Complete synchronization, phase synchronization and parameters estimation in a realistic chaotic system

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\textbf{A B S T R A C T}

The two-parameter phase space in certain nonlinear system is investigated and the chaotic region of parameters are measured to show its chaotic properties. Within the chaotic parameter region, the complete synchronization, phase synchronization and parameters estimation are discussed in detail by using adaptive synchronization scheme and Lyapunov stability theory. Two changeable gain coefficients are introduced into the controllable positive Lyapunov function and thus the parameter observers. It is found that complete synchronization or phase synchronization occurs with different controllers being used though the parameter observers are the same. Phase synchronization is observed when zero eigenvalue of Jacobi matrix, which is composed of the errors of corresponding variables in the drive and driven chaotic systems. The optimized selection of controllers can induce transition of phase synchronization and complete synchronization.

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1. Introduction

Chaos and spatiotemporal chaos are found in the biological, physical and chemical systems, the study of chaos and spatiotemporal chaos have been paid much attention in the last decades \cite{1-22}. For example, Misra et al. \cite{1} discussed the role of phase synchronization in information process, Pikovsky et al. \cite{2} investigated the phase synchronization of chaotic systems due to external forcing, Shuai et al. \cite{3} studied the phase synchronization in certain neuron models, detailed review about different kinds of synchronization was reported in \cite{4}, Ávila et al. \cite{5} measured the phase synchronization in the lighted-controlled oscillators, Perc et al. \cite{6,7} gave excellent explanation about regular and chaotic calcium oscillations and control unstable orbits outside the chaotic attractor, Wu et al. \cite{9} detected the phase synchronization and coherence of calcium oscillations in coupled hepatocytes, Perc et al. \cite{10,11} constructed the visualization of chaotic attractors and analyzed the time series of human electrocardiogram, Alatriste et al. \cite{12} checked the phase synchronization in tilted deterministic ratchets, Nikulin et al. \cite{13} observed the alpha and beta oscillations in the human electroencephalogram, Wei et al. \cite{14,15} simulated the adaptive control of chaos in power system and synchronization in permanent magnet synchronous motor, Liu et al. \cite{16} constructed a new chaotic system with three variables and give detailed presentation about its realization in circuit, Miliou et al. \cite{17} researched the effectiveness of secure communication based on chaos synchronization in the presence of noise, Denker et al. \cite{18} discovered the phase synchronization of the local field potential in motor cortex during movement preparation, Kim et al. \cite{19} suggested that phase synchronization is useful to detect biological associations between genes, Bob et al. \cite{20} confirmed the EEG phase synchronization in patients with paranoid schizophrenia.

\begin{thebibliography}{22}
\bibitem{1} Misra et al.
\bibitem{2} Pikovsky et al.
\bibitem{3} Shuai et al.
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\bibitem{18} Denker et al.
\bibitem{19} Kim et al.
\bibitem{20} Bob et al.
\end{thebibliography}
2. Problems and scheme

Erjaee and Momani [21] discussed the phase synchronization in fractional differential chaotic systems, Choi et al. [22] explored the phase synchronization between Kuramoto oscillators with finite inertia. Within the dynamical theory, nonlinear differential equations and coupled map lattices are often used to model these systems. Particularly, readers can refer to the elaborate works in this field as reported in Refs. [23,24]. Chaos and hyperchaos are observed in nonlinear differential equations when appropriate parameters are used while spatiotemporal pattern are observed in reaction–diffusion systems and coupled oscillators or networks of neurons [25–29]. It is important to measure the collective behaviors of the oscillators or neurons in the networks, particularly; the synchronization and/or the desynchronization of neuronal activities in networks of neurons could give useful clues to understand the mechanism of certain neuronal disease [30]. Within this topic, some important works should be cared, for example, Wang et al. gave elaborate presentation and investigation about time-delay and rewiring probability induced transition of synchronization in the networks of neurons with topological structure as scale-free and small-world topology type [31,32]. Gosak and Perc illustrated some new ways to induce coherence resonance and stochastic resonance in the excitable networks and chaotic systems [33,34].

In recent years, many interesting chaotic systems are observed and simulated in circuits, these chaotic and/or hyperchaotic systems show different dynamical properties. These models give important information to study synchronization and secure communication [35–37]. The synchronization of chaos or hyperchaos is classified as complete synchronization [38,39], lag synchronization [40–43], generalized synchronization [44] and phase synchronization [45–50]. Transition of synchronization [51–56] can be induced in the chaotic systems and burst synchronization [57] can occur in neurons, particularly, the time-delay induced synchronization in neurons and networks reported by Wang et al. [53–56] can give good clues to understand the mechanism of information encoding and wave propagation among neurons. Furthermore, other topics about synchronization were also presented as the anti-phase synchronization [58–65] and cluster synchronization [66] and others [67]. Therefore, so many schemes have been proposed to study and detect the synchronization [47–49]. It is very critical to estimate unknown parameters in the system by using adaptive synchronization [68–70] and Lyapunov stability theory within the study and control of chaos, hyperchaos, pattern formation and transition of spatiotemporal pattern within the networks of chaotic oscillators.

In fact, some practical and realistic systems should be checked and investigated within this topic about chaos and control, for example, Krese et al. studied the dynamics of laser droplet generation in experimental and theoretical ways and it could give useful guidance for accurate welding procedures [71]. In this paper, the two-parameter region supporting chaotic state in a three-variable realistic system [16], which could be realized in circuit and give practical design for signal generator with wide bandwidth, wave carrier and secure keys for application in secure communication. The dynamical properties of this circuit are measured and detected by calculating the Lyapunov exponent spectrum extensively. Within the chaotic parameter region, adaptive synchronization scheme is used to study the transition of phase synchronization and complete synchronization, and the four parameters in the drive system are unknown. It is found that parameters are identified well and complete synchronization occurs with appropriate controllers and parameter observed being constructed. Phase synchronization is observed when partial unknown parameters are estimated exactly.

