the following lemma.

\textbf{Lemma 20:} For any \((x, z)\) satisfying (WZ1)

\[p_{ABC}(\{(A, B, c) : (WZ2), \text{not (WZ3)}\}) \leq \frac{2^{-n|\varepsilon_B - 2\zeta_{YZ}(3\varepsilon_A)|} + \beta_B}{|\operatorname{Im} A| |\operatorname{Im} B|}.\]

\textbf{Proof:} If \((x, z, A, B, c)\) satisfies (WZ1) and (WZ2) but not (WZ3), there is \(y' \in C_{AB}(c, Bg_A(c|x))\) such that \(y' \neq g_A(c|x)\). Then, from (66) and Lemmas 23 and 25, we have \((x, g_A(c|x), z) \in T_{Y|XZ,3\varepsilon_A}\) and

\[\mu_{Y|Z}(y'|z) \geq \mu_{Y|Z}(g_A(c|x), z) = \frac{\mu_{YZ}(g_A(c|x), z)}{\mu_{Z}(z)} \geq \frac{2^{-n[H(Y,Z) - \zeta_{YZ}(3\varepsilon_A)]}}{2^{-n[H(Y,Z) + 2\zeta_{YZ}(3\varepsilon_A)]}}.
\]

This implies that

\[\mathcal{G} \setminus \{g_A(c|x)\} \cap C_A(c, Bg_A(c|x)) \neq \emptyset,\]

where

\[\mathcal{G} \equiv \{y' : \mu_{Y|Z}(y'|z) \geq 2^{-n[H(Y|Z) + 2\zeta_{YZ}(3\varepsilon_A)]}\}.
\]

February 12, 2013 DRAFT
Let \( y_{A,e} \equiv g_A(c|x) \). From Lemma 4, we have

\[
p_{ABC} \left( \{(A, B, c) : (WZ2), \not (WZ3)\} \right) \leq p_{ABC} \left( \{ (A, B, c) : [G \setminus \{y_{A,e}\}] \cap C_{AB}(c, By_{A,e}) \neq \emptyset \} \right)
\]

\[
\leq \frac{|G|}{|\text{Im}A||\text{Im}B|} + \beta_B \\
\leq 2^{n[H(Y|Z)+2\zeta YZ(3\epsilon A)]|A|_A+|A|_B} + \beta_B
\]

where the second inequality comes from Lemma 4 and the third inequality comes from the fact that \(|G| \leq 2^{n[H(Y|Z)+2\zeta YZ(3\epsilon A)]}\).

Proof of Theorem 7: Let \( \text{Error}_{XZ}(A, B, c) \) be defined as

\[
\text{Error}_{XZ}(A, B, c) \equiv \mu_{XZ} \left( \{ (x, z) : (x, \varphi^{-1}(\varphi(x), z), z) \not \in T_{XY Z, 3\epsilon A} \} \right).
\]

Since

\[
(x, \varphi^{-1}(\varphi(x), z), z) = (x, g_{AB}(c, Bg_A(c|x)), z)
\]

\[
= (x, g_A(c|x), z)
\]

\[
\in T_{XY Z, 3\epsilon A}.
\]

under conditions (WZ1)–(WZ3), then we have

\[
\text{Error}_{XZ}(A, B, c) \leq \mu_{XZ}(S_1^e) + \mu_{XZ}(S_1 \cap S_2^c) + \mu_{XZ}(S_1 \cap S_2 \cap S_3^c),
\]

where

\[
S_i \equiv \{ (x, z) : (WZi) \}.
\]

Let \( \delta > 0 \) be an arbitrary positive number.

First, we evaluate \( E_{ABC} [\mu_{XZ}(S_1^e)] \). From Lemma 26, we have

\[
E_{ABC} [\mu_{XZ}(S_1^e)] \leq \frac{\delta}{3}
\]

for all sufficiently large \( n \).

