BLOW-UP FOR PARABOLIC AND HYPERBOLIC PROBLEMS
WITH VARIABLE EXponents

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Abstract. In this paper we study the blow up problem for positive solutions of parabolic and hyperbolic problems with reaction terms of local and nonlocal type involving a variable exponent. We prove the existence of initial data such that the corresponding solutions blow up at a finite time.

1. Introduction

In this paper we will study the following parabolic problem

\[
\begin{aligned}
\begin{cases}
  u_t &= \Delta u + f(u) & (x,t) \in \Omega \times [0,T), \\
  u(x,0) &= u_0(x) & x \in \Omega \\
  u(x,t) &= 0 & (x,t) \in \partial \Omega \times [0,T),
\end{cases}
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^n \) is a bounded domain with smooth boundary \( \partial \Omega \), and the source term is of the form

\[
f(u) = a(x)u^{p(x)}
\]
or

\[
f(u) = a(x) \int_{\Omega} u^{q(y)}(y,t)dy.
\]

We will impose the following conditions on the functions \( p(x), q(x) : \Omega \to (1, +\infty) \) and the continuous function \( a(x) : \Omega \to \mathbb{R} \):

\[
\begin{aligned}
1 < p^- &\leq p(x) \leq p^+ < +\infty, \\
1 < q^- &\leq q(x) \leq q^+ < +\infty, \\
0 < c_a &\leq a(x) \leq C_a < +\infty.
\end{aligned}
\]

The assumptions on \( a(x) \) can be relaxed, but we do not strive for generality here.

Parabolic problems with sources like the ones in (1.1) appear in several branches of applied mathematics, and they have been used to model chemical reactions, heat transfer or population dynamics. We refer the interested reader to [13] and [15], and the references therein. Moreover, sources of type (1.2), with \( a(x) < 0 \), and \( 1 < p(x) < 2 \) arise in the mathematical models which describe the processes of absorption, see [1], [2] and [3] for the questions of existence, uniqueness and localization of solutions.

It is well known that the solution can become unbounded at a finite time \( T_f \) when \( p(x) \equiv p > 1 \), i. e., the solution blows up, in the sense that \( \lim_{t \to T_f} \| u(\cdot, t) \|_{\infty} = +\infty \) for initial data large enough. There exists a huge literature in this problem.

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and several methods were used, let us cite among them the eigenfunction argument of Kaplan [12], and we will follow his ideas in this work. Also, more general nonlinearities were considered (such as exponential growth, moving sources, and highly concentrated sources), and specially nonlocal reaction terms containing $L^q$ norms like condition (1.3) with $q(x) \equiv q > 1$, see [16], and again we can have blow up for sufficiently large initial data.

However, nonconstant powers seems to be new in the literature. The variable $L^p(x)$ spaces are of interest for their applications to modelling in a wide variety of physical problems; on the theoretical side, there are many interesting features of $L^p(x)$ spaces which present difficult challenges (among them, the problem of the density of continuous functions; or the lack of homogeneity and similarity variables for our problem). We refer the interested reader to the survey [9] and the references therein for their mathematical properties, see also Halsey [8], Ruzicka [14], and Zhikov [17] for different applications.

In section 2 we will study the local existence of positive solutions, and the existence of solutions which blows up in finite time for initial data sufficiently large. Our main result is the following theorem:

**Theorem 1.1.** Let $\Omega \in \mathbb{R}^n$ be a bounded smooth domain and let $u$ be a positive solution of equation (1.1), with $p(x)$, $q(x)$ and $a(x)$ satisfying conditions (1.4). Then, for sufficiently large initial datum $u_0(x)$, there exist a finite time $T_f > 0$ such that

$$\sup_{0 \leq t \leq T_f} \|u(x,t)\|_{L^\infty(\Omega)} = +\infty.$$  

**Remark 1.2.** We will see that $u_0$ must satisfy

$$\int_{\Omega} u_0 \varphi dx \geq C,$$

where $\varphi$ is the first eigenfunction of the Dirichlet laplacian on $\Omega$ (scaled to have $\int_{\Omega} \varphi dx = 1$) for certain fixed positive constant $C$ which will depends only on the domain $\Omega$ and the bounds $c_a, C_a$ given in condition (1.4).