2. Problems and scheme

The three-variable chaotic system reported in [16] is described by

\[
\begin{align*}
\frac{dx}{dt} &= a(z - x), \\
\frac{dy}{dt} &= bx - xz, \\
\frac{dz}{dt} &= cy - dz.
\end{align*}
\]

where \(a, b, c, d\) are parameters and two nonlinear terms exist in Eq. (1), chaotic state occurs when the parameters are selected with appropriate values. For example, \(a = 8, b = 40, c = 10/3, d = 4,\) Eq. (1) shows chaotic state and one positive Lyapunov exponent is approached. Extensive numerical studies are given to measure the chaotic parameter region within Eq. (1) by calculating the maximal Lyapunov exponent in the two-parameter space \(a\) vs. \(c\) and \(b\) vs. \(c,\) and the results are shown in Fig. 1(a and b). Then two group of parameters are given to illustrate the chaotic attractors, the three Lyapunov exponents are 0.38833, 1.62E−4, −10.48635 at \(a = 3, b = 40, c = 1.6\) and \(d = 4;\) and another group exponents are 0.98943, 1.52E−4, −13.97254 at \(a = 5, c = 40, b = 2.5, d = 4.\)

The results in Fig. 1(a) show that chaos can be induced in Eq. (1) as appropriate parameters \(a, c\) are selected at fixed parameters \(b = 40, d = 4,\) and strange chaotic attractor are observed at corresponding parameter regions supporting chaos. The extensive studies are shown in Fig. 1(d) to confirm that chaos can occur by selecting appropriate parameters in the two-parameter phase space \(b\) vs. \(c\) at fixed parameter \(a = 8, d = 4.\)

The results in Fig. 1 show appropriate parameters could be selected to induce chaos in Eq. (1), it also indicates that chaotic time series are much different when different groups of parameters are selected to induce chaos in Eq. (1). Therefore, it is critical to detect these unknown parameters within Eq. (1) for further application and study. An improved scheme is used to study the parameter estimation, phase synchronization and complete synchronization. The corresponding driven system (response system) is given with
\[
\begin{align*}
\frac{dx}{dt} &= \hat{a}(\hat{z} - \hat{x}) + u_1, \\
\frac{dy}{dt} &= \hat{b} \hat{x} - \hat{x} \hat{z} + u_2, \\
\frac{dz}{dt} &= \hat{x} \hat{y} / C_0 - \hat{c} \hat{y} - \hat{d} \hat{z} + u_3,
\end{align*}
\]

where \( u_1, u_2, u_3 \) are controllers to be constructed, and the number of controllers are uncertain. In a practical way, smaller number of controllers and simpler form of controllers are practical greatly. The positive Lyapunov function with two controllable gain coefficients are presented by

\[
V = \alpha \left( e_x^2 + e_y^2 + e_z^2 \right) + \left( e_a^2 + e_b^2 + e_c^2 + e_d^2 \right) / \beta,
\]

where positive gain coefficients \( \alpha, \beta \) are selectable and the error variables in Eq. (3) denote

\[
\begin{align*}
& e_x = x - \hat{x}, \quad e_y = y - \hat{y}, \quad e_z = z - \hat{z}; \\
& e_a = a - \hat{a}, \quad e_b = b - \hat{b}, \quad e_c = c - \hat{c}, \quad e_d = d - \hat{d}; \\
& \theta(e_x, e_y, e_z) = \sqrt{\left( e_x^2 + e_y^2 + e_z^2 \right)}.
\end{align*}
\]

Furthermore, the differential coefficients of the error variables in Eq. (4) vs. time are deduced as

\[
\begin{align*}
\dot{e}_x &= de_x / dt = \hat{a} e_z - \hat{a} e_x + e_a(z - x) - u_1; \\
\dot{e}_y &= de_y / dt = \hat{b} e_x + e_b x - xz + \hat{x} \hat{z} - u_2; \\
\dot{e}_z &= de_z / dt = xy - \hat{x} \hat{y} - e_x y - e_y z - \hat{c} e_y - \hat{d} e_z - u_3.
\end{align*}
\]
The differential coefficient of Lyapunov function in Eq. (3) vs. time is simplified when the definitions in Eqs. (4) and (5) are considered.

\[ V = e_0 [2\dot{e}_x/\beta + 2x(z - x)] + e_1 [2\dot{e}_y/\beta + 2xye_y] + e_c [2\dot{e}_z/\beta - 2xze_z] + 2z\left(\dot{ae}_e x - \dot{ae}_c x - u_1 e_x + be_x e_y - xze_y + \dot{xe}_y - u_2 e_y + xye_z - \dot{xe}_z - e_c e_z - de_z - u_3 e_z\right). \]  

(6)

The parameter observers are deduced and the differential coefficient of Lyapunov function in Eq. (6) are presented by

\[ \dot{a} = -e_a = \alpha\beta(z - x)e_x, \quad \dot{b} = -e_b = \alpha\beta ye_y, \]
\[ \dot{c} = -e_c = -\alpha\beta ye_y, \quad \dot{d} = -e_d = -\alpha\beta ze_z; \]

(7)

\[ V = 2z\left(\dot{ae}_e x - \dot{ae}_c x - u_1 e_x + be_x e_y - xze_y + \dot{xe}_y - u_2 e_y + xye_z - \dot{xe}_z - e_c e_z - de_z - u_3 e_z\right). \]

(8)

where the known parameters \(a, b, c, d\) in drive system are supposed to be constant, and it is confirmed that parameter observers in Eq. (7) are also effective when the unknown parameters jump suddenly vs. time.