Next, we evaluate \( E_{ABC} [\mu_{XZ}(S_1 \cap S_2^c)] \). From (20), we have

\[
\mu_{XYZ}(x, y, z) = \mu_{XZ}(x, z)\mu_{Y|X}(y|x)
\]

\[
= \frac{\mu_{XZ}(x, z)\mu_{XY}(x, y)}{\mu_X(x)}
\]

\[
= \mu_{XY}(x, y)\mu_{Z|X}(z|x)
\]

and

\[
\arg \max_{y' \in C_n(e)} \mu_{XY}(x, y') = \arg \max_{y' \in C_n(e)} \mu_{XY}(x, y')\mu_{Z|X}(z|x).
\]
from Lemma 27, we have
\[ \gamma \]
the fact that there is \( n \) for all sufficiently large \( T \). We assume that \( n \) is sufficiently large \( \alpha \) and sufficiently large \( \beta \), where the last inequality comes from (21) and the properties of \( (\alpha, \beta) \) and \( |Y|^{l_A}/|\text{Im} A| \).

Finally, we evaluate \( E_{ABC}[\mu_{XZ}(S_1 \cap S_2 \cap S_3)] \). From Lemma 20, we have
\[
E_{ABC}[\mu_{XZ}(S_1 \cap S_2 \cap S_3)] \leq \sum_{(x,z) \in T_{XZ,\gamma}} \mu_{XZ}(x, z) p_{AB}(\{(A, B, \epsilon) : (WZ2), \text{not (WZ3)}\})
\]
\[
\leq \frac{2^{-n} \epsilon A - 2\gamma \mu_{XZ}(x, z)}{|\text{Im} A||\text{Im} B|} + \beta_B
\]
\[
\leq \frac{\delta}{3}
\]
From (67)–(71), we have the fact that for any \( \delta > 0 \) and for all sufficiently large \( n \) there are \( A, B, \) and \( c \) such that

\[
\text{Error}_{XY}(A, B, c) \leq \delta.
\]

From Lemma 24, we have

\[
\rho_n(x, f_n(y, z)) = \sum_{(x, y, z) \in X \times Y \times Z} \nu_{xyz}(x, y, z) \rho(x, y, z) \leq \sum_{(x, y, z) \in X \times Y \times Z} \mu_{XYZ}(x, y, z) \rho(x, f(y, z)) + |X||Y||Z| \rho_{\text{max}} \sqrt{6 \varepsilon A}
\]

for \( (x, y, z) \in T_{XYZ,3\varepsilon A} \). Then we have

\[
E_{XZ} \left[ \rho_n(X^n, f_n(\varphi^{-1}(\varphi(X^n), Z^n))) \right] \leq E_{XZ} [\rho(X, f(Y, Z))] + |X||Y||Z| \rho_{\text{max}} \sqrt{6 \varepsilon A} + \delta \rho_{\text{max}}
\]

for all sufficiently large \( n \) by letting

\[
\delta \leq 3 - \sqrt{6}|X||Y||Z| \sqrt{\varepsilon A}.
\]

G. Proof of Theorem 8

We define

- \( (x, y) \in T_{XY, \gamma} \) (OHO1)
- \( g_A(c|y) \in T_{Z,X,2\varepsilon A}(x, y) \) (OHO2)
- \( g_{AB}(c, Bg_A(c|y))) = g_A(c|y) \) (OHO3)
- \( g_B(\hat{\beta}x|g_{AB}(c, Bg_A(c|y))) = x \) (OHO4)

and assume that \( \gamma > 0 \) satisfies

\[
\gamma + \sqrt{2\gamma} \log |X||Y| \leq \varepsilon A.
\]

We prove the following lemma.

**Lemma 21:** For any \((x, y)\) satisfying (OHO1),

\[
p_{AB\hat{B}C} \left( \left\{ (A, B, \hat{B}, c) : \text{(OHO2), (OHO3), not (OHO4)} \right\} \right) \leq \frac{2^{-n||x| - 2\varepsilon\alpha A(3\varepsilon A)}}{|\text{Im}\hat{B}|} + \beta_{\hat{B}}.
\]

