Vector Spaces and Binary Quantifiers

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1 Introduction  Caicedo [1] and others [3] have observed that monadic quantifiers cannot count the number of classes of an equivalence relation. This implies the existence of a binary quantifier which is not definable by monadic quantifiers. The purpose of this paper is to show that binary quantifiers cannot count the dimension of a vector space. Thus we have an example of a ternary quantifier which is not definable by binary quantifiers.

The general form of a binary quantifier is

\[ Q_{x_1} y_1 \ldots x_n y_n \phi_1(x_1, y_1) \ldots \phi_n(x_n, y_n). \]

An example of such a quantifier is (in addition to all monadic quantifiers) the similarity quantifier:

\[ S_{x_1} y_1 x_2 y_2 \phi_1(x_1, y_1) \phi_2(x_2, y_2) \iff \phi_1(\cdot, \cdot) \text{ and } \phi_2(\cdot, \cdot) \text{ are isomorphic as binary relations.} \]

We let \( \mathcal{L}(Q) \) denote the extension of first-order logic by the quantifier \( Q \). Recall the definition of \( \Delta(\mathcal{L}(Q)) \) from [2]. It is proved in [4] that \( \Delta(\mathcal{L}(S)) \) is equivalent to second-order logic. Even monadic quantifiers can have very powerful \( \Delta \)-extensions. Thus, simple syntax (such as \( \mathcal{L}(Q) \)) is no guarantee for simple model theory.

2 Vector spaces—the main lemma  Let \( K \) be an infinite field. We shall consider vector spaces

\[ \mathcal{V} = \langle V, +, \cdot, 0; K \rangle \]

over \( K \). Here + denotes addition of vectors, \( \cdot \) denotes multiplication of vectors by an element of the field, and 0 is the zero vector. Thus \( \mathcal{V} \) should be considered as a two-sorted structure. Let \( L \) denote the language associated with \( \mathcal{V} \).

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consisting of symbols \(+\), \(\cdot\), \(0\) for the vector operations, a constant symbol \(c\) for each \(c \in K\), and symbols for the field operations. The \textit{linear type} of an \(n\)-tuple \(a_1, \ldots, a_n\) of elements of \(V\) is the set of linear equations

\[ c_1x_1 + \ldots + c_nx_n = 0 \]

satisfied by \(a_1, \ldots, a_n\) \((c_1, \ldots, c_n \in K)\).

**Main Lemma** Let \(\mathcal{V}\) and \(\mathcal{W}'\) be two vector spaces over \(K\) of dimensions \(d\) and \(d'\) respectively. Let \(a_1, \ldots, a_n\) be an \(n\)-tuple from \(\mathcal{V}\) and \(a_1', \ldots, a_n'\) an \(n\)-tuple from \(\mathcal{W}'\) of the same linear type. Suppose

\[ n + 2 < d, d' \leq |K| \]

Then there is a bijection \(f: \mathcal{V} \to \mathcal{W}'\) such that \((x, y, a_1, \ldots, a_n)\) has the same linear type in \(\mathcal{V}\) as \((fx, fy, a_1', \ldots, a_n')\) in \(\mathcal{W}'\), whatever \(x, y \in V\).

**Proof:** Let \(H\) be the subspace of \(\mathcal{V}\) generated by \(a_1, \ldots, a_n\) and \(H'\) the respective subspace of \(\mathcal{W}'\). Let \(G\) be a subspace of \(\mathcal{V}\) such that \(\mathcal{V} = H \oplus G\) and \(G'\) a similar subspace of \(\mathcal{W}'\). Note that \(G\) and \(G'\) have dimensions of at least 2, since \(d, d' \geq n + 2\). Let \(W\) be a maximal subset of \(G\) with respect to the property

\[ x \neq y \& x, y \in W \Rightarrow \{x, y\} \text{ free}. \]

Then every vector in \(G\) has the representation \(\lambda w\) for unique \(\lambda \in K\) and \(w \in W\). Let \(W'\) be defined similarly in \(G'\).

From \(d \leq |K|\) it follows that \(|V| = |K|\) (recall that \(K\) is infinite). Similarly \(|H| = |G| = |K|\). Clearly \(|W| \geq |K|\). Thus \(|W| = |K|\). By symmetry, \(|W'| = |K|\).

