Controllability and observability of switched Boolean control networks

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Abstract: The controllability and observability are investigated for a class of switched Boolean control networks (SBCNs). Using semi-tensor product of matrices, the dynamics of an SBCN can be transformed into an algebraic form. The model-input-state (MIS) matrix of an SBCN is introduced and studied for the first time. This matrix contains complete information of the MIS mapping. A necessary and sufficient condition for the controllability of SBCN is obtained. The corresponding control and switching law, which drive a point to a given reachable point are designed. A sufficient condition for the observability of an SBCN is given. Moreover, under the assumption of controllability, one necessary and sufficient condition is derived for the observability.

1 Introduction

The Boolean network that was firstly introduced by Kauffman [1] becomes a hot topic in system biology, physics and system science. The authors of [2–4] have studied the topological structure of a Boolean network. Another important topic is the control of Boolean networks. Boolean control networks (BCNs) were proposed in [5] to describe gene-regulatory networks. The authors [6, 7] have studied the controllability of BCNs. Li and Sun [8] analyse the observability analysis of BCNs with impulsive effects. Necessary and sufficient conditions for the observability of the BCNs with impulsive effects are obtained. Using the similar method in [8], one can study the observability of the BCN with impulsive effects. In fact, the models of an SBN can be switched according to a defined probability, we call this kind of networks probabilistic Boolean networks (PBNs). Further works currently under submission by Zhao and Cheng have already given a strict proof for controllability of PBCNs.

Recently, a new matrix product, namely, the semi-tensor product (STP) of matrices, was proposed in [9], which generalises the conventional matrix product to two arbitrary matrices. Using STP, a logical equation can be expressed as an algebraic one and the dynamics of a Boolean (control) network can be converted into a linear (bilinear) discrete-time (control) system [10]. For finite value systems, Boolean network and its generalisations are proper models to describe them.

On the other hand, switched systems are very important in control theory. Many natural and engineering systems appear with different models according to the environment changes. For instance, transmission systems, aircraft formation control, robot movement and traffic control are all based on switched control network models. Recently, a new model switched Boolean network is proposed. Switched Boolean networks (SBCNs) have been used to biology systems [11]. In [11], an SBCN model is constructed for the acute myeloid leukaemia (AML) signaling network. AML is characterised by the rapid growth of abnormal white blood cells, which accumulate in the patient bone marrow and perturb the production of normal blood cells. It constructs a model of the AML signalling network by combining the signalling pathways involved in either myeloid differentiation or cell proliferation. However, there is not much theoretical results on SBCNs. Hwang and Lee [11] just uses a simple SBCN model to calculate the output: frequency of the activation. Therefore it is worthwhile to study SBCNs. This paper considers the controllability and observability of SBCNs. The key tools used in this paper are the STP of matrices and the model-input-state (MIS) matrix of SBCNs. The latter one is firstly introduced in this paper.

This MIS matrix idea comes from the input-state incidence matrix of BCN in [12]. The form of the results in [12] is concise and the algorithm proposed can be easily programmed by computer. Motivated by [12], the MIS matrix is introduced to analyse the controllability and observability of SBCNs.

This paper is organised as follows. Some necessary preliminaries including notations and the matrix expression of SBCNs are presented in Section 2. Section 3 introduces the MIS matrix and its some useful properties. Controllability, trajectory tracking and observability of SBCNs are investigated in Sections 4–6, respectively. An illustrative example is given in Section 7. Finally, Section 8 concludes the paper briefly.
2 Preliminaries

2.1 Notations

- \( M_{m \times n} \) is the set of \( m \times n \) real matrices.
- \( \text{Col}(M) \) (\( \text{Row}(M) \)) is the \( i \)th column (row) of matrix \( M \).
- \( \text{Col}(M) \) (\( \text{Row}(M) \)) is the set of columns (rows) of matrix \( M \).
- \( D := \{0, 1\} \).
- \( \delta_i \) is the \( i \)th column of the identity matrix \( I_a \).
- \( \Delta_x := [\delta_1, \delta_2, \ldots, \delta_n] \). When \( n = 2 \) we simply use \( \Delta := \Delta_2 \).
- Assume a matrix \( M = [\delta_1, \delta_2, \ldots, \delta_n] \in M_{m \times n} \). Then \( M \) is called a logical matrix, and denoted simply by \( M = \delta_a[i_1, i_2, \ldots, i_s] \).

The set of \( m \times n \) logical matrices is denoted by \( L_{m \times n} \).