**Proof:** We define

\[
x_{A,B,\hat{B},c} \equiv g_B(\hat{B}x|g_{AB}(c, Bg_A(c, x)))
\]
\[
z_{A,B,c} \equiv g_{AB}(c, Bg_A(c, x)).
\]
Assume that conditions (OHO1)-(OHO3) are satisfied but (OHO4) is not. From Lemma 23 and (72), we have 
\((x, y, g_A(c, y)) \in T_{XYZ, 3e_A}\) and there is \(x' \in C_\hat{B} (\hat{B} x)\) such that \(x' \neq g_\hat{B}(\hat{B} x, g_{AB}(c, B g_A(c|x)))\). From Lemma 26, we have

\[
\mu_{X|Z}(x'|z_{A,B,c}) \geq \mu_{X|Z}(x_{A,B,\hat{B},c}|z_{A,B,c}) = \frac{\mu_{XZ}(x_{A,B,\hat{B},c}, z_{A,B,c})}{\mu_Z(z)} \geq \frac{2^{-n[H(XZ)+\epsilon x z(3\varepsilon_A)]}}{2^{-n[H(Z)+\epsilon z(3\varepsilon_A)]}} = 2^{-n[H(X|Z)+2\epsilon x z(3\varepsilon_A)]}.
\]

This implies that

\[
\left[ \mathcal{G}(z_{A,B,c}) \setminus \{x_{A,B,\hat{B},c}\} \right] \cap C_\hat{B}(\hat{B} x) \neq \emptyset,
\]

where

\[
\mathcal{G}(z) \equiv \left\{ x' : \mu_{X|Z}(x'|z) \geq 2^{-n[H(X|Z)+2\epsilon x z(3\varepsilon_A)]} \right\}.
\]

From Lemma 1, we have

\[
p_{AB\hat{B}C} \left( \left\{ (A, B, \hat{B}, c) : \text{(OHO2), (OHO3), not (OHO4)} \right\} \right) \leq p_{AB\hat{B}C} \left( \left\{ (A, B, \hat{B}, c) : \left[ \mathcal{G}(z_{A,B,c}) \setminus \{x_{A,B,\hat{B},c}\} \right] \cap C_\hat{B}(\hat{B} x_{A,B,\hat{B},c}) \neq \emptyset \right\} \right) = \sum_{A, B, c} p_{A,B,c}(A, B, c)p_{\hat{B}} \left( \hat{B} : \left[ \mathcal{G}(z_{A,B,c}) \setminus \{x_{A,B,\hat{B},c}\} \right] \cap C_\hat{B}(\hat{B} x_{A,B,\hat{B},c}) \neq \emptyset \right) \leq \sum_{A, B, c} p_{A,B,c}(A, B, c) \left[ \frac{\left[ \mathcal{G}(z_{A,B,c}) \setminus \{x_{A,B,\hat{B},c}\} \right] \alpha_{\hat{B}}}{|\text{Im}\hat{B}|} + \beta_{\hat{B}} \right] \leq \frac{2^n[H(X|Z)+2\epsilon x z(3\varepsilon_A)]\alpha_{\hat{B}}}{|\text{Im}\hat{B}|} + \beta_{\hat{B}},
\]

where the last inequality comes from the fact that

\[
|\mathcal{G}(z_{A,B,c})| \leq 2^n[H(X|Z)+2\epsilon x z(3\varepsilon_A)],
\]

for all \(A, B\) and \(c\).

**Proof of Theorem 8:** Under the conditions (OHO1)-(OHO4), we have

\[
\varphi^{-1}(\varphi_X(x), \varphi_Y(y)) = g_{\hat{B}}(\hat{B} x, g_{AB}(c, B g_A(c|x))) = x.
\]

Then the decoding error probability is upper bounded by

\[
\text{Error}_{XY}(A, B, \hat{B}, c) \leq \mu_{XY}(S_1') + \mu_{XY}(S_2') + \mu_{XY}(S_1 \cap S_2 \cap S_3') + \mu_{XY}(S_1 \cap S_2 \cap S_3 \cap S_4') \tag{73}
\]

where we define

\[
S_i \equiv \{(v, x) : \text{(OHOi)}\}.
\]
From (22), we have

\[ \mu_{XYZ}(x, y, z) = \mu_{XY}(x, y)\mu_{Z|Y}(z|y) = \frac{\mu_{XY}(x, y)\mu_{YZ}(y, z)}{\mu_Y(y)} = \mu_{X|Y}(x|y)\mu_{YZ}(y, z) \]

and

\[
\arg \max_{z' \in C_A(c)} \mu_{Z|Y}(z'|y) = \arg \max_{z' \in C_A(c)} \mu_{Y|Z}(y, z') = \arg \max_{z' \in C_A(c)} \mu_{XY}(x, y, z') = \arg \max_{z' \in C_A(c)} \mu_{Z|XY}(z'|x, y)
\]

