Linear fractional differential equations with variable coefficients

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Abstract

This work is devoted to the study of solutions around an \( \alpha \)-singular point \( x_0 \in [a, b] \) for linear fractional differential equations of the form \( [L_{\alpha}(y)](x) = g(x, \alpha) \), where

\[
[L_{\alpha}(y)](x) = y^{(\alpha)}(x) + \sum_{k=0}^{n-1} a_k(x) y^{(k\alpha)}(x)
\]

with \( \alpha \in (0, 1] \). Here \( n \in \mathbb{N} \), the real functions \( g(x) \) and \( a_k(x) \) (\( k = 0, 1, \ldots, n-1 \)) are defined on the interval \([a, b]\), and \( y^{(\alpha)}(x) \) represents sequential fractional derivatives of order \( k\alpha \) of the function \( y(x) \). This study is, in some sense, a generalization of the classical Frobenius method and it has applications, for example, in obtaining generalized special functions. These new special functions permit us to obtain the explicit solution of some fractional modeling of the dynamics of many anomalous phenomena, which until now could only be solved by the application of numerical methods.1

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1. Introduction

Let \( \Omega = [a, b] \) be a finite interval on the real axis \( \mathbb{R} \). Let \( x \in \Omega \) and \( \alpha \in \mathbb{R} \) (\( 0 < \alpha \leq 1 \)). The Riemann–Liouville fractional integrals \( I_{a+}^{\alpha} \) and derivative \( D_{a+}^{\alpha} \) are defined by (see, for example, [10])

\[
(I_{a+}^{\alpha} y)(x) := \frac{1}{\Gamma(\alpha)} \int_{a}^{x} \frac{y(t) dt}{(x-t)^{1-\alpha}} \quad \text{and} \quad (D_{a+}^{\alpha} y)(x) := \left( \frac{d}{dx} \right) (I_{a+}^{1-\alpha} y)(x).
\] (1.1)

Additionally, we have the Caputo fractional derivatives (\( C_{a+}^{\alpha} y)(x) := (I_{a+}^{1-\alpha} D_{a+} y)(x) \) (see [3]). It is well known that the two derivatives are related, for suitable functions (see, for example, [10]).
We will work here following the definition of sequential fractional derivative presented by Miller and Ross [8]

\[ D^{\alpha}_a y := \sum_{k=0}^{n-1} a_k (y^{(k)} + \frac{(t-a)^{\alpha-k}}{\Gamma(\alpha-k)}) \]

\[ (D^{\alpha}_a y)(x) = \left( \sum_{k=0}^{n-1} a_k (y^{(k)} + \frac{(t-a)^{\alpha-k}}{\Gamma(\alpha-k)}) \right)(x). \]  

(1.2)

We will use the following notation to write a linear fractional differential equation (LFDE) of order \( n\alpha \) (\( n \in \mathbb{N} \) and \( 0 < \alpha \leq 1 \)):

\[ [L_{n\alpha}(y)](x) := (D^{n\alpha}_a y)(x) + \sum_{k=0}^{n-1} a_k (x) \left( D^{\alpha}_a y \right)(x) \equiv y^{(n\alpha)}(x) + \sum_{k=0}^{n-1} a_k(x)y^{(k\alpha)}(x) = f(x), \]  

(1.3)

where \( y^{(0)}(x) := y(x) \), and \( y^{(k\alpha)}(x) := (D^{k\alpha}_a y)(x) (k = 1, \ldots, n) \) represents a fractional sequential derivative.

Fractional order differential equations, that is, those which involve real or complex order derivatives, have become a very important tool for modeling the anomalous dynamics of numerous processes involving complex systems found in many diverse fields of science and engineering (see, for example, Kilbas et al. [5]). We should point out that since about 1990 there has been a tremendous increase in the use of fractional models for simulating the dynamics of various anomalous processes, especially those involving ultraslow diffusion; see in this regard the extraordinary monograph by Metzler and Klafter [7]. To give a wider view of the large different fields where fractional models have found application, among many others, we note as relevant the following references: Podlubny [9], Hilfer [4], Zaslavsky [11], and Kilbas et al. [5].