According to the Lyapunov stability theory, the errors of corresponding variables and the positive Lyapunov function shown in Eq. (3) are stabilized to certain threshold and the two chaotic systems reach complete synchronization only when the differential coefficient of Lyapunov function in Eq. (8) are negative completely

\[ V = -2\dot{a}e_x^2 - 2\dot{b}e_y^2 + 2z\left(\dot{ae}_e x - \dot{ae}_c x - u_1 e_x + be_x e_y - xze_y + \dot{xe}_y - u_2 e_y + xye_z - \dot{xe}_z - e_c e_z - u_3 e_z\right) < 0. \]

(9)

It could be satisfied by selecting different groups of controllers with appropriate forms. It indicates that two controllers are required at least within this chaotic model. The first case can be described by

\[ V = -2\dot{a}e_x^2 - 2\dot{b}e_y^2 < 0, \]
\[ u_1 = 0, \]
\[ 2z(\dot{ae}_e x - \dot{ae}_c x - u_1 e_x + be_x e_y - xze_y + \dot{xe}_y - u_2 e_y + xye_z - \dot{xe}_z - e_c e_z - u_3 e_z) = 0. \]

(10a, 10b, 10c)

In fact, the inequality condition is robust only when the identified results \(\dot{a}, \dot{d}\) are positive all the time, otherwise, the divergence of variables between the two chaotic systems occurs and synchronization could be destructed. Surely, some specific initial values for the six variables and four parameter observers could support the condition in Eq. (10) while it is so difficult to detect these specific initial values. Therefore, the inequality condition in Eq. (9) could be reformed by

\[ V = -2\dot{a}e_x^2 - 2\dot{b}e_y^2 - 2\dot{k}_1 e_x^2 - 2\dot{k}_2 e_x^2 - 2\dot{k}_3 e_x^2 + 2z\left(k_1 e_x^2 + \dot{ae}_e x - u_1 e_x + be_x e_y - xze_y + \dot{xe}_y + k_2 e_y^2 - u_2 e_y + xye_z - \dot{xe}_z - e_c e_z - u_3 e_z\right) < 0. \]

(11)

Different group of appropriate controllers could be deduced from the following condition

\[ \dot{k}_1 e_x^2 + \dot{ae}_e x - u_1 e_x + \dot{be}_x e_y - xze_y + \dot{xe}_y + k_2 e_y^2 - u_2 e_y + xye_z - \dot{xe}_z - e_c e_z - u_3 e_z = 0. \]

(12)

The differential coefficient of Lyapunov function vs. time is negative when appropriate positive values are given to \(k_1, k_2, k_3\) and it reads

\[ V = -2\dot{a}e_x^2 - 2\dot{b}e_y^2 - 2\dot{k}_1 e_x^2 - 2\dot{k}_2 e_x^2 - 2\dot{k}_3 e_x^2 < 0. \]

(13)

At first, we discuss the case that two controllers are imposed on the driven system as shown in Eq. (2), it has no choice but to require

\[ u_1 = 0, \quad k_1 = 0. \]

(14)

\[ V = -2\dot{a}e_x^2 - 2\dot{b}e_y^2 - 2\dot{k}_1 e_x^2 - 2\dot{k}_2 e_x^2 + 2z\left(\dot{ae}_e x + \dot{be}_x e_y - xze_y + k_2 e_y^2 - u_2 e_y + xye_z - \dot{xe}_z - e_c e_z - u_3 e_z\right) < 0. \]

(15)

For simplicity, the negative differential coefficient of Lyapunov function vs. time is proposed by

\[ V = -2\dot{a}e_x^2 - 2\dot{k}_2 e_y^2 - 2\dot{k}_3 e_y^2 - 2\dot{b}e_y^2 = -2\dot{a}e_x^2 - 2\dot{x}ye_y^2 - 2\dot{xe}_y^2 - 2\dot{de}_z^2 < 0. \]

(16)

where \(k_2 = k_3 = \beta\), and the controllers could be selected freely by satisfying the following criterion

\[ \dot{ae}_e x + \dot{be}_x e_y - xze_y + \dot{xe}_y + \dot{k}_2 e_y^2 - u_2 e_y + \dot{be}_x e_y + xye_z - \dot{xe}_z - e_c e_z - u_3 e_z = 0. \]

(17)

In the following section, different groups of controllers are selected to estimate the four unknown parameters in the drive system, and reach complete synchronization, phase synchronization, respectively.
3. Numerical simulation results and discussions

In the numerical simulation studies, the fourth order Runge–Kutte algorithm is used to calculate the nonlinear equations with time step \( h = 0.001 \). Two controllers are selected to control the driven system to synchronize the drive system and the four unknown parameters are desired to be identified exactly with much short transient period. The initial values for the drive system are given with \( (0.1,0.2,0.3) \), initial values of variable for the driven system are presented with \( (0.2,0.1,0.5) \), and the initial values for the identified results in Eq. (7) are \( (4,5,6,3) \). The complete synchronization and phase synchronization will be discussed and investigated, respectively. In the case of phase synchronization, the phases of the two chaotic systems are measured by using the Fast Fourier Transform Algorithm (FFTA) to analyze the corresponding variables in the two chaotic systems because it is so costly and time-consuming to calculate the phase by Hilbert transformation [72].