This implies that ML coding by using \( \mu_{Z|Y} \) is equivalent to that using \( \mu_{Z|XY} \). By applying a similar argument to that in the proof of Theorem 7, we have

\[
E_{ABBC}[\mu_{XY}(S_1)] \leq \frac{\delta}{4} \tag{74}
\]

\[
E_{ABBC}[\mu_{XY}(S_2)] \leq \alpha_A - 1 + 2^{-n} [\beta_A + 1] + \frac{2^{-n\varepsilon_A} |Z|^{l_A}}{|\text{Im}A|} \leq \frac{\delta}{4} \tag{75}
\]

\[
E_{ABBC}[\mu_{XY}(S_1 \cap S_2 \cap S_3)] \leq \frac{2^{-n[\varepsilon_B - \varepsilon_A - \varepsilon_C(3\varepsilon_A)]} |Z|^{l_A + l_B} \alpha_{AB}}{|\text{Im}A||\text{Im}B|} + \beta_{AB} \leq \frac{\delta}{4} \tag{76}
\]

for all sufficiently large \( n \) by assuming (23) and (72). Furthermore, from Lemma 21, we have

\[
E_{ABBC}[\mu_{XY}(S_1 \cap S_2 \cap S_3 \cap S_4)] \leq \sum_{(x, y) \in T_{XY}} \mu_{XY}(x, y) p_{ABBC} \{(A, B, c) : (\text{OH2}), (\text{OH3}), \text{not (OH4)}\}\]

\[
\leq \sum_{(x, y) \in T_{XY}} \mu_{XY}(x, y) \left[ 2^{-n[\varepsilon_B - 2\varepsilon_C(3\varepsilon_A)]} |X|^{l_A} \tilde{\alpha_B} + \beta_{\tilde{B}} \right] \leq \frac{2^{-n[\varepsilon_B - 2\varepsilon_C(3\varepsilon_A)]} |X|^{l_A} \tilde{\alpha_B}}{|\text{Im}\tilde{B}|} + \beta_{\tilde{B}} \leq \frac{\delta}{4} \tag{77}
\]

for all sufficiently large \( n \), where the last inequality comes from (24) and the properties of \( (\alpha_{\tilde{B}}, \beta_{\tilde{B}}) \) and \( |X|^{l_\alpha}/|\text{Im}\tilde{B}| \).

From (73)–(77), we have the fact that for all \( \delta > 0 \) and sufficiently large \( n \) there are \( A, B, \tilde{B}, \) and \( c \) such that

\[
\text{Error}_{XY}(A, B, \tilde{B}, C) \leq \delta.
\]
VII. CONCLUSION

In this paper we introduced the notion of the hash property of an ensemble of functions and proved that an ensemble of $q$-ary sparse matrices satisfies the hash property. Based on this property, we proved the achievability of the coding theorems for the Slepian-Wolf problem, the Gel’fand-Pinsker problem, the Wyner-Ziv problem, and the One-helps-one problem. This implies that the rate of codes using sparse matrices combined with ML coding can achieve the optimal rate. We believe that the hash property is essential for coding problems and our theory can also be applied to other ensembles of functions suitable for efficient coding algorithms. In other words, it is enough to prove the hash property of a new ensemble to obtain several coding theorems. It is a future challenge to derive the performance of codes when ML coding is replaced by one of these efficient algorithms given in [1][18][11]. It is also a future challenge to apply the hash property to other coding problems. For example, there are studies of the fixed-rate universal source coding and the fixed-rate universal channel coding [31], and the wiretap channel coding and the secret key agreement [32].

APPENDIX

Method of Types

We use the following lemmas for a set of typical sequences. It should be noted that our definition of a set of typical sequences is introduced in [15][41] and differs from that defined in [6][4][16][48].

Lemma 22 ([6, Lemma 2.6]):

$$\frac{1}{n} \log \frac{1}{\mu_U(u)} = H(\nu_U) + D(\nu_U \| \mu_U)$$

$$\frac{1}{n} \log \frac{1}{\mu_{U|V}(u|v)} = H(\nu_{U|V}) + D(\nu_{U|V} \| \mu_{U|V})$$

Lemma 23 ([41, Theorem 2.5]): If $v \in T_{V,\gamma}$ and $u \in T_{U|V,\gamma'}(v)$, then $(u, v) \in T_{U,V,\gamma+\gamma'}$. If $(u, v) \in T_{U,V,\gamma}$, then $u \in T_{U,\gamma}$.