- \( B = [b_{ij}] \in M_{m \times n} \) is called a Boolean matrix if \( b_{ij} \in D \), for all \( i \) and \( j \). The set of \( m \times n \) Boolean matrices is denoted by \( B_{m \times n} \).
- \( A \in M_{m \times n} \). Equally split the columns of \( A \) into \( s \) blocks, then \( \text{Blk}(A) \) is the \( i \)th \( m \times n \) block of \( A, i = 1, 2, \ldots, s \), that is

\[
A = \begin{bmatrix}
\text{Blk}_1(A) & \text{Blk}_2(A) & \cdots & \text{Blk}_s(A)
\end{bmatrix}
\]

- \( A \in M_{m \times n}, B \in M_{p \times q} \). The Kronecker product of matrices is

\[
A \otimes B := \begin{bmatrix}
a_{11}B & a_{12}B & \cdots & a_{1p}B \\
a_{21}B & a_{22}B & \cdots & a_{2p}B \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1}B & a_{m2}B & \cdots & a_{mp}B
\end{bmatrix}
\]

- \( A \in M_{m \times n}, B \in M_{p \times q} \). Then the STP of \( A \) and \( B \) is

\[
A \times B := (A \otimes I_{j/p})(B \otimes I_{i/j})
\]

where \( s = \text{lcm}(n, p) \) is the least common multiple of \( n \) and \( p \). The matrix product in this paper is assumed to be STP.

- Define Boolean addition \( +_B \) and Boolean product \( \times_B \) as

\[
a +_B b := a \lor b, \quad a \times_B b := a \land b, \quad a, b \in D
\]

If \( A, B \in B_{m \times n} \), then

\[
A +_B B := (a_{ij} +_B b_{ij}) \in B_{m \times n}
\]

If \( A \in B_{m \times n} \) and \( B \in B_{n \times p} \), then

\[
A \times_B B = (c_{ij}) \in B_{m \times p}
\]

where

\[
c_{ij} = a_{i1} \times_B b_{1j} +_B \cdots +_B a_{im} \times_B b_{mj} := \sum_{k=1}^m a_{ik} \times_B b_{kj} \]

If \( A \in B_{n \times n} \), we denote

\[
A^{(k)} = A \times_B \cdots \times_B A \in B_{n \times n}
\]

2.2 Matrix expression of SBCNs

A BCN with \( n \)-network nodes, \( m \) inputs, and \( p \) outputs can be described (as see (5))

where \( x_i, u_i \in D, i = 1, 2, \ldots, n, l = 1, \ldots, m \), \( f_i : D^{2m} \rightarrow D, i = 1, 2, \ldots, n \) and \( h_j : D^p \rightarrow D, j = 1, 2, \ldots, p \) are logical functions.

Let

\[
X(t) = (x_1(t) \cdots x_n(t))^T, \quad U(t) = (u_1(t) \cdots u_m(t))^T, \quad Y(t) = (y_1(t) \cdots y_p(t))^T
\]

Then (5) can be briefly expressed as

\[
\begin{align*}
X(t+1) &= F(U(t), X(t)) \\
Y(t) &= H(X(t))
\end{align*}
\]

where

\[
F(U(t), X(t)) = \begin{bmatrix}
f_1(u_1(t), \ldots, u_m(t), x_1(t), \ldots, x_n(t)) \\
f_2(u_1(t), \ldots, u_m(t), x_1(t), \ldots, x_n(t)) \\
\vdots \\
f_m(u_1(t), \ldots, u_m(t), x_1(t), \ldots, x_n(t))
\end{bmatrix}
\]

\[
H(X(t)) = \begin{bmatrix}
h_1(x_1(t), \ldots, x_n(t)) \\
h_2(x_1(t), \ldots, x_n(t)) \\
\vdots \\
h_p(x_1(t), \ldots, x_n(t))
\end{bmatrix}
\]

Consider a switched BCN

\[
\begin{align*}
X(t+1) &= F_0(U(t), X(t)) \\
Y(t) &= H(X(t))
\end{align*}
\]

where

\[
\sigma(t) : \mathbb{Z}_+ \rightarrow \Lambda = \{1, 2, \ldots, N\}
\]

is the switching law. The switching models \( F_i, i = 1, 2, \ldots, N \) are determined by

\[
F_i(U(t), X(t)) = \begin{bmatrix}
f^{(i)}_1(u_1(t), \ldots, u_m(t), x_1(t), \ldots, x_n(t)) \\
f^{(i)}_2(u_1(t), \ldots, u_m(t), x_1(t), \ldots, x_n(t)) \\
\vdots \\
f^{(i)}_m(u_1(t), \ldots, u_m(t), x_1(t), \ldots, x_n(t))
\end{bmatrix}
\]

\[
\begin{bmatrix}
x_1(t+1) \\
x_2(t+1) \\
\vdots \\
x_n(t+1)
\end{bmatrix} = \begin{bmatrix}
f_1(u_1(t), u_2(t), \ldots, u_m(t), x_1(t), x_2(t), \ldots, x_n(t)) \\
f_2(u_1(t), u_2(t), \ldots, u_m(t), x_1(t), x_2(t), \ldots, x_n(t)) \\
\vdots \\
f_m(u_1(t), u_2(t), \ldots, u_m(t), x_1(t), x_2(t), \ldots, x_n(t))
\end{bmatrix}
\]

\[
y_j(t) = h_j(x_1(t), x_2(t), \ldots, x_n(t)), \quad j = 1, \ldots, p
\]
Assume the switching law (10) is designable. Let $\Sigma$ denote the model corresponding to $F_i$.