According to the conditions in Eqs. (16) and (17), controllers are selected with

\[
\begin{align*}
    u_2 &= \hat{b}e_x - xz + \hat{k}z + \beta e_y, \\
    u_3 &= xy - \hat{k}y - \beta e_y + \hat{a}e_z + \beta e_z.
\end{align*}
\]  

(18)

The differential coefficient of corresponding variables are deduced by

\[
\begin{pmatrix}
    \dot{e}_x \\
    \dot{e}_y \\
    \dot{e}_z
\end{pmatrix} =
\begin{pmatrix}
    -\hat{a} & 0 & \hat{a} \\
    0 & -\beta & 0 \\
    -\hat{a} & 0 & -\hat{a} - \beta
\end{pmatrix}
\begin{pmatrix}
    e_x \\
    e_y \\
    e_z
\end{pmatrix} +
\begin{pmatrix}
    e_x (z-x) \\
    e_x y + e_x z \\
    -e_x y - e_z z
\end{pmatrix}.
\]  

(19)

The Eq. (19) can deduce three negative eigenvalue of Jacobi matrix at the fixed equilibrium point \( (0,0,0) \), it just indicates that the controllers in Eq. (18) are effective to control the driven system to synchronize the drive system instantaneously. In Fig. 2, it gives the evolution of error function in Eq. (4c) and the identified results at \( x = 6, \beta = 3 \), and the parameters in the drive system are selected with \( a = 8, b = 40, c = 10/3, d = 4 \).

The results in Fig. 2 confirm that the two chaotic systems reach complete synchronization and the four unknown parameters are measured and stabilized to their corresponding real values. Furthermore, it is interesting to check the effectiveness of the controllers as shown in Eq. (18) and the parameter observers in Eq. (7) when other gain coefficients are used, and it is also important to check its robustness within other chaotic parameter regions. The extensive numerical results confirm that
the above-mentioned scheme still are effective to estimate the unknown parameters with high accuracy and complete synchronization could be reached greatly, and moderate gain coefficients will induce a shorter transient period. In Fig. 3, another group of parameters are given to check the effectiveness of the above mentioned scheme.

The results in Fig. 3 confirm that the controllers in Eq. (18) and parameter observers in Eq. (17) are still effective to estimate the unknown parameters and realize complete synchronization between the two chaotic systems even if other group parameters are used. Furthermore, it is important to study the role of gain coefficients in the parameter observers and the negative differential coefficient of Lyapunov function vs. time. In Fig. 4, it gives the distribution of error function of corresponding variables and the divergence function between the identified parameters and the real value of unknown parameters. For simplicity, the summation of errors for corresponding variables and parameters are calculated from $t = 100$ time units to 500 time units.

\[
\theta(e_x, e_y, e_z) = \sum_{i=100}^{500} \sqrt{e_x^2(i) + e_y^2(i) + e_z^2(i)}; \quad (20a)
\]

\[
\theta(e_a, e_b, e_c, e_d) = \sum_{i=100}^{500} \sqrt{e_a^2(i) + e_b^2(i) + e_c^2(i) + e_d^2(i)}; \quad (20b)
\]

where the summation is calculated from $t = 100$ time units ($t = i \times h$, $h$ is time step) to $t = 500$ time units, in this way, a transient period about 400 time units are used to check the convergence of the error functions. In fact, the results are independent of the transient period to calculate the summation of error function.

The results in Fig. 4 confirm that much too small or big value for the two gain coefficients will destruct synchronization of the two chaotic systems, let alone the realization of parameter estimation, while moderate and appropriate values of gain coefficients will make the controllers and parameters work well to realize complete synchronization and parameter identification with high accuracy.

In the following, phase synchronization will be discussed by selecting different controllers and it is found that just some of the unknown parameters could be identified. Then another group of two controllers are constructed according to Eqs. (16) and (17), it is found that only three unknown parameters are estimated while the rest keeps uncertain.

\[
\begin{align*}
  u_2 &= b e_x - x z + \dot{x} z, \\
  u_3 &= x y - \dot{x} y - c e_y + \dot{a} e_x + \beta e_z.
\end{align*}
\]

(21)

Fig. 3. Time series of the identified results for the four unknown parameters in the drive system in Eq. (1) for (a) and (b); the evolution of error function in Eq. (4c) for (c) and phase portrait of $x$ vs. $\dot{x}$ for (d). The gain coefficient are selected with $\alpha = 6, \beta = 3$ in controllers as shown in Eq. (18), and the parameter observers are constructed as shown in Eq. (7), the real value of the drive system $a = 3, b = 40, c = 1.6, d = 4.$
Furthermore, we also check the results when the controllers are selected by

\[ \dot{\mathcal{V}} = -2\dot{x}\dot{a}e_x^2 - 2\dot{x}\dot{k}_x e_x^2 - 2\dot{x}\dot{k}_y e_y^2 - 2\dot{x}\dot{k}_z e_z^2 = -2\dot{x}\dot{a}e_x^2 - 2\dot{x}\dot{p}e_y^2 - 2\dot{x}\dot{d}e_z^2 < 0. \]  

(22)

An zero eigenvalue of Jacobi matrix is developed at the fixed equilibrium point \((0, 0, 0)\) according to Eq. (23), and the other eigenvalues are \[-(\dot{a} + \dot{d} + \beta) \pm \sqrt{(\dot{a} + \dot{d} + \beta)^2 - 4\dot{a}(\dot{a} + \dot{d} + \beta)} / 2, \] as a result, the astringency of error series of \(e_y\) could become uncertain and complete synchronization could be destructed in certain way. And the error function as shown in Eq. (4c) will oscillate within certain scope. In fact, phase synchronization occurs at this case. Therefore, it is important to study the evolution of phase corresponding to the second variables \(y\) and \(\dot{y}\). And the numerical results are illustrated in Fig. 5.