We should note that fractional modeling in applied fields is encountering serious difficulties in advancing since only integral transforms are being used. This is because many of the mathematical resources available for ordinary cases do not have fractional equivalents. In this sense we can assure that the method of separation of variables or the use of special functions which proceed from the corresponding generalizations of differential equations associated with classical special functions, which has had such an important role in developing the ordinary models, will aid in advancing the application of fractional differential equations to the modeling of certain complex systems.

We shall use the following concept of an \( \alpha \)-analytic function:

**Definition 1.1.** Let \( \alpha \in (0, 1] \), \( f(x) \) be a real function defined on the interval \([a, b]\), and \( x_0 \in [a, b] \). Then \( f(x) \) is said to be \( \alpha \)-analytic at \( x_0 \) if there exists an interval \( N(x_0) \) such that, for all \( x \in N(x_0) \), \( f(x) \) can be expressed as a series of power of \( (x-x_0)^{\alpha} \). That is, \( f(x) \) can be expressed as \( \sum_{n=0}^{\infty} c_n (x-x_0)^{n\alpha} \) \((c_n \in \mathbb{R})\), this series being absolutely convergent for \( |x-x_0| < \rho \) \((\rho > 0)\).

According to this definition we can classify the points in the interval \([a, b]\) for the homogeneous equation \( [L_{n\alpha}(y)](x) = 0 \) as \( \alpha \)-ordinary and \( \alpha \)-singular points, and these latter points as regular \( \alpha \)-singular and essential \( \alpha \)-singular as follows:

**Definition 1.2.** A point \( x_0 \in [a, b] \) is said to be an \( \alpha \)-ordinary point of the equation \( [L_{n\alpha}(y)](x) = 0 \) if the functions \( a_k(x) (k = 0, 1, \ldots, n-1) \) are \( \alpha \)-analytic in \( x_0 \). A point \( x_0 \in [a, b] \) which is not \( \alpha \)-ordinary will be called \( \alpha \)-singular.

**Definition 1.3.** Let \( x_0 \in [a, b] \) be an \( \alpha \)-singular point of the equation \( [L_{n\alpha}(y)](x) = 0 \). Then \( x_0 \) is said to be a regular \( \alpha \)-singular point of this equation if the functions \( (x-x_0)^{(n-k)\alpha} a_k(\alpha) \) are \( \alpha \)-analytic in \( x_0 \) \((k = 0, 1, \ldots, n-1)\). Otherwise, \( x_0 \) is said to be an essential \( \alpha \)-singular point.

We will apply in this work the general theory for sequential LFDE of order \( n\alpha \) \((0 < \alpha \leq 1)\), see [2], to study the solutions of LFDE with variable coefficients \([L_{n\alpha}(y)](x) = 0\), around a regular \( \alpha \)-singular point. In some sense we present in this work a generalization of the Frobenius theory, which is strongly connected to the study of many important generalized special functions and with the explicit solution of linear fractional partial differential equations with no constant coefficients. We point out here that the series method, based on the expansion of the unknown solution \( y(x) \) in a fractional power series to obtain solutions of fractional differential equations, was first suggested by Al-Bassam (see, for example, [1,6]).
2. Solution around an $\alpha$-singular point to a fractional differential equation of order $\alpha$

In this section we obtain the fractional power series solutions to a homogeneous LFDE of order $\alpha$ ($0 < \alpha \leq 1$), around a regular $\alpha$-singular point $x_0$ of the equation. It will be more convenient, for our purposes, to consider the differential equation written as

$$[L_\alpha(y)](x) := (x - x_0)^\alpha y^{(\alpha)}(x) + q(x)y(x) = 0,$$

where $q(x)$, due to the regular $\alpha$-singular character of $x_0$, is an $\alpha$-analytic function around $x_0$.