In order to use matrix expression, we identify $1 \sim \delta_1$ and $0 \sim \delta_2$, equivalently, $D \sim \Delta$. Thus, we can equivalently consider the mapping $F : D^s \rightarrow D^k$ as a mapping $F : \Delta^s \rightarrow \Delta^k$. Consequently, the BCN (5) can be expressed into its algebraic form [13] as

$$
\begin{bmatrix}
    x(t+1) \\
    y(t)
\end{bmatrix} =
\begin{bmatrix}
    L_{\alpha(t)}(u(t))x(t) \\
    Hx(t)
\end{bmatrix}
$$

(12)

where $x = \kappa(x, x_1) \in \Delta^s$, $y = \kappa(y, y_1) \in \Delta^s$, $u = \kappa(u, u_1) \in \Delta^s$ and $L \in \mathcal{L}_{\Delta^s, \Delta^s}$ is the structure matrix of system (5). Let $L_1$ be the structure matrix corresponding to the model $\Sigma_1$. Then the algebraic form of SBCN (9) can be expressed as

$$
\begin{bmatrix}
    x(t+1) \\
    y(t)
\end{bmatrix} =
\begin{bmatrix}
    L_{\alpha(t)}(u(t))x(t) \\
    Hx(t)
\end{bmatrix}
$$

(13)

where $[L_{\alpha(t)}, t = 1, 2, \ldots]$ is a sequence of logical matrices. $L_{\alpha(t)}$ is the structure matrix corresponding to the model $\Sigma(t)$. Here, we have an example of the algebraic form of an SBCN.

**Example 1:** Let $X(t) = (x_1(t) x_2(t))^T$, $U(t) = u(t)$. Assume the dynamics of an SBCN is

$$X(t+1) = F_{\alpha(t)}(U(t), X(t))$$

(14)

where $\alpha(t) : \mathbb{Z} \rightarrow \{1, 2, 3, 4\}$. The switching models $F_i, i = 1, 2, 3, 4$ are

$$
F_1 = \begin{bmatrix} u \land x_1 & u \rightarrow x_1 \\ u & \end{bmatrix},
F_2 = \begin{bmatrix} u \land x_1 & u \rightarrow x_1 \\ u \land x_2 & u \rightarrow x_2 \\ u & \end{bmatrix},
F_3 = \begin{bmatrix} u \land x_2 & u \rightarrow x_2 \\ u & \end{bmatrix},
F_4 = \begin{bmatrix} u \land x_2 & u \rightarrow x_2 \\ u \land x_1 & u \rightarrow x_1 \\ u & \end{bmatrix}
$$

The corresponding structure matrices are

$$
L_1 = \delta_1[1 \ 1 \ 4 \ 3 \ 3 \ 3 \ 3], \ L_2 = \delta_1[1 \ 2 \ 3 \ 4 \ 3 \ 3 \ 3], \ L_3 = \delta_1[1 \ 3 \ 2 \ 4 \ 3 \ 3 \ 3], \ L_4 = \delta_1[1 \ 4 \ 1 \ 4 \ 3 \ 3 \ 3]
$$

(15)

### 3 MIS matrices of SBCNs

We introduce the concept of MIS matrix to describe dynamic process of an SBCN in this section. It is a generalisation of incidence matrix of BCN in [12].

Denote $\mathcal{U}$ as the corresponding input set and $\chi$ as the state set of (13). First, we define the set of points in the product space as $\{Q_i\} = \mathcal{U} \times \chi$, where $i = 1, \ldots, N_0, N_0 = N \times 2^m \times 2^m$ and $N, m, n$ are defined in Section 2.

For instance, consider Example 1. We obtain $N_0 = 4 \times 2^1 \times 2^2 = 32$ and the corresponding points $Q_i, i = 1, \ldots, 32$ are as follows:

$$
\begin{align*}
Q_1 : \Sigma_1, u(t) = \delta_1, x(t) = \delta_1 & := (1, 1, 1) \\
Q_2 : \Sigma_1, u(t) = \delta_2, x(t) = \delta_1 & := (1, 1, 2) \\
& \vdots \ \\
Q_4 : \Sigma_2, u(t) = \delta_2, x(t) = \delta_2 & := (a, b, c) \\
& \vdots \ \\
Q_{32} : \Sigma_4, u(t) = \delta_2, x(t) = \delta_1 & := (4, 2, 4)
\end{align*}
$$

(16)

According to (14) we draw the flow of $(\Sigma(t), u(t), x_1(t), x_2(t))$ on the product space $\{\Sigma_i\} \times \mathcal{U} \times \chi$, which is called the MIS dynamic graph. Since the graph is too large to be drawn here, we only give the neighbourhood of $Q_1$ as in Fig. 1.