Then, in section 3 we prove a similar nonexistence theorem for the following hyperbolic problem:

\begin{equation}
\begin{cases}
  u_{tt} &= \Delta u + f(u) & (x,t) \in \Omega \times [0,T), \\
  u(x,0) &= u_0(x) & x \in \Omega \\
  u_t(x,0) &= u_1(x) & x \in \Omega \\
  u(x,t) &= 0 & (x,t) \in \partial \Omega \times [0,T),
\end{cases}
\end{equation}

with $f(u)$ as before. The proof is based on the one in [7], and we extend it to nonlocal sources as well.

Finally, in section 4 we will discuss briefly some open problems.

2. **Parabolic Problems**

Let us prove the local existence in time and regularity of solutions of

$$u_t = \Delta u + a(x)u^p(x) + b(x) \int_\Omega u^q(y,t)dy$$
with initial datum \(u_0(x)\) and homogeneous Dirichlet boundary conditions. We assume that \(0 < c \leq a, b \leq C\). Equation (1.1) could be written as

\[
u(x, t) = \int_{\Omega} G(x, z, t)u_0(z)dz + \int_{\Omega} G(x, z, \tau - s)(a(z)u_0^{p(z)} + b(z)\int_{\Omega} u^{q(y)}(y, t)dy) dzds,
\]

where \(G(x, z, t)\) is the Green function. Now, the existence and uniqueness of solutions for a given \(u_0(x)\) could be obtained by a fixed point argument.

We define inductively

\[
\begin{align*}
u_1(x, t) &= 0 \\
u_{n+1}(x, t) &= \int_{\Omega} G(x, z, t)u_0(x) + \int_{\Omega} G(x, z, \tau - s)(a(z)u_n^{p(z)} + b(z)\int_{\Omega} u^{q(y)}(y, t)dy) dzds,
\end{align*}
\]

and the convergence of the sequence \(\{u_n\}\) follows by showing that

\[
Q(u) = \int_{\Omega} G(x, z, \tau - s)(a(z)u_n^{p(z)} + b(z)\int_{\Omega} u^{q(y)}(y, t)dy) dzds
\]

is a contraction in

\[
E = \{C^{1, 2}(\Omega_T) \cap C(\overline{\Omega_T}) : \|u\|_{\infty} \leq M\}
\]

where \(\Omega_T = \Omega \times [0, T]\), \(M > M_0\) is a fixed positive constant, and \(M_0 = \|u_0(x)\|_{\infty}\) (in order to get \(\|u_0(x)\|_{\infty} < M\)).

Let us note first that, for any \(x \in \Omega\) fixed, we have

\[
u^{p(x)} - v^{p(x)} = p(x)\nu^{p(x)-1}(u - v)
\]

with \(w = su + (1 - s)v, s \in (0, 1)\). Although \(s\) depends on \(x\), we always have

\[
(2.1) \quad \|p(x)\nu^{p(x)-1}(u - v)\|_{\infty} \leq p^{+}(2M)^{p^{-1}}\|u - v\|_{\infty},
\]

and a similar inequality is valid for \(q(x)\).

Now, let us define \(\mu(t)\) as

\[
\mu(t) = \sup_{\Omega_T \cap 0 \leq \tau < t} \int_{\Omega} G(x, z, t-s)dzds,
\]

clearly \(\mu(t) \to 0\) when \(t \to 0^+\).

It remains to prove that, for sufficiently small \(\mu(t)\), \(Q\) is a contraction, that is, there exists \(k < 1\) such that

\[
\|Q(u) - Q(v)\|_{\infty} \leq k\|u - v\|_{\infty}
\]

for every \(u, v \in E\).