**Theorem 2.1.** Let $x_0 \geq a$ be a regular $\alpha$-singular point of (2.1) of order $\alpha$, and let

$$q(x) = \sum_{n=0}^{\infty} q_n (x - x_0)^{n\alpha}$$

be the power series expansion of the $\alpha$-analytic function $q(x)$. Then there exists the solution

$$y(x; \alpha, s_1) = (x - x_0)^{s_1} \sum_{n=0}^{\infty} a_n (x - x_0)^{n\alpha}$$

to Eq. (2.1) on a certain interval to the right of $x_0$. Here $a_0$ is a non-zero arbitrary constant, $s_1 > -1$ is the uniqueness real solution to the equation

$$\frac{\Gamma(s + 1)}{\Gamma(s - \alpha + 1)} + q_0 = 0,$$

and the coefficients $a_n$ ($n \geq 1$) are given by $a_n = -\frac{\Gamma(n\alpha + s - \alpha + 1)}{\Gamma(n\alpha + s + 1)} \sum_{l=0}^{n-1} a_l q_{n-l}$. Moreover, if the series (2.2) converges for all $x$ in a semi-interval $0 < x - x_0 < R$ ($R > 0$), then the series solution (2.3) of Eq. (2.1) is also convergent in the same interval.

**Proof.** Seeking a solution to Eq. (2.1) of the form (2.3), fractionally differentiating $y(x; \alpha, s_1)$ and substituting the result into (2.1), if we set $f_0(s) = \frac{\Gamma(s + 1)}{\Gamma(s - \alpha + 1)} + q_0$, then we obtain

$$f_0(s) = 0 \quad \text{and} \quad a_n = -\frac{\sum_{l=0}^{n-1} a_l q_{n-l}}{f_0(n\alpha + s)}.$$  

(2.4)

Thus, if $s_1$ is the only real root of the equation $f_0(s) = 0$, then the second expression in (2.4) provides, by recurrence, the coefficients $a_n$ of (2.3) in terms of $a_0$.

Now we prove the convergence of the series. Let $0 < r < R$. Since the series (2.2) is convergent, there exists a constant $M > 0$ such that for all $n \in \mathbb{N}$

$$|q_{n-l}| \leq \frac{M \Gamma(n\alpha)}{r^{n\alpha}} \quad \text{and} \quad |a_n| \leq \frac{M}{|f_0(n\alpha + s)|} \sum_{k=1}^{n} \frac{|a_{n-k}|}{r^{k\alpha}}.$$  

(2.5)

So using the asymptotic representation

$$\frac{\Gamma(z + a)}{\Gamma(z + b)} = z^{a-b} \left[ 1 + O \left( \frac{1}{z} \right) \right] \quad (|\arg(z + a)| < \pi; |z| \to \infty),$$

(2.6)

we get that the series $\sum_{n=0}^{\infty} c_n (x - x_0)^{n\alpha}$ converges for all $x$ such that $0 < |x - x_0| < r$. From this we conclude that (2.3) converges for $0 < x - x_0 < R$. \(\square\)

**Example 2.2.** Consider the LFDE

$$(x - 1)^\alpha y^{(\alpha)}(x) - y(x) = 0,$$

(2.7)
where \( y^{(\alpha)}(x) \) represents either the Caputo or the Riemann–Liouville fractional derivative. Then, since the point \( x = 1 \) is a regular \( \alpha \)-singular point of (2.7), we shall seek a solution to this equation around the point \( x = 1 \) of the form (2.3). As we have seen earlier, \( s \) must be a real solution (which always exists) of the corresponding fractional index equation
\[
\frac{\Gamma(s + 1)}{\Gamma(s - \alpha + 1)} = 1 \quad (s > -1).
\]

Suppose that the above-mentioned solution is \( s = \beta \ ( \beta > 0 ) \). It then holds that \( a_n = 0 \ (n \in \mathbb{N}) \). Then the general solution to Eq. (2.7) will be the
\[
y(x) = a_0(x - 1)^\beta \ (a_0 \neq 0).
\]
Lastly, let us see the values of \( \beta \) corresponding to some values of \( \alpha \) in the following table:

\[
\alpha \quad 0.01 \quad 0.1 \quad 0.3 \quad 0.5 \quad 0.6 \quad 0.9 \quad 0.95 \quad 0.99
\]
\[\beta \quad 0.46664 \quad 0.51201 \quad 0.61506 \quad 0.72118 \quad 0.77539 \quad 0.94267 \quad 0.97124 \quad 0.99423\]