Now we construct MIS matrix $J \in \mathbb{B}_{N_0 \times N_0}$ of (13) in the following way

$$
J_{ij} = \begin{cases} 
1 & \text{there exists an edge from } Q_i \text{ to } Q_j \\
0 & \text{otherwise}
\end{cases}
$$

(17)

If $J$ is the MIS matrix of (14), we can write the first column of $J$ from Fig. 1 as

$$
\begin{bmatrix}
    J_{01} \\
    J_{02} \\
    J_{03} \\
    \vdots \\
    J_{0N_0}
\end{bmatrix}
$$

(18)

From the constructing procedure of $J$, it is easy to see that property (18) is also true for general case. That is, the MIS matrix of system (13) is

$$
\begin{bmatrix}
    J_{01} \\
    J_{02} \\
    J_{03} \\
    \vdots \\
    J_{0N_0}
\end{bmatrix}
$$

(19)

$J_0$ is called the basic block of $J$. From Proposition 2.5 in [12], we obtain the following result.

**Proposition 1:**

$$(J^{i+1})_0 = M^i J_0$$

(20)

where

$$M = \sum_{i=0}^{2^m} B_{j_k} (J_0)$$

(21)

**Proof:** Replacing $2^m$ with $2^m N$ in Proposition 2.5 of [12], we can easily obtain the proposition. $\square$

In fact, we have the following proposition, which shows the physical meaning of MIS matrix.

**Proposition 2:** Consider system (13). Denote the $(i,j)$th element of the $s$th power of its MIS matrix as $(J^{s})_{ij} = c$. Then there are $c$ paths from point $Q_i$ to $Q_j$ at $s$th step with proper controls and models.

![Fig. 1 Neighbourhood of $Q_1$](image-url)
Proof: When \( s = 1 \) the conclusion is obvious. From the constructing procedure of the MIS matrix, we know that \( J_i \) means whether there exists a set of controls and a model such that \( Q_i \) is reachable from \( Q_j \) in one step by judging if \( J_i \) = 1 or not.

Now assume \((J_s)_{ij}\) is the number of paths from point \( Q_j \) to \( Q_i \) at \( s \)th step. Since a path from \( Q_j \) to \( Q_i \) at \((s + 1)\)th step can be considered as a path from \( Q_j \) to \( Q_i \) at \( s \)th step and from \( Q_k \) to \( Q_i \) at one step. Thus

\[
c = \sum_{k=1}^{N} J_{ik}(J_s^s)_{ij}
\]

which is exactly \((J_s^{s+1})_{ij}\). The proof is completed. \( \square \)

From the proposition above the following result is obvious.

**Corollary 1:** Consider the MIS matrix \( J \) of SBCN (13). \( Q_i \) is reachable from \( Q_j \) at \( s \)th step, iff \( J_i > 0 \).

**4 Controllability of SBCNs**

In this section, we consider the reachability and controllability of SBCNs. We give the following two new definitions first.

**Definition 1:** Consider the SBCN (9).

1. A state \( X_i \in D^s \) is said to be reachable from \( X_s \), if there exist a switching law \( \sigma(t) : Z_s \rightarrow \{1, 2, \ldots\} \) and a control sequence \( \{u(t), t = 0, 1, 2, \ldots, s-1\} \), such that the corresponding model-input pairs \( \{\Sigma(t), u(t), t = 0, 1, 2, \ldots, s-1\} \), where \( 0 < s < \infty \), makes the trajectory of the controlled network reaching \( X_i \) from \( X_s \) at time \( s \);
2. \( X_i \in D^s \) is said to be reachable, if it is reachable from any \( X_j \in D^s \);
3. The SBCN (9) is said to be reachable, if any \( X_i \in D^s \) is reachable.

**Definition 2:** Consider the SBCN with form (9).

1. The SBCN is said to be controllable from \( X_i \) to \( X_j \in D^s \), if there exist a switching law \( \sigma(t) : Z_s \rightarrow \{1, 2, \ldots\} \) and a control sequence \( \{u(t), t = 0, 1, 2, \ldots, s-1\} \), such that the corresponding model-input pairs \( \{\Sigma(t), u(t), t = 0, 1, 2, \ldots, s-1\} \), where \( 0 < s < \infty \), drive the trajectory from \( X(0) = X_i \) to \( X(s) = X_j \);
2. The SBCN is said to be controllable at \( X_i \), if it is controllable from \( X_i \) to any \( X_j \in D^s \);
3. The SBCN is said to be controllable, if it is controllable at any \( X_i \in D^s \).

**Remark 1:** The SBCNs are finite-state systems, the reachability is obviously equivalent to the controllability.

**Remark 2:** The \( X_0 \in D^s \) in system (9) corresponds to \( X_0 \in \Delta_{2^n} \) in the equivalent system (13).