Now we shall define the mapping \(f\). We let \(f\) be the identity on \(K\). Let \(f\) map \(W\) one-one onto \(W'\). As \(\tilde{a}\) and \(\tilde{a}'\) have the same linear type, we have

\[ (H, \tilde{a}) \cong (H', \tilde{a}') \]

and we can let \(f\) map \(H\) isomorphically onto \(H'\) such that \(f(a_i) = a_i'\) \((i = 1, \ldots, n)\).

Now if \(v \in V\), then \(v\) has a unique representation

\[ v = \lambda w + h, \]

where \(\lambda \in K\), \(w \in W\), and \(h \in H\), and we can define

\[ f(v) = \lambda f(w) + f(h). \]

This clearly makes \(f\) onto. To prove the claim concerning linear type, let

\[ \mu_1x_1 + \mu_2x_2 + \mu_3z_1 + \ldots + \mu_{n+2}z_n = 0 \]

be an equation satisfied by \((b_1, b_2, a_1, \ldots, a_n)\) in \(\mathcal{V}\). Let

\[ b_i = \lambda_iw_i + h_i, \quad (i = 1, 2). \]

Thus

\[ \mu_1\lambda_1w_1 + \mu_2\lambda_2w_2 + \mu_1h_1 + \mu_2h_2 + \mu_3a_1 + \ldots + \mu_{n+2}a_n = 0. \]

As \(G \cap H = \{0\}\), we must have

\[ \mu_1\lambda_1w_1 + \mu_2\lambda_2w_2 = 0. \]
By the very definition of $W$, either $\mu_1\lambda_1 = \mu_2\lambda_2 = 0$ or $w_1 = w_2$ (and $\mu_1\lambda_1 + \mu_2\lambda_2 = 0$). We also have

$$\mu_1 h_1 + \mu_2 h_2 + \mu_3 a_1 + \ldots + \mu_{n+2} a_n = 0.$$ 

Now in any case

$$\mu_1 \lambda_1 f(w_1) + \mu_2 \lambda_2 f(w_2) = 0$$

and

$$\mu_1 f(h_1) + \mu_2 f(h_2) + \mu_3 a_1' + \ldots + \mu_{n+2} a_n' = 0,$$

whence

$$\mu_1 f(b_1) + \mu_2 f(b_2) + \mu_3 a_1' + \ldots + \mu_{n+2} a_n' = 0,$$

as desired. The converse is entirely similar.

3 Equivalence of vector spaces

We show that the dimension of vector spaces cannot be distinguished in certain logics.

Let $Q$ be a binary quantifier, that is, a quantifier of type

$$(*) \quad Q x_1y_1 \ldots x_ny_n \phi_1(x_1, y_1, z) \ldots \phi_n(x_n, y_n, z).$$

Let $L_{\infty\omega}$ denote the infinitary language over the language $L$ defined in Section 2. If $\phi(z)$ is a formula of type $(*)$, where each $\phi_i(x_i, y_i, z)$ is a quantifier-free formula of $L_{\infty\omega}$, and $T$ a linear type of $m$-tuples, let

$$\pi_K(\phi(z), T)$$

be the true propositional symbol, if the statement $(**)$ below holds, and the falsity symbol otherwise:

$$(**) \quad \text{There is a vector space } V \text{ over } K \text{ of dimension } d, m + 2 < d \leq |K| (\hat{z} = (z_1, \ldots, z_m)) \text{ which satisfies } \phi(\hat{a}) \text{ for some } m\text{-tuple } \hat{a} \text{ of linear type } T.$$ 

Let $L_{\infty\omega}(Bin)$ be the extension of $L_{\infty\omega}$ by all binary generalized quantifiers.

Elimination Lemma

Suppose $\phi(x)$ is in $L_{\infty\omega}(Bin)$ and $\alpha$ is a cardinal exceeding the number of free variables in any subformula of $\phi(x)$. Then there is a quantifier free $\phi^*(\hat{x})$ in $L_{\infty\omega}$ such that

$$\forall \hat{x}(\phi(\hat{x}) \leftrightarrow \phi^*(\hat{x}))$$

holds in any vector space over $K$ of dimension $d$, $\alpha + 1 \leq d \leq |K|$.