3. Solution around an \( \alpha \)-singular point of a fractional differential equation of order \( 2\alpha \)

We now focus our attention on the following homogeneous LFDE of order \( 2\alpha \) (\( 0 < \alpha \leq 1 \)):
\[
[L_{2\alpha}(y)](x) := (x - x_0)^{2\alpha} y^{(2\alpha)}(x) + (x - x_0)^{\alpha} p(x) y^{(\alpha)}(x) + q(x) \quad (3.1)
\]
\[
p(x) = \sum_{n=0}^{\infty} p_n (x - x_0)^{n\alpha} \quad \text{and} \quad q(x) = \sum_{n=0}^{\infty} q_n (x - x_0)^{n\alpha}. \quad (3.2)
\]
valid on a semi-interval \( 0 < x - x_0 < R \) for some \( R > 0 \), and with \( x_0 \geq a \). Our goal is to find a solution to (3.1) of the form
\[
y(x; \alpha, s) = (x - x_0)^s \sum_{n=0}^{\infty} a_n (x - x_0)^{n\alpha} \quad (3.3)
\]
with \( a_0 \neq 0 \), and \( s \) being a number to be determined.

Differentiating (3.3) we get, for \( \Re(s) > \alpha - 1 \) and \( s \notin \mathbb{Z}^- \), and \( f_0(s) = \frac{\Gamma(s + 1)}{\Gamma(s - 2\alpha + 1)} + \frac{\Gamma(s + 1)}{\Gamma(s - \alpha + 1)} p_0 + q_0 \),
\[
f_0(s) = 0 \quad \text{and} \quad f_k(s) = \frac{\Gamma(s + 1)}{\Gamma(s - \alpha + 1)} + q_k. \quad (3.4)
\]
If \( f_0(s) = 0 \) has two complex roots, they must be conjugates, because \( \Gamma(z) = \overline{\Gamma(\overline{z})} \) for all \( z \in \mathbb{C} \).

If \( s_1 \) is the larger real root of \( f_0(s) = 0 \), in the case where such an equation has two real solutions, then we could get a solution to (3.1) of the form (3.3), which we denote by \( y(x; \alpha, s_1) \). The coefficients \( a_n \), for any \( n \geq 1 \), have the following explicit representations:
\[
(-1)^n \begin{vmatrix}
f_1(s_1) & f_0(s_1 + \alpha) & 0 & \cdots & 0 \\
f_2(s_1) & f_1(s_1 + \alpha) & f_0(s_1 + 2\alpha) & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
f_{n-1}(s_1) & f_{n-2}(s_1 + \alpha) & f_{n-3}(s_1 + 2\alpha) & \cdots & f_0(s_1 + (n-1)\alpha) \\
f_n(s_1) & f_{n-1}(s_1 + \alpha) & f_{n-2}(s_1 + 2\alpha) & \cdots & f_1(s_1 + (n-1)\alpha) \\
\end{vmatrix}
= \frac{f_0(s_1 + \alpha) f_0(s_1 + 2\alpha) \cdots f_0(s_1 + n\alpha) a_0}{f_0(s_1) f_0(s_1 + 2\alpha) \cdots f_0(s_1 + n\alpha)}. \quad (3.5)
\]
If \( s_2 \) is a second root of the indicial equation \( (s_1 - s_2 \neq n\alpha, n \in \mathbb{N}_0) \), then the expression (3.5) with \( s_1 \) replaced by \( s_2 \) yields a second solution to Eq. (3.1), except for the case \( s_1 = s_2 + n\alpha \) for all \( n \geq 0 \).

The proof of the convergence, for \( 0 < x - x_0 < R \), of both solutions, linearly independent \( y(x; \alpha, s_1) \) and \( y(x; \alpha, s_2) \), is analogous to that used for Theorem 2.3, if we take into account the above-mentioned asymptotic relation (2.6).