Equally split the basic block of \( J_s \) of (13) into \( 2^nN \) blocks as

\[
J_s = [Blk_1(J_0), Blk_2(J_0), \ldots, Blk_{2^nN}(J_0)]
\]

where \( m, n, N \) are same as those defined in Section 2.

Using Boolean algebra, from Proposition 1, we know \( Blk_i(J_0) = M_l^{(i-1)}Blk_i(J_0) \). Convert the two-index \((i,j) = (M_l^{(j))}, 2^N) \) into a single-index \((\mu(i,j); 2^N) \) [14]. Then \( \mu(i,j) = 1, \ldots, 2^N \). By the constructing process, it is clear that \( Blk_i(J_0) = M_l^{(i)} \) corresponds to the \( \mu(i,j) \)th model-input pair \((\Sigma = \Sigma_i, u = \delta_0^{(i)}) \). Moreover, each \( Col_i[Blk_i(J_0)](J_0) \) corresponds to the state \( x = \delta_0^{(i)} \). Based on this observation, we can prove the following two theorems.

**Theorem 1:** Assume the MIS matrix of system (13) is \( J_0 \) and \( J_0 \) is the basic block of \( J \). System (13) is reachable, iff

\[
C_0 := \sum_{i=1}^{2^nN-1} \beta_i M_{ij}^{(0)} > 0, \quad M_{ij}^{(0)} = \sum_{i=1}^{2^nN} \beta_i Blk_i(J_0)
\]

**Proof:** From Corollary 1 and (21), we know that \( x(s) = \delta_x^{(i)} \) is reachable from \( x(0) = \delta_x^{(0)} \) at \( s \)th step, iff

\[
\sum_{i=1}^{2^nN-1} \beta_i (Blk_i(J_0))_{ij} = (M_{ij}^{(0)})_{ij} > 0
\]

Then, \( x = \delta_x^{(i)} \) is reachable from \( x(0) = \delta_x^{(0)} \), iff

\[
\sum_{i=1}^{2^nN-1} \beta_i (M_{ij}^{(0)})_{ij} > 0
\]

Thus, \( x = \delta_x^{(i)} \) is reachable (from any initial states), iff

\[
\sum_{i=1}^{2^nN-1} \beta_i \text{Row}_{a}(M_{ij}^{(0)}) > 0
\]

Hence we can conclude that system (13) is reachable iff \( C_0 > 0 \). \( \square \)

Furthermore from Remark 1, we obtain the following theorem.

**Theorem 2:** System (13) is controllable, iff

\[
C_0 > 0
\]

**5 Trajectory tracking and control-switching-law design**

The purpose of this section is finding a control and a proper switching law, which drive \( x_0 \) to \( x_2 \). The trajectory from \( x_0 \) to \( x_2 \) is not unique in general. In the following, we propose an algorithm to find a shortest trajectory. Let \( x_0 = \delta_x^{(i)} \) and \( x_2 = \delta_x^{(2)} \). We give the following algorithm. \( J_0 \) is the basic block of MIS matrix \( J \).

**Algorithm 1:** Let the \((i,j)\)th element of the controllability matrix \( C_{ij} > 0 \).

- **Step 1:** Find the smallest \( s \), such that in the block decomposed form

\[
J_s^{(i)} = [Blk_1(J_0^{(i)}), \ldots, Blk_{2^nN}(J_0^{(i)})]
\]

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Fig. 2 Algorithm 5.1 Step 1

there exists a block, $\text{Blk}_u(J_0^{(0)})$, which has its $(i,j)$th element

$$[\text{Blk}_u(J_0^{(0)})]_{ij} > 0$$  \hspace{1cm} (28)

Change single-index $(\alpha(\alpha_1, \alpha_2); 2mN)$ into double-index $(\alpha_1, \alpha_2; N, 2^m)$. Then, set $\Sigma(0) = \Sigma_0$, $u(0) = \delta_2^{(0)}$ and $x(s) = \delta_2^{(0)}$. If $s = 1$, stop. Else, go to next step (see Fig. 2).

- Step 2: Find $k, \beta$, such that

$$[\text{Blk}_\beta(J_0)]_{ik} > 0; \ [\text{Blk}_u(J_0^{(s-1)})]_{ij} > 0$$  \hspace{1cm} (29)

Change single-index $(\beta(\beta_1, \beta_2); 2mN)$ into two-index ($\beta_1, \beta_2; N, 2^m$). Then, set $\Sigma(s - 1) = \Sigma_\beta$, $u(s - 1) = \delta_2^{(0)}$, and $x(s) = \delta_2^{(0)}$. If $s = 1$, stop. Else, replace $s$ by $s - 1$ and replace $i$ by $k$, and go back to Step 2.

Proposition 3: As long as $x_t$ is reachable from $x_0$, the control sequence $\{u(0), \ldots, u(s-1)\}$ and model sequence $\{\Sigma(0), \ldots, \Sigma(s-1)\}$ generated by Algorithm 1 can drive the trajectory from $x_0$ to $x_t$.