We have

\[
\begin{align*}
\|Q(u) - Q(v)\|_{\infty} &\leq \|\int_{\Omega} G(x, z, \tau - s)a(z)(u_n^{p(z)} - v_n^{p(z)})dzds\|_{\infty} + \\
&\|\int_{\Omega} G(x, z, \tau - s)b(z)\left(\int_{\Omega} u^{q(y)}(y, t)dy - v^{q(y)}(y, t)dy\right)dzds\|_{\infty}.
\end{align*}
\]

By applying inequality (2.1) we obtain

\[
\|Q(u) - Q(v)\|_{\infty} \leq \mu(t)C(2M)^{p^{-1}}\|p\|_{\infty}\|u - v\|_{\infty} + \mu(t)C(\int_{\Omega} dy)(2M)^{p^{-1}}\|q\|_{\infty}\|u - v\|_{\infty} \leq \mu(t)C(2M)^{\max(p^+, q^+)-1}(\|p\|_{\infty} + \|q\|_{\infty})\|u - v\|_{\infty}
\]

where \(|\Omega|_n\) denotes the \(n\)-dimensional measure of \(\Omega\).

Hence, if \(0 \leq t \leq \delta\), \(\mu(t)\) is small enough and \(Q\) is a contraction.
Remark 2.1. The eigenvalue method of Kaplan leads to an ordinary differential inequality of first order which blows up in finite time. We recall it since we will need here this result.

Lemma 2.2. Let \( y(t) \) be the solution of
\[
y'(t) \geq cy^r(t), \quad y(0) > 0,
\]
where \( r > 1 \), and \( c > 0 \). Then, \( y(t) \) cannot be globally defined, and
\[
y(t) \geq \left( y(0)^{1-r} - \frac{r-1}{c} t \right)^{-1/(r-1)}.
\]

The lemma follows by direct integration, and gives an upper bound for the blow up time.

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let \( \lambda_1 \) be the first eigenvalue of the Laplacian in \( \Omega \) with zero Dirichlet boundary conditions,
\[
-\Delta \varphi = \lambda \varphi,
\]
and let \( \varphi \) be the corresponding eigenfunction. We can choose the eigenfunction \( \varphi \) positive on \( \Omega \) and also we assume that
\[
\int_{\Omega} \varphi = 1,
\]
which is always possible due to the linearity of the eigenvalue problem, by multiplying the eigenfunction by an appropriate constant.

Let us consider first \( f(u) = u^p(x) \).

We introduce the function \( \eta(t) = \int_{\Omega} u \varphi \), and we have:
\[
\eta'(t) = \int u_t \varphi = \int \Delta u \varphi + \int a(x)u^p(x)\varphi = -\lambda_1 \eta + \int a(x)u^p(x)\varphi,
\]
where in the last step we applied the Green formula twice, and then replaced \( \Delta \varphi \) by \( -\lambda_1 \varphi \).

We consider now the term \( \int_{\Omega} a(x)u^p(x)\varphi \). For each \( t > 0 \), we divide \( \Omega \) in two sets,
\[
\Omega_{(<1)} = \{ x \in \Omega : u(x, t) < 1 \}, \quad \Omega_{(\geq1)} = \{ x \in \Omega : u(x, t) \geq 1 \}.
\]

Now, we have
\[
\int_{\Omega} a(x)u^p(x)\varphi = \int_{\Omega_{(\geq1)}} a(x)u^p(x)\varphi + \int_{\Omega_{(<1)}} a(x)u^p(x)\varphi
\geq \int_{\Omega_{(\geq1)}} a(x)u^p - \varphi
= \int_{\Omega_{(\geq1)}} a(x)u^p - \varphi + \int_{\Omega_{(<1)}} a(x)u^p - \varphi - \int_{\Omega_{(<1)}} a(x)u^p - \varphi
\geq \int_{\Omega} a(x)u^p - \varphi - \int_{\Omega_{(<1)}} a(x)u^p - \varphi
\geq c_a \int_{\Omega} u^p - \varphi - C_a \int_{\Omega} \varphi,
\]
where in last step we use the bounds for \( a(x) \) and for the second integral we also used the bound \( u \leq 1 \) on \( \Omega_{(<1)} \), and then we changed \( \Omega_{(<1)} \) by a larger domain.