There still remains the problem of how to find a second solution to Eq. (3.1) for the case when \( s_1 - s_2 = n\alpha(n \in \mathbb{N}_0) \). The following theorem gives an answer for these special cases.
Theorem 3.1. Let \( x_0 \geq a \) be a regular \( \alpha \)-singular point of Eq. (3.1), and let the series (3.2) be convergent on a semi-interval \( 0 < x - x_0 < R \) with \( R > 0 \). Let \( s_1 \) and \( s_2 \) be two real roots of the fractional indicial Eq. (3.4) with \( s_1, s_2 > \alpha - 1 \) and \( s_1 \geq s_2 \). Then, in the interval \( 0 < x - x_0 < R \), Eq. (3.1) has one solution of the form

\[
y_1(x; \alpha, s_1) = (x - x_0)^{s_1} \sum_{n=0}^{\infty} a_n(s_1)(x - x_0)^{n\alpha} \quad (a_0(s_1) \neq 0),
\]

(3.6)

where the coefficients \( a_n(s_1) \) are given in terms of \( a_0(s_1) \) by the formula (3.5) with \( s \) replaced by \( s_1 \). The following assertions also hold:

(a) If \( s_1 = s_2 \), then a second solution to (3.1) has the following form:

\[
y_2(x; \alpha, s_1) = y_1(x; \alpha, s_1) \log(x - x_0) + \sum_{n=0}^{\infty} b_n(x - x_0)^{n\alpha+s_1},
\]

(3.7)

where \( b_n = \frac{\partial}{\partial s} (a_n(s)) |_{s=s_1} \), and \( a_n(s) \) is given by (3.5).

(b) If \( s_1 - s_2 = n\alpha \) with \( n \in \mathbb{N} \), then a second solution to Eq. (3.1) is given by

\[
y_2(x; \alpha, s_2) = y_1(x; \alpha, s_1) \cdot A(x) \cdot \log(x - x_0) + \sum_{n=0}^{\infty} c_n(s_2)(x - x_0)^{n\alpha+s_2},
\]

(3.8)

where \( A(x) \) is a function obtained by evaluating the derivative \( A(x) = \frac{\partial}{\partial s} [(s - s_2)y(x; \alpha, s)] |_{s=s_2} \).

Moreover, the series (3.6) and (3.7) are convergent for all \( x \) on the semi-interval \( 0 < x - x_0 < R \).

Proof. The proof is analogous to that for the ordinary case. \( \square \)

Remark 3.2. Let us emphasize the fact that, for the case when the Riemann–Liouville fractional derivative is replaced by the Caputo derivative, the results obtained for solutions around regular \( \alpha \)-singular points coincide exactly with those in the Riemann–Liouville case.

Example 3.3. Consider the following generalized Bessel equation of order \( \nu = 0 \):

\[
x^{2\alpha}y^{(2\alpha)}(x) + x^{\alpha}y^{(\alpha)}(x) + x^{2\alpha}y(x) = 0 \quad (0 < \alpha < 1).
\]

(3.9)

Let us find two linearly independent solutions to (3.9) around the regular \( \alpha \)-singular point \( x = 0 \). The indicial equation associated with (3.9) is given by

\[
\frac{\Gamma(s + 1)}{\Gamma(s - 2\alpha + 1)} + P_0 \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} + q_0 = 0.
\]

This yields the following solutions for different values of \( \alpha \):

\[
\begin{array}{cccccc}
\alpha & 0.99 & 0.8 & 0.65 & 0.628 & 10^{-4} \\
\beta_1 & -0.459 & -0.6784 & -0.9629 & -0.1312 & -0.9999 \\
\beta_2 & 0.08194 & 0.0535 & -0.1051 & - & -
\end{array}
\]

This table lets us obtain a solution around \( x = 0 \) for (3.9) for \( \alpha = 0.628 \) and for \( \alpha = 10^{-4} \), and two linearly independent solutions in the cases \( \alpha = 0.99 \), \( \alpha = 0.8 \), and \( \alpha = 0.65 \).

Remark 3.4. The theory presented above also can be applied to cases which include non-sequential Riemann–Liouville or Caputo derivatives by using the corresponding relations of the type (1.2).

References