Proof: Since $x_t$ is reachable from $x_0$, there exists the smallest $s$ such that $[\text{Blk}_u(J_0^{(0)})]_{ij} > 0$. That means if $\Sigma(0) = \Sigma_u$, $u(0) = \delta_2^{(0)}$, and $x(0) = \delta_2^{(0)}$, there exists at least one path from $x(0)$ to $x(s) = \delta_2^{(0)}$. Hence, it is obvious that there must exist $k$ such that $x_0$ can reach $\delta_2^{(0)}$ at step $s$ with $u(0) = \delta_2^{(0)}$, and $\beta(\beta_1, \beta_2)$ such that $u(s - 1) = \delta_2^{(0)}$, which makes $\text{L}_k \delta_2^{(0)} = \delta_2^{(0)}$. Equivalently, we can find $k, \beta$, such that

$$[\text{Blk}_\beta(J_0)]_{ik} > 0; \ [\text{Blk}_u(J_0^{(s-1)})]_{ij} > 0$$

In the same way, we can find $\beta'$ and $k'$ such that $\delta_2^{(0)}$ can be reached at step $(s - 2)$ with $\text{L}_k \delta_2^{(0)} = \delta_2^{(0)}$. Continuing this process, the switching law and the sequence of controls and states from $x_0$ to $x_t$ can be obtained.

6 Observability of SBCNs

In this section, we consider the observability of SBCNs. To this end, we first introduce the following definitions.

Definition 3: Consider SBCN system (9). Let $X^0_1, X^0_2 \in \mathbb{D}^n$ be two initial states and $U(t) \in \mathbb{D}^m$ be inputs.

1. $X^0_1, X^0_2$ are said to be distinguishable, if there exist a switching law $\sigma(t) : \mathbb{Z}_+ \rightarrow \{1, 2, \ldots, s\}$, an integer $s \geq 0$, and a control sequence $\{U(t), t = 0, 1, 2, \ldots, s\}$, such that the corresponding model-input sequence

$$\{(\Sigma(0), U(0)), \ldots, (\Sigma(s), U(s))\}, \ s \geq 0$$

leads the two outputs at time $s + 1$ different. That is

$$Y_1(s + 1) \neq Y_2(s + 1)$$

2. The system is said to be observable, if any two initial states $X^0_1, X^0_2 \in \mathbb{D}^n$ are distinguishable.

Consider MIS matrix $J$. Blk$_{u(i,j)}(J_0^{(0)})$ corresponds to the $\mu(i,j)$th model-input pair ($\Sigma = \Sigma_i, u = \delta_u$). Moreover, each Col$_{l}[\text{Blk}_u(J_0^{(0)})]$ corresponds to the state $x = \delta_x$. To exchange the running order of the indices between $(\Sigma(t), u(t))$ and $(x(t))$, we use swap matrix to define

$$\bar{J}_0^{(s)} := J_0^{(s)} W_{2^s \times 2^m}$$  \hspace{1cm} (30)

Denote the $j$th block of $J_0^{(s)}$ as $\text{Blk}_k(J_0^{(s)}) \in B_{2^s \times 2^m}, j = 1, 2, \ldots, 2^s$.

Now each block $\text{Blk}_k(J_0^{(s)})$ corresponds to $x = \delta_x$, and each block Col$_{i}[\text{Blk}_k(J_0^{(s)})]$ corresponds to the model-input pair $(\Sigma = \Sigma_i, u = \delta_u)$. Then we get the following lemma obviously.

Lemma 1: The set Col$[\text{Blk}_k(J_0^{(s)})]$ consists of all the state $x(t + s)$ corresponding to $x(t) = \delta_x$ by choosing $2^m$ kinds of possible model-input pairs.

Using Lemma 1 we have the following sufficient condition for the jointly observability. It has the similar form as the SBN situation in [12].

Theorem 3: Consider system (13) and its MIS matrix $J$, $J_0$ is the basic block of $J$. If

$$\bigvee_{i=1}^{2^m \times 2^m} \left( (H \times \text{Blk}_k(J_0^{(s)})) \sigma (H \times \text{Blk}_k(J_0^{(s)})) \right) \neq 0$$  \hspace{1cm} (31)

then system (13) is observable.

Proof: According to Lemma 1, Col$[\text{Blk}_k(J_0^{(s)})]$ consists of all the possible state $x(t + s)$ corresponding to $x(t) = \delta_x$. Thus, the columns of $H \times \text{Blk}_k(J_0^{(s)})$ and $H \times \text{Blk}_k(J_0^{(s)})$ are all the possible outputs corresponding to two different states $\delta_x$ and $\delta_x$, respectively. It is easy to see that (31) implies that at least one step the outputs corresponding to $\delta_x$ and $\delta_x$ are distinct. The proof is complete.

In the following we give a necessary and sufficient condition for the observability of SBCNs under the assumption of controllability. The constructing technique comes from [15].