Finally, by using Jensen’s inequality, and the fact that \( \int_{\Omega} \varphi = 1 \), we obtain
\[
c_a \int_{\Omega} u^p - \varphi - C_a \int_{\Omega} \varphi \geq c_1 \eta^p(t) - C_a,
\]
which gives
\[ \eta'(t) \geq -\lambda_1 \eta(t) + c_1 \eta^{p^-}(t) - C_a, \]
and now the result follows from Lemma 2.2 for \( \eta(0) \) big enough, since
\[ \eta(t) = \int_\Omega u(x,t) \varphi(x) \leq \| u(.,t) \|_{L^\infty(\Omega)} \int_\Omega \varphi(x) = \| u(.,t) \|_{L^\infty(\Omega)}. \]

Let us consider now the case \( f(u) = a(x) \int_\Omega u^q(y,t)dy \). After defining \( \eta(t) = \int_\Omega u \varphi \) we repeat the previous argument and we obtain in much the same way
\[ \eta'(t) \geq -\lambda_1 \eta(t) + \int_\Omega a(x) \left( \int_\Omega u^q(y,t)dy \right) \varphi(x)dx. \]

Since \( \varphi \) is regular, we have
\[ \int_\Omega u^q(y,t)dy \left( \int_\Omega a(x) \varphi(x)dx \right) \geq c_a \int_\Omega u^q(y,t) \frac{\varphi(y)}{\| \varphi \|_\infty} dy. \]

By considering again \( \Omega_{<1} \) and \( \Omega_{\geq 1} \) as before, and by applying Jensen’s inequality we obtain
\[ c_a \int_\Omega u^q(y,t) \frac{\varphi(y)}{\| \varphi \|_\infty} dy \geq c_1 \eta^{q}(t) - C_a, \]
where \( c_1 \) depends only on \( c_a \) and \( \| \varphi \|_\infty \). Hence,
\[ \eta'(t) \geq -\lambda_1 \eta(t) + c_1 \eta^{q-}(t) - C_a, \]
and the result follows by Lemma 2.2.

3. Nonlinear Wave Equations

We begin by stating the following lemma, its proof can be found in [7]:

Lemma 3.1. ([7], Lemma 1.1) Let \( y(t) \in C^2 \) satisfying
\[ y''(t) \geq h(y(t)), \]
\( y(0) = \alpha > 0, \ y'(0) = \beta > 0, \) and \( h(s) \geq 0 \) for all \( s \geq a \). Then, \( y'(t) > 0 \) whenever \( y \) exists; and
\[ t \leq \int_a^{y(t)} \left( \beta^2 + 2 \int_a^s h(x)dx \right)^{-1/2} ds. \]
Theorem 3.2. Let \( u \in C^2 \) be a solution of problem (3.2), with \( p(x), q(x) \) and \( a(x) \) satisfying conditions (1.4). Then, there exist sufficiently large initial data \( u_0, u_1 \) such that

\[
\sup_{0 \leq t \leq T_f} \|u(x, t)\|_{L^\infty(\Omega)} = +\infty.
\]

We sketch now the main steps of the proof, which is similar to the one of Theorem 1.1, by using now Lemma 3.2.

**Proof of Theorem 3.2.** Let \((\lambda_1, \varphi)\) be the first eigenpair of the laplacian in \( \Omega \) with zero Dirichlet boundary conditions, and let \( \varphi \) be normalized such that

\[
\int_\Omega \varphi = 1.
\]

Let us suppose that \( f(u) = u^{p(x)} \), the other is similar. We define the function \( \eta(t) = \int_\Omega u \varphi \), and we have:

\[
\eta''(t) = \int_\Omega u_{tt} \varphi = \int_\Omega \Delta u \varphi + \int_\Omega a(x)u^{p(x)} \varphi = -\lambda_1 \eta + \int_\Omega a(x)u^{p(x)} \varphi,
\]

(we applied the Green formula twice, and then replaced \( \Delta \varphi \) by \(-\lambda_1 \varphi\)).