The results in Fig. 5(a–c) show that only three unknown parameters are estimated exactly, the second unknown parameter \(b\) cannot be stabilized and the identified results oscillate vs. time in random. The error function as shown in Eq. (4c) is restricted within certain thresholds and never decreases to zero vs. time. To the best of our knowledge, the error of the second corresponding variables develops vs. time in random due to the zero eigenvalue of Jacobi matrix in Eq. (23). The results in Fig. 5(d) confirm that the corresponding phase position of the second variables in the two chaotic systems reach phase synchronization though the amplitudes of the corresponding second variables never become close to each other. Within the time series of error function as shown in Eq. (4c), the results in Fig. 5(c) could also result from the case that the identified results \(\dot{a}, \dot{d}\) could be detected with negative values in random, it will induce breakdown of synchronization according to the condition as shown in Eq. (23). Furthermore, other groups of gain coefficients are presented to check the case as shown in Eqs. (21)–(23), the similar results are approached that phase synchronization is induced and only three unknown parameters could be estimated exactly.

Within the problem of phase synchronization, phase of the chaotic or hyperchaotic system should be defined beforehand. A definition of more general validity could be the Hilbert transformation [72]. According to this definition, the analytical signals could be reduced from any time series \(g(t)\) from the dynamical system as \(s_i(t) = g_i(t) + ig_i(t)\), where pure imaginary number is denoted by \(”\imath”\) and \(g_i(t) = PV \int_0^t g(\tau)(t - \tau) d\tau / \pi\), here PV denotes the principal value of the integral. As a result, the phase is often described by \(\phi_i = \arctan(g_i(t)/\dot{g}_i(t))\). In fact, it is very costly to perform the convolution involved in the Hilbert transformation. Within the phase synchronization of the three-variable chaotic systems, all the corresponding variables could be converted to phases based on the Hilbert transformation and the similar Lyapunov function as shown in Eq. (3) to measure the optimized gain coefficient areas in the two-gain coefficients phase space but it is so time-consuming. Therefore, the Fast Fourier Transform Algorithm (FFTA) is used to measure the phase synchronization when the time series of corresponding variables are calculated.

Then, another group of parameters, which induces chaotic state in Eq. (1), are selected as \(a = 3, b = 40, c = 1.6, d = 4\) to check the effectiveness of controllers as shown in Eq. (21), and the results are plotted in Fig. 6.

The results in Fig. 6 confirm that phase synchronization still is approached and three parameters are estimated, while the rest one are not identified completely because the corresponding error of variables \((y, \dot{y})\) cannot be stabilized to zero. Furthermore, we also check the results when the controllers are selected by

Fig. 4. The distribution of the summation of error function for corresponding variables as defined in Eq. (20a) in the two-gain coefficients phase space \((x vs. \beta)\) is illustrated in (a), and the summation of error function for identified results and real values of unknown parameters in the two-gain coefficients phase space \((x vs. \beta)\) is plotted in (b), with a transient period about 400 time units and the real value of the drive system \(a = 3, b = 40, c = 1.6, d = 4\), the controllers are shown in Eq. (18).
\[ u_2 = \frac{b e_x}{C_0} x z + \frac{\bar{x} \bar{z}}{C_0}, \]
\[ u_3 = \frac{xy}{C_0} - \frac{\bar{x} \bar{y}}{C_0} e_y + \beta e_z - \bar{d} e_z, \]
\[ V = -2 \bar{a} \bar{e}_y^2 - 2 \bar{x} k_2 e_x^2 - 2 \bar{x} k_4 e_z^2 = -2 \bar{a} \bar{e}_y^2 - 2 \bar{x} \beta e_z^2 < 0, \]
\[ \begin{pmatrix} \dot{e}_x \\ \dot{e}_y \\ \dot{e}_z \end{pmatrix} = \begin{pmatrix} -\bar{a} & 0 & \bar{a} \\ 0 & 0 & 0 \\ -\bar{a} & 0 & -\beta \end{pmatrix} \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix} + \begin{pmatrix} e_x (z - x) \\ e_y x \\ -e_z y - e_y z \end{pmatrix}. \]

It is found that the eigenvalues are \[ -\bar{a} + \beta \pm \sqrt{(\bar{a} + \beta)^2 - 4 \bar{a} (\bar{a} + \beta)}/2 \] and another one is zero. Eq. (26) becomes stable synchronization as appropriate gain coefficient \( \beta \) is used. And extensive numerical results just demonstrate that phase synchronization between the second variables and the corresponding unknown parameter cannot be measured while the other three unknown parameters are identified exactly. In other words, mismatch of parameters between the two chaotic systems can induce phase synchronization when indirectional control is imposed on the driven system. In this case, two corresponding variables reach complete synchronization, and the rest one reach phase synchronization due to the mismatch of corresponding parameters.

When two chaotic or hyperchaotic systems reach complete synchronization, the errors of corresponding variables decrease to zero vs. time with certain transient period. The dynamical system, which is made up of errors of the corresponding variables, will hold negative eigenvalue of Jacobi matrix and the conditional Lyapunov exponents are negative. Desynchronization or destruction of synchronization occurs when the eigenvalue of Jacobi matrix or conditional Lyapunov exponents are positive. In fact, phase synchronization is certain state between complete synchronization and desynchronization, therefore, it is interesting to investigate the case that zero eigenvalue of Jacobi matrix are observed. The above results in Figs. 5 and 6 presented some discussion about the mixed phase synchronization and partial synchronization, that one corresponding variable reach phase synchronization, and other two corresponding variables reach complete synchronization as well.
Furthermore, the cases that two zero eigenvalues of Jacobi matrix for the dynamical system, which is constructed by the corresponding variables, will be discussed in detail. To present a distinct presentation and understanding, three controllers are introduced into the driven system with different forms according to the condition as shown in Eq. (11). The diagonal elements of the dynamical system for the error variables are described by $(J_{xx}, J_{yy}, J_{zz})$, the case for $(0,0,J_{zz})$, $(0,J_{yy},0)$ and $(J_{xx},0,0)$ will be discussed, respectively. Within the case for $(0,0,J_{zz})$, it makes