Split $J_0$ into $2^mN$ equal blocks as

$$J_0 = [B_1, B_2, \ldots, B_{2^mN}]$$  \hspace{1cm} (32)

where $[B_1, B_2, \ldots, B_{2^mN}] := [\text{Blk}_1(J_0), \text{Blk}_2(J_0), \ldots, \text{Blk}_{2^mN}(J_0)], B_i \in \mathcal{L}_{2^s \times 2^s}, i = 1, 2, \ldots, 2^mN$.

Define a sequence of sets of matrices $\Gamma_i \in \mathcal{L}_{2^s \times 2^s}, i = 0, 1, \ldots$, as

$$\Gamma_0 = \{H\}$$
$$\Gamma_i = \{HB_i | i = 1, \ldots, 2^mN\}$$

$$\Gamma_i = \{HB_{i_1}B_{i_2} \cdots B_{i_t} | i_1, \ldots, i_t = 1, \ldots, 2^mN\}$$  \hspace{1cm} (33)
Note that $\Gamma_j \subset \mathcal{L}_{2^m,2^n}$ is a finite set. Thus, there exists the smallest $s^* > 0$ such that

$$\Gamma_j \subset \bigcup_{i=1}^{s^*} \Gamma_i, \forall j > s^*$$

We also use $\Gamma_j$ as the matrix, consisting of its elements and arranging in a column. For instance

$$\Gamma_1 = \begin{bmatrix} HB_1 \\ HB_2 \\ \vdots \\ HB_{2^mN} \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} HB_1 B_1 \\ HB_1 B_2 \\ \vdots \\ HB_{2^mN} B_{2^mN} \end{bmatrix}, \ldots$$ (34)

Set the observability matrix for SBCNs as

$$O_s = \begin{bmatrix} \Gamma_0 \\ \Gamma_1 \\ \vdots \\ \Gamma_s \end{bmatrix}$$ (35)

Then we obtain the following theorem.

**Theorem 4:** Assume that system (13) is controllable. Then it is observable, iff

$$\text{rank}(O_s) = 2^s$$ (36)

**Proof:** (Sufficiency) Let the initial state be $x_0$. Define a time sequence $t^i_i | i = 1, 2, \ldots, 2^mN$ satisfying

$$t^i_{i+1} > t^i_i + 1, \quad i = 0, 1, \ldots, 2^mN - 1 \text{ with } t^i_0 := 0$$

Using $\Sigma(t^i_{i+1}) = \Sigma(i), u(t^i_{i+1}) = \delta_{2^m}$, $\mu(i, j) = 1, \ldots, 2^mN$, $i = 1, \ldots, N$, $j = 1, \ldots, 2^m$, it is easy to see that

$$\begin{bmatrix} y(t^i_1 + 1) \\ y(t^i_2 + 1) \\ \vdots \\ y(t^i_{2^mN} + 1) \end{bmatrix} = \begin{bmatrix} HB_1 x(t^i_1) \\ HB_1 x(t^i_2) \\ \vdots \\ HB_{2^mN} x(t^i_{2^mN}) \end{bmatrix}$$

Since system (13) is controllable there exists proper input such that $x(t^i_1) = x_0$. Thus

$$\begin{bmatrix} y(t^i_1 + 1) \\ y(t^i_2 + 1) \\ \vdots \\ y(t^i_{2^mN} + 1) \end{bmatrix} = \Gamma_s x_0$$

In general, assume that we have a time sequence $t^i_{i+1}, t^i_i = 1, \ldots, 2^mN, k = 1, \ldots, s$. Convert the multi-index $(i_1, i_2, \ldots, i_r; 2^mN, \ldots, 2^mN)$ into a single-index $(v(i_1, \ldots, i_r); 2^mN^r)$. We assume that this time sequence satisfies

$$t^i_{i+1} > t^i_i + 1, \quad i = 0, 1, \ldots, 2^mN - 1 \text{ with } t^i_0 := 0$$

Let $i_r = \mu(i^r, j^r), r = 1, \ldots, s$, we use

$$\Sigma(t^i_{i-r+1}) = \Sigma(i^r), \quad u(t^i_{i-r+1}) = \delta_{2^m}^i$$

Then we obtain the following theorem.

$$\text{rank}(O_s) = 2^s$$ (36)

**Proof:** (Sufficiency) Let the initial state be $x_0$. Define a time sequence $t^i_i | i = 1, 2, \ldots, 2^mN$ satisfying

$$t^i_{i+1} > t^i_i + 1, \quad i = 0, 1, \ldots, 2^mN - 1 \text{ with } t^i_0 := 0$$