The term \( \int_\Omega a(x)u^{p(x)} \varphi \) is handled as before, by considering for each \( t > 0 \) the sets, \( \Omega_{\{<1\}} = \{ x \in \Omega : u(x, t) < 1 \} \) and \( \Omega_{\{\geq 1\}} = \{ x \in \Omega : u(x, t) \geq 1 \} \).

Hence, we get

\[
\int_\Omega a(x)u^{p(x)} \varphi \geq c_a \int_\Omega u^{p^-} \varphi - C_a \int_\Omega \varphi,
\]

depending only on the bounds for \( a(x) \) and the fact that \( u \leq 1 \) on \( \Omega_{\{<1\}} \) for the second integral. Moreover, Jensen’s inequality gives

\[
c_a \int_\Omega u^{p^-} \varphi - C_a \int_\Omega \varphi \geq c_1 \eta^{p^-}(t) - C_a,
\]

and we obtain the differential inequality

\[
\eta''(t) \geq -\lambda_1 \eta(t) + c_1 \eta^{p^-}(t) - C_a.
\]

We can apply now the previous Lemma 3.2 for \( \alpha = \eta(0) \) big enough such that 

\(-\lambda_1 \eta(t) + c_1 \eta^{p^-}(t) - C_a > 0\), and let us observe that

\[
\alpha = \eta(0) = \int_\Omega u_0 \varphi,
\]

\[
\beta = \eta'(0) = \int_\Omega u_1 \varphi.
\]

Hence,

\[
\eta(t) = \int_\Omega u(x, t) \varphi(x) \leq \|u(\cdot, t)\|_{L^\infty(\Omega)} \int_\Omega \varphi(x) = \|u(\cdot, t)\|_{L^\infty(\Omega)},
\]

and \( u \) blows up before the maximal time of existence defined in equation (3.1) is reached. \( \square \)
4. Critical Exponents and Other Problems

Given a blowing up initial datum \( u_0 \) and its blow up time \( T_f \), when \( p(x) \equiv p \) is a constant greater than one, the profile of a blowing up solution and the rate of blow up depends on \( T_f \) and \( p \), namely
\[
\|u(\cdot, t)\|_{\infty} \sim (T_f - t)^{-1/(p-1)}.
\]

However, this kind of results were obtained by using self similar solutions and similarity variables, dynamical systems arguments, or the behavior of an associated ordinary differential equation, see [4, 5, 6, 10, 11, 15] and the references therein. Hence, an interesting question arises in this context: how they depend on the function \( p(x) \)?

For example, Lemma 2.2 gives a bound for the blow up time which depends on \( p^− \), and also from Lemma 3.2, it is easy to see that a similar situation holds for hyperbolic problems. However, the proof of Theorem 1.1 shows that for a blowing up initial datum \( u_0 \) it is enough to consider the minimum of \( p(x) \) on certain subset of \( \Omega \), namely
\[
p^∗ = \inf \{ p(x) : x \in \Omega \text{ and } u(x, t) \geq 1 \text{ for some } t < T_f \}
\]
which gives a different bound for the blow up time. Hence, we may have a stronger dependence of the blow up time on the initial data. We conjecture that \( p^∗ \) could be made arbitrarily close to \( p^+ \).

The same reasoning shows the difficulty to define the notion of critical exponent. Our main theorem consider only the case \( p(x) > 1 \), although the existence of blowing up solutions for \( p(x) \) satisfying \( p^− < 1 < p^+ \) cannot be disregarded.

Moreover, also the blow up rate, the blow up set of a solution, and the blow up profile seems to depend on a nontrivial way on both the exponent \( p(x) \) and the initial data \( u_0(x) \). We believe that those problems deserves more attention and further work will be needed to settle those questions.

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