Proof: The first statement can be proved from the fact that

$$D(\nu_{uv} \| \mu_{UV}) = D(\nu_v \| \mu_V) + D(\nu_u \| \mu_U|V)\nu_v).$$

The second statement can be proved from the fact that

$$D(\nu_v \| \mu_V) \leq D(\nu_{uv} \| \mu_{UV}),$$

which is derived from (78) and the non-negativity of the divergence.

Lemma 24 ([41, Theorem 2.6]): If $u \in T_{U,\gamma}$, then

$$|\nu_u(u) - \mu_u(u)| \leq \sqrt{2\gamma}, \text{ for all } u \in U,$$

$$\nu_u(u) = 0, \text{ if } \mu_u(u) = 0.$$

Proof: The lemma can be proved directly from the fact that

$$\sum_{u \in U} |\nu_u - \mu_u| \leq \sqrt{2D(\nu \| \mu_U)} \frac{1}{\log_2 e},$$

where $e$ is the base of the natural logarithm (see [4, Lemma 12.6.1]).
Lemma 25 ([41, Theorem 2.7]): Let $0 < \gamma \leq 1/8$. Then,
\[
\left| \frac{1}{n} \log_2 \frac{1}{\mu_U(u)} - H(U) \right| \leq \zeta_U(\gamma)
\]
for all $u \in T_{U,\gamma}$, and
\[
\left| \frac{1}{n} \log_2 \frac{1}{\mu_{U|V}(u|v)} - H(U|V) \right| \leq \zeta_{U|V}(\gamma')
\]
for $v \in T_{V,\gamma}$ and $u \in T_{U|V,\gamma'}(v)$, where $\zeta_U(\gamma)$ and $\zeta_{U|V}(\gamma'|\gamma)$ are defined in (2) and (3), respectively.

Proof: From Lemma 22, we have
\[
\left| \frac{1}{n} \log_2 \frac{1}{\mu_V(v)} - H(V) \right| \leq D(\nu_v \parallel \mu_V) + |H(\nu_v) - H(V)|.
\]
We have (79) from [6, Lemma 2.7].

From Lemmas 22 and 24, we have
\[
\left| \frac{1}{n} \log_2 \frac{1}{\mu_{U|V}(u|v)} - H(U|V) \right| \\
\leq D(\nu_{u|v} \parallel \mu_{U|V}) + |H(\nu_{u|v}) - H(\mu_{U|V})| + |H(\mu_{U|V}) - H(U|V)|,
\]
and
\[
|H(\mu_{U|V}) - H(U|V)| \leq \sqrt{2} \log_2 |U|,
\]
respectively. We have (80) from the above inequalities and [6, Lemma 2.7].

Lemma 26 ([41, Theorem 2.8]): For any $\gamma > 0$, and $v \in V^n$,
\[
\mu_U([T_{U,\gamma}]^c) \leq 2^{-n[\gamma - \lambda_U]}
\]
\[
\mu_{U|V}([T_{U|V,\gamma}(v)]^c|v) \leq 2^{-n[\gamma - \lambda_{U|V}]},
\]
where $\lambda_U$ and $\lambda_{U|V}$ are defined in (1).

Proof: The lemma can be proved from [6, Lemma 2.2] and [6, Lemma 2.6].

Lemma 27 ([41, Theorem 2.9]): For any $\gamma > 0$, $\gamma' > 0$, and $v \in T_{V,\gamma}$,
\[
\left| \frac{1}{n} \log_2 |T_{U,\gamma}| - H(U) \right| \leq \eta_U(\gamma)
\]
\[
\left| \frac{1}{n} \log_2 |T_{U|V,\gamma}(v)| - H(U|V) \right| \leq \eta_{U|V}(\gamma'|\gamma),
\]
where $\eta_U(\gamma)$ and $\eta_{U|V}(\gamma'|\gamma)$ are defined in (4) and (5), respectively.

Proof: The lemma can be proved in the same way as the proof of [6, Lemma 2.13].

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