Using $\Sigma(t^i_{i+1}) = \Sigma(i), u(t^i_{i+1}) = \delta_{2^m}$, $\mu(i, j) = 1, \ldots, 2^mN$, $i = 1, \ldots, N$, $j = 1, \ldots, 2^m$, it is easy to see that

$$\begin{bmatrix} y(t^i_1 + 1) \\ y(t^i_2 + 1) \\ \vdots \\ y(t^i_{2^mN} + 1) \end{bmatrix} = \begin{bmatrix} HB_1 x(t^i_1) \\ HB_1 x(t^i_2) \\ \vdots \\ HB_{2^mN} x(t^i_{2^mN}) \end{bmatrix}$$

Since system (13) is controllable there exists proper input such that $x(t^i_1) = x_0$. Thus

$$\begin{bmatrix} y(t^i_1 + 1) \\ y(t^i_2 + 1) \\ \vdots \\ y(t^i_{2^mN} + 1) \end{bmatrix} = \Gamma_s x_0$$

Note that here we assume that the sets of time sequences are apart from each other far enough. Precisely

$$t^i_1 1 \ldots 1 > (2^mN 2^mN \ldots 2^mN)^{\delta_{2^m}} + s$$

Finally, we have

$$\Theta_s x_0 = \begin{bmatrix} \Gamma_0 x_0 \\ \Gamma_1 x_0 \\ \vdots \\ \Gamma_s x_0 \end{bmatrix} = \begin{bmatrix} y(0) \\ y(t^i_1 + 1) \\ y(t^i_2 + 1) \\ \vdots \\ y(t^i_{2^mN} + 1) \\ \vdots \\ y(t^i_1 + s^*) \\ y(t^i_2 + s^*) \\ \vdots \\ y(t^i_{2^mN} + s^*) \end{bmatrix} := Y$$ (37)

Note that $Y$ is a set of observed data. Then since $\text{rank}(\Theta_s) = 2^s$, $x_0$ can be uniquely solved out as

$$x_0 = (\Theta_s^T \Theta_s)^{-1} \Theta_s^T Y$$

(Necessity) By the definition of $s^*$ it is easy to see that $Y$ contains all possible outputs. Now if $\text{rank}(\Theta_s) < 2^s$, one sees easily that in addition to $x_0$ there exists at
least another solution \( x'_0 \) of (37). Then \( x_0 \) and \( x'_0 \) are not distinguishable. \( \square \)

7 Illustrative example

Let \( X(t) = (x_1(t)x_2(t))^T, U(t) = u(t) \). Assume the dynamics of an SBCN is

\[
\begin{align*}
X(t + 1) &= F_{\sigma(t)}(U(t), X(t)) \\
y(t) &= x_1(t) \lor x_2(t)
\end{align*}
\]

(38)

where \( \sigma(t): \mathbb{Z}_+ \rightarrow \{1, 2\} \). The switching models \( F_i, i = 1, 2 \) are

\[
F_1 = \begin{bmatrix} u \land x_1 \\ u \rightarrow x_1 \end{bmatrix}, \quad F_2 = \begin{bmatrix} u \land x_2 \\ u \rightarrow x_1 \end{bmatrix}
\]

Then we obtain the structure matrices of \( F_1 \) and \( F_2 \), respectively, as

\[
L_1 = \delta_4[1 1 4 4 3 3 3 3], \quad L_2 = \delta_4[1 3 2 4 3 3 3 3]
\]

The MIS matrix of the system is

\[
J = \begin{bmatrix} J_0 \\ J_0 \\ J_0 \\ J_0 \end{bmatrix}
\]

(39)

where \( J_0 = \delta_4[1 1 4 4 3 3 3 3 1 3 2 4 3 3 3 3] \).

Then, we have

\[
M_B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}
\]

(40)

It follows that

\[
M_B^{(2)} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad M_B^{(3)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}
\]

(41)

Finally, we obtain

\[
C_p = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \succ 0
\]

(42)

According to Theorem 2 system (38) is controllable. From (34), we know\[
\Gamma_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \in B_{32 \times 4}
\]

(43)

and

\[
\text{rank}(\Gamma_2) = 4
\]

Since

\[
\text{rank}(\Gamma_2) \leq \text{rank}(O_r) \leq 4
\]

we obtain

\[
\text{rank}(O_r) = 4
\]

Thus, system (38) is observable.

8 Conclusions

In this paper, the controllability, trajectory tracking and observability of SBCNs have been investigated. A new concept, namely, the MIS matrix for SBCN, is introduced. In the light of MIS matrix, the complete information about possible trajectories has been put into a certain matrix. A necessary and sufficient condition for controllability and reachability has been derived. Then a necessary condition for observability is revealed for the first time. Moreover, under the assumption of controllability a necessary and sufficient condition for the observability is also obtained.

The weakness of the approach proposed in this paper is the computational complexity. Since the MIS matrix is of dimension \( 2^{m+n}N \times 2^{m+n}N \) and the observability matrix is also not small, especially when \( N \) is large. We can use STP toolbox based on MATLAB to calculate these matrices in PC. However, even in STP toolbox we cannot deal with the matrix with dimension bigger than \( 2^{25} \).
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10 References