Proof: The proof proceeds by induction on the length of $\phi(\hat{x})$. To prove the quantifier step, consider a formula $\phi(\hat{x})$ of type $(*)$ above. Let $\mathcal{F}$ be the set of all linear types of $m$-tuples. If $T \in \mathcal{F}$, let $P_T(\hat{z})$ be the conjunction of all equations

$$(+) \quad c_1z_1 + \ldots + c_mz_m = 0$$

which belong to $T$ as well as of all

$$c_1z_1 + \ldots + c_mz_m \neq 0$$
such that $(+)$ is not in $T$. Finally, let

$$
\phi^*(\vec{z}) = \bigvee_{T \in \mathcal{S}} (P_T(\vec{z}) \land \pi_K(\phi(\vec{z}), T)).
$$

To prove the claimed equivalence of $\phi(\vec{z})$ and $\phi^*(\vec{z})$, let $\mathcal{V}'$ be a vector space over $K$ of dimension $\geq \alpha$. For a start, suppose $\mathcal{V}'$ satisfies $\phi(\vec{a}')$ where $\vec{a}'$ is an $m$-tuple from $\mathcal{V}'$. As it turns out in a while, we may assume the $\vec{a}'$ are all from $V$ (and not from $K$). Let $T \in \mathcal{S}$ be the linear type of $\vec{a}'$. Thus $\mathcal{V}'$ satisfies $P_T(\vec{a}')$. By definition, $\pi_K(\phi(\vec{z}), T)$ is true (take $\mathcal{V} = \mathcal{V}'$ in (**)). Therefore $\phi^*(\vec{a}')$ holds in $\mathcal{V}'$. For the converse, suppose $\mathcal{V}'$ satisfies $\phi^*(\vec{a}')$. There are a $T \in \mathcal{S}$, and an $m$-tuple $\vec{a}$ as in (**). Now $\mathcal{V}$ satisfies $\phi(\vec{a})$ and $\vec{a}$ and $\vec{a}'$ have the same linear type $T$. Let $\mathcal{V} \rightarrow \mathcal{V}'$ be as in the Main Lemma. If there happened to be elements of $K$ in $\vec{a}'$, $f$ would be fixed on them, so they would cause no trouble.

By the conclusion of the Main Lemma, the sequences $(x, y, \vec{a})$ and $(fx, fy, \vec{a}')$ have the same linear type whatever $x, y \in V$. This implies

$$
\mathcal{V} \models \phi_i(x, y, \vec{a}) \iff \mathcal{V}' \models \phi_i(fx, fy, \vec{a}')
$$

for all $i = 1, \ldots, m$ and $x, y \in V$. By the closure of $Q$ under isomorphisms, we get

$$
\mathcal{V} \models \phi(\vec{a}) \iff \mathcal{V}' \models \phi(\vec{a}').
$$

We have already observed that $\phi(\vec{a})$ holds in $\mathcal{V}$. Therefore $\mathcal{V}' \models \phi(\vec{a}')$ as desired.

Corollary 1 Let $\phi$ be a sentence in $\mathcal{L}_{\infty\omega}(\text{Bin})$ and let $\alpha$ be a cardinal greater than the number of free variables in any subformula of $\phi$. Then either $\phi$ is true in all vector spaces over $K$ of dimension $d$, $\alpha + 1 < d < |K|$, or true in none.

This result shows that $\mathcal{L}_{\infty\omega}(\text{Bin})$ cannot distinguish two infinite-dimensional vector spaces over $\mathbb{R}$, and $\mathcal{L}_{\omega\omega}(\text{Bin})$ cannot distinguish finite-dimensional vector spaces over, say, $\mathbb{Q}$ from the infinite dimensional one.

Proposition Suppose $\mathcal{V}$ and $\mathcal{V}'$ are two vector spaces over an uncountable field $K$ of different infinite dimensions. Suppose $\mathcal{W}$ and $\mathcal{W}'$ are $\text{PC}(\mathcal{L}(Q_1))$-classes such that $\mathcal{V} \in \mathcal{W}$ and $\mathcal{V}' \in \mathcal{W}'$. Then $\mathcal{W} \cap \mathcal{W}' \neq \emptyset$.