$$V = -2αk_2e_y^2 - 2αk_3e_z^2 < 0,$$

$$u_1 = ae_z - ae_x,$$

$$u_2 = -xz + \hat{x} + be_x,$$

$$u_3 = xy - \hat{x} - ce_y + \hat{c}e_z - de_z,$$

\[
\begin{pmatrix}
\dot{e}_x \\
\dot{e}_y \\
\dot{e}_z
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & -\beta & 0
\end{pmatrix}
\begin{pmatrix}
e_x \\
e_y \\
e_z
\end{pmatrix} +
\begin{pmatrix}
e_0(z - x) \\
e_0x \\
-e_y - e_z
\end{pmatrix}.
\]  

Or

$$V = -2αk_2e_y^2 - 2αk_3e_z^2 - 2x\beta e_x^2 - 2x\beta e_z^2 < 0,$$

$$u_1 = ae_z - ae_x,$$

$$u_2 = -xz + \hat{x} + be_x,$$

$$u_3 = xy - \hat{x} - ce_y + \hat{c}e_z,$$

\[
\begin{pmatrix}
\dot{e}_x \\
\dot{e}_y \\
\dot{e}_z
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & -\beta - \hat{d} & 0
\end{pmatrix}
\begin{pmatrix}
e_x \\
e_y \\
e_z
\end{pmatrix} +
\begin{pmatrix}
e_0(z - x) \\
e_0x \\
-e_y - \hat{d}e_z
\end{pmatrix}.
\]  


**Fig. 6.** Time series of the identified results for the four unknown parameters in the drive system in Eq. (1) for (a) and (b); the evolution of error function in Eq. (4c) for (c) and phase portrait of $\phi(y)$ vs. $\phi(y)$ for (d). The gain coefficient are selected with $α = 6$, $β = 3$ in controllers as shown in Eq. (21), and the parameter observers are constructed as shown in Eq. (7). The time series of variable $y$, $\hat{y}$ are converted to phase position, and the phase position could be calculated by using Hilbert transformation or Fast Fourier Transform Algorithm (FFT), the real values for the four unknown parameters in the drive system are given with $a = 3$, $b = 40$, $c = 1.6$, $d = 4$.

Furthermore, the cases that two zero eigenvalues of Jacobi matrix for the dynamical system, which is constructed by the corresponding variables, will be discussed in detail. To present a distinct presentation and understanding, three controllers are introduced into the driven system with different forms according to the condition as shown in Eq. (11). The diagonal elements of the dynamical system for the error variables are described by $(J_{xx}, J_{yy}, J_{zz})$, the case for $(0,0,J_{zz})$, $(0,J_{yy},0)$ and $(J_{xx},0,0)$ will be discussed, respectively. Within the case for $(0,0,J_{zz})$, it makes

$$V = -2αk_2e_y^2 - 2αk_3e_z^2 < 0,$$

$$u_1 = ae_z - ae_x,$$

$$u_2 = -xz + \hat{x} + be_x,$$

$$u_3 = xy - \hat{x} - ce_y + \hat{c}e_z - de_z,$$

\[
\begin{pmatrix}
\dot{e}_x \\
\dot{e}_y \\
\dot{e}_z
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & -\beta & 0
\end{pmatrix}
\begin{pmatrix}
e_x \\
e_y \\
e_z
\end{pmatrix} +
\begin{pmatrix}
e_0(z - x) \\
e_0x \\
-e_y - e_z
\end{pmatrix}.
\]  

Or

$$V = -2αk_2e_y^2 - 2αk_3e_z^2 - 2x\beta e_x^2 - 2x\beta e_z^2 < 0,$$

$$u_1 = ae_z - ae_x,$$

$$u_2 = -xz + \hat{x} + be_x,$$

$$u_3 = xy - \hat{x} - ce_y + \hat{c}e_z,$$

\[
\begin{pmatrix}
\dot{e}_x \\
\dot{e}_y \\
\dot{e}_z
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & -\beta - \hat{d} & 0
\end{pmatrix}
\begin{pmatrix}
e_x \\
e_y \\
e_z
\end{pmatrix} +
\begin{pmatrix}
e_0(z - x) \\
e_0x \\
-e_y - \hat{d}e_z
\end{pmatrix}.
\]
It is confirmed that exact solution of eigenvalue of Jacobi matrix is fixed at \((0,0,-\beta)\) in Eqs. (27b) and (27c), while the solution of eigenvalue of Jacobi matrix in Eqs. (27e) and (27f) is stabilized at \((0,0,-\beta-d)\), and the negative eigenvalue of Jacobi matrix is helpful to induce stable phase synchronization.

The results in Fig. 7 show that partial unknown parameters \((c,d)\) are identified while the parameter \(a, b\) cannot be measured and stabilized to corresponding real values. The error of corresponding third variables \((z,\hat{z})\) can synchronize each other completely and thus the corresponding parameters \((c,d)\) are estimated; the other two corresponding variables \((y,\hat{y})\) and \((x,\hat{x})\) just reach phase synchronization as well and the phase synchronization is stable because the solution of eigenvalue of Jacobi matrix in Eq. (27) is negative completely. Extensive numerical studies have also been carried out and similar phase synchronization is stabilized as above mentioned.

Within the case for \((0,J_{yy},0)\), it also makes
\[
V = -2xz\beta e_z^2 - 2xz\beta e_z^2 = -2z\beta e_z^2 < 0,
\]
\[
u_1 = \hat{a} e_z - \hat{a} e_z,
\]
\[
u_2 = -xz + \dot{x}z + \beta e_x + \beta e_y,
\]
\[
u_3 = xy - \dot{x}y - c e_y - \dot{c} e_z,
\]
\[
\begin{pmatrix}
\dot{e}_x \\
\dot{e}_y \\
\dot{e}_z
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
e_x \\
e_y \\
e_z
\end{pmatrix}
+ \begin{pmatrix}
ed_x(z-x) \\
ed_y \\
ed_z(-e_y - e_z)
\end{pmatrix}.
\]

It is confirmed that exact solution of eigenvalue of Jacobi matrix as shown in Eq. (28c) is fixed at \((0,-\beta,0)\), phase synchronization occurs between the corresponding variables \((z,\hat{z})\) and \((x,\hat{x})\), and the results are illustrated in Fig. 8.

The results in Fig. 8 confirm that the second corresponding variables \(y,\hat{y}\) reach complete synchronization and the corresponding unknown parameters \(b\) is estimated exactly, the other three unknown parameters \(a, c, d\) cannot be stabilized to their real values and the phase synchronization occurs between the corresponding variables as well. Extensive numerical results confirm that the controllers as shown in Eq. (28) keeps robust and effective to reach mixed phase synchronization and complete synchronization between the two chaotic systems in other chaotic parameter regions with other appropriate gain coefficients being used.

Then it will investigate the case for \((J_{xx},0,0)\), and the conditions are described by
\[
V = -2z\beta e_z^2 < 0,
\]
\[
u_1 = \hat{a} e_z - \hat{a} e_z,
\]
\[
u_2 = -xz + \dot{x}z + \beta e_x + \beta e_y,
\]
\[
u_3 = xy - \dot{x}y - c e_y - \dot{c} e_z,
\]
\[
\begin{pmatrix}
\dot{e}_x \\
\dot{e}_y \\
\dot{e}_z
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
e_x \\
e_y \\
e_z
\end{pmatrix}
+ \begin{pmatrix}
ed_x(z-x) \\
ed_y \\
ed_z(-e_y - e_z)
\end{pmatrix}.
\]

Or
\[
V = -2z\beta e_z^2 - 2z\hat{a} e_z^2 < 0.
\]
\[
u_1 = \hat{a} e_z + \beta e_z,
\]
\[
u_2 = -xz + \dot{x}z + \beta e_x,
\]
\[
u_3 = xy - \dot{x}y - c e_y - \dot{c} e_z,
\]
\[
\begin{pmatrix}
\dot{e}_x \\
\dot{e}_y \\
\dot{e}_z
\end{pmatrix}
= \begin{pmatrix}
-\hat{a} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
e_x \\
e_y \\
e_z
\end{pmatrix}
+ \begin{pmatrix}
ed_x(z-x) \\
ed_y \\
ed_z(-e_y - e_z)
\end{pmatrix}.
\]

It is confirmed that exact solution of eigenvalue of Jacobi matrix as shown in Eq. (29c) is fixed at \((-\beta,0,0)\) (its eigenvalues of Jacobi matrix in Eq. (29f) are \((-\beta - a, 0, 0)\)), phase synchronization occurs between the corresponding variables \((y,\hat{y})\) and \((x,\hat{x})\), and the results are illustrated in Fig. 9.

The results in Fig. 9 show that the first group of variables \(x,\hat{x}\) reaches complete synchronization and the corresponding unknown parameter \(a\) is measured and estimated to its real value. Phase synchronization is also observed between the other two groups of variables when the corresponding variables in time domain are converted to frequency domain by using FFT or Hilbert transformation.
Fig. 7. Time series of the identified results for the four unknown parameters in the drive system in Eq. (1) for (a)–(c), and error of the third variables $e_z$ for (d); the phase portrait of $x$ vs. $\dot{x}$ for (e), $\psi(x)$ vs. $\dot{\psi}(x)$ for (f), $y$ vs. $\dot{y}$ for (g), $\psi(y)$ vs. $\dot{\psi}(y)$ for (h). The gain coefficient are selected with $a = 6$, $b = 2$ in controllers as shown in Eq. (27b), and the parameter observers are constructed as shown in Eq. (7). The time series of variable $x$, $y$, $\dot{x}$, $\dot{y}$ are converted to phase positions, and the phase position could be calculated by using Hilbert transformation or Fast Fourier Transform Algorithm (FFT), the real values for the four unknown parameters in the drive system are given with $a = 3$, $b = 40$, $c = 1.6$, $d = 4$. 
Fig. 8. Time series of the identified results for the four unknown parameters in the drive system in Eq. (1) for (a)–(c), and error of the second group of variables $e_y$ for (d); the phase portrait of $x$ vs. $\dot{x}$ for (e), $\phi(x)$ vs. $\psi(x)$ for (f), $z$ vs. $\dot{z}$ for (g), $\phi(z)$ vs. $\psi(z)$ for (h). The gain coefficient are selected with $a = 6$, $b = 2$, and the parameter observers are constructed as shown in Eq. (7). The time series of variable $x$, $z$, $\dot{x}$, $\dot{z}$ are converted to phase positions, and the phase position could be calculated by using Hilbert transformation or Fast Fourier Transform Algorithm (FFT), the real values for the four unknown parameters in the drive system are given with $a = 3$, $b = 40$, $c = 1.6$, $d = 4$. 
Fig. 9. Time series of the identified results for the four unknown parameters in the drive system in Eq. (1) for (a)–(c), and error of the first group of variables $e_i$ for (d); the phase portrait of $y$ vs. $\dot{y}$ for (e), $\phi(y)$ vs. $\dot{\phi(y)}$ for (f), $z$ vs. $\dot{z}$ for (g), $\phi(z)$ vs. $\dot{\phi(z)}$ for (h). The gain coefficients are selected with $a = 6$, $b = 2$ in controllers as shown in Eq. (29b), and the parameter observers are constructed as shown in Eq. (7). The time series of variable $y$, $z$, $\dot{y}$, $\dot{z}$ are converted to phase positions, and the phase position could be calculated by using Hilbert transformation or Fast Fourier Transform Algorithm (FFT), the real values for the four unknown parameters in the drive system are given with $a = 3$, $b = 40$, $c = 1.6$, $d = 4$. 
Compared the results in Fig. 7 and the results in Figs. 8 and 9, it is found that perfect phase synchronization is perturbed when more unknown parameters are not identified exactly, for example, two unknown parameters are identified in Fig. 7 while only one unknown parameter can be estimated, thus much mismatch of parameters between the drive system and the driven system occurs due to the great divergence between the identified parameters and the unknown parameters in the drive system.

Furthermore, we also investigate the case that there are three zero eigenvalues in the Jacobi matrix from the error dynamical system, and the problems are described by

\[ V = 0, \]  

\[ \begin{align*} 
  u_1 &= \hat{a}e_z - \hat{a} e_x, \\
  u_2 &= -xz + \hat{x}z + \hat{b}e_x, \\
  u_3 &= xy - \hat{x}y - \hat{c}e_y - \hat{d}e_z, 
\end{align*} \]

\[ \begin{pmatrix} \dot{e}_x \\ \dot{e}_y \\ \dot{e}_z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix} + \begin{pmatrix} e_o(z - x) \\ e_x \\ -e_y - e_o z \end{pmatrix}. \]

As a result, the error function of the corresponding variables and parameters develops at critical state vs. time, and the results are plotted in Fig. 10.

Phase synchronization still is observed and all the unknown parameters cannot be estimated. Perfect phase synchronization occurs between two groups of corresponding variables \( x \) vs. \( \hat{x} \), \( y \) vs. \( \hat{y} \), and it is confirmed when the time series are converted into phase portion based on Hilbert or FFT. Much mismatch of parameters is induced when the identified results for parameters show certain divergences from the real value of unknown parameters to be identified. Clearly, perfect phase synchronization could be approached when all the parameters are matched completely.

**Fig. 10.** The phase portrait of \( x \) vs. \( \hat{x} \) for (a), \( \phi(x) \) vs. \( \phi(\hat{x}) \) for (b), \( y \) vs. \( \hat{y} \) for (c), \( \phi(y) \) vs. \( \phi(\hat{y}) \) for (d), \( z \) vs. \( \hat{z} \) for (e), \( \phi(z) \) vs. \( \phi(\hat{z}) \) for (f). The gain coefficient are selected with \( \alpha = 6, \beta = 2 \) in controllers as shown in Eq. (30b). The time series of variable \( x, y, z, \hat{x}, \hat{y}, \hat{z} \) are converted to phase positions, and the phase position could be calculated by using Hilbert transformation or Fast Fourier Transform Algorithm (FFT), the real values for the four unknown parameters in the drive system are given with \( a = 3, b = 40, c = 1.6, d = 4 \).
Within the three-variable chaotic system, only one error function \((e_x, e_y, e_z)\) with certain gain ratio is presented in the differential coefficient of positive Lyapunov functions vs. time to induce negative state. The controllers and parameter observers are deduced from the negative differential coefficient of positive Lyapunov functions vs. time analytically, it induces transition from complete synchronization (all the unknown parameters are estimated) to perfect phase synchronization (most of the unknown parameters are identified), then to imperfect phase synchronization (or phase lock) when most of the unknown parameters are not estimated. It could be due to the much mismatch of parameters between the drive and the response chaotic system because the identified parameters oscillate around certain thresholds but never succeed to be stabilized at the real value of the corresponding unknown parameters. On the other hand, the dynamical equations described by the Jacobi matrix as shown in Eqs. (23), (26), (27c), (28c), (29c), (29f) and (30c) indicate that synchronization rate and the convergence of error function also depend on the matching degree of parameters in the two chaotic systems, perfect phase synchronization could be approached distinctly and stronger gain coefficient is helpful to reach this target well.

4. Conclusions

In this paper, an improved scheme is used to realize stable phase synchronization, complete synchronization and parameters identification, two controllable gain coefficients are introduced into the Lyapunov function, which is constructed by all the corresponding variables and parameters, thus the convergence rate of the errors of corresponding variables and the divergence between the identified parameter and the real values of corresponding unknown parameters could be controllable. Based on the Lyapunov stability theory, the two chaotic can reach complete synchronization, stable phase synchronization by deducing different kind of controllers from the stable criterion, which requires the differential coefficient of positive Lyapunov function vs. time become negative. Within complete synchronization, all the unknown parameters could be measured and identified well, while partial parameters could be estimated in the case of phase synchronization. Within the phase synchronization, the time series of the corresponding variables diverge from each other, and the corresponding parameters could not be estimated, while the time series of phase position of the corresponding variables keep pace with each other when the time series of corresponding variables in time domain is converted to phase position in the frequency domain by using the Hilbert transformation or Fast Fourier Transform Algorithm (FFT). To measure the phase synchronization directly, controllers could be deduced from the negative inequality of the differential coefficient of positive Lyapunov function vs. time and the eigenvalues of Jacobi matrix for the dynamical system that consists of the corresponding variables and parameters should be fixed zero at least. And the phase synchronization is stable only if other eigenvalues of Jacobi matrix are negative. Comparing the results in this work with the previous works in this topic, the scheme is improved by introducing controllable gain coefficients to find optimized parameter region to realize parameter estimation and synchronization, particularly, phase synchronization is investigated analytically by using this scheme and it is confirmed to be effective as well.

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