Proof: By compactness there are vector spaces $\mathcal{W} \in \mathcal{W}$ and $\mathcal{W}' \in \mathcal{W}'$ over a field $K'$ such that $\mathcal{W}$ and $\mathcal{W}'$ have uncountable dimension. This depends on the fact that in any vector space over an uncountable field of dimension $\geq n$ there are uncountably many vectors, no $n$ of which are linearly dependent ($n \geq 2$). (Consider vectors with coordinates $(x, x^2, x^3, \ldots, x^n)$ where $x$ belongs to the field. No $n$ of these vectors are linearly dependent because

$$
\begin{vmatrix}
1 & x_2 & \ldots & x_n \\
x_1^2 & x_2^2 & \ldots & x_n^2 \\
\vdots & \vdots & \ddots & \vdots \\
x_1^n & x_2^n & \ldots & x_n^n
\end{vmatrix}
= \prod_{1 \leq i < j \leq n} x_i(x_i - x_j) \neq 0
$$
if \( x_1, \ldots, x_n \) are nonzero and different.) We may assume \( |\mathcal{W}| = |\mathcal{W}'| = |\mathcal{K}'| = \aleph_1 \). Thus \( \dim(\mathcal{W}) = \dim(\mathcal{W}') = \aleph_1 \) whence \( \mathcal{W} \cong \mathcal{W}' \). This implies \( \aleph_0 \cap \aleph_1 \neq \emptyset \).

This proposition shows that we cannot hope to separate the vector spaces, which were proved to be inseparable by binary quantifiers, by \( PC \)-classes of \( \mathcal{L}(Q_1) \). Other examples have to be used if one wants to show the undefinability of \( \Delta(\mathcal{L}(Q_1)) \) by binary quantifiers. The same applies to such extensions of \( \mathcal{L}(Q_1) \) as \( \mathcal{L}^{\text{Pos}} \) and \( \mathcal{L}(aa) \). Thus we have:

**Corollary 2** We can replace \( \mathcal{L}_{\omega\omega}(\text{Bin}) \) in Corollary 1 by \( \Delta(\mathcal{L}(Q_1)), \Delta(\mathcal{L}^{\text{Pos}}) \) and \( \Delta(\mathcal{L}(aa)) \).

4 **Logics which can separate vector spaces** The most straightforward example of a logic capable of distinguishing infinite dimensional vector spaces from finite dimensional ones is \( \mathcal{L}_{\omega_1\omega} \): consider the sentence

\[
\bigwedge_{n<\omega} \exists x_1 \ldots x_n \forall f_1 \ldots f_n \in K(f_1 x_1 + \ldots + f_n x_n = 0 \iff f_1 = \ldots = f_n = 0).
\]

This sentence is in fact in the fragment \( \mathcal{L}_{\text{HYP}} \) where \( \text{HYP} \) is the smallest admissible language containing \( \omega \). Thus we have\(^{1}\):

**Proposition** \( \Delta(\mathcal{L}(Q_0)) \not\subseteq \mathcal{L}_{\omega\omega}(\text{Bin}) \).

By considering the sentences

\[
Q_1 x B(x) \land \bigwedge_{n<\omega} \forall x_1 \ldots x_n \in B \forall f_1 \ldots f_n \in K
\]

\[
\left( \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j \rightarrow (f_1 x_1 + \ldots + f_n x_n = 0 \iff f_1 = \ldots = f_n = 0) \right)
\]

\[
\neg Q_1 x B(x) \land \forall x \bigvee_{n<\omega} \exists x_1 \ldots x_n \in B \exists f_1 \ldots f_n \in K(x = f_1 x_1 + \ldots + f_n x_n),
\]

and bearing in mind that \( \mathcal{L}_{\text{HYP}} \not\subseteq \Delta(\mathcal{L}(Q_0, Q_1)) \), one gets:

**Proposition** \( \Delta(\mathcal{L}(Q_0, Q_1)) \not\subseteq \mathcal{L}_{\omega\omega}(\text{Bin}) \).

**Corollary 3** \( \mathcal{L}_{\omega_1\omega}(Q_1) \not\subseteq \mathcal{L}_{\omega\omega}(\text{Bin}) \).

We shall now introduce a ternary quantifier \( Q \) which is not definable in \( \mathcal{L}_{\omega\omega}(\text{Bin}) \). For a ternary predicate \( D(x, y, z) \), constants \( c_0, c_1 \), and a unary predicate \( B(x) \) consider the formulas:

\[
\phi_0(x, y, u, v) \iff x \neq u \land x \neq v \land y \neq u \land y \neq v \land
\]

\[
((x = y \land u = v) \lor (x \neq y \land u \neq v \land \exists z (D(x, y, z) \land D(u, v, z)) \land \exists z ((D(x, u, z) \land D(u, y, z)) \lor (D(x, u, z) \land D(y, v, z)))))
\]

\[
\phi_1(x, y, u, v) \iff (\phi_0(x, y, u, v) \land (\exists z (D(x, u, z) \land D(y, v, z))) \rightarrow x = y)
\]

\[
\phi_2(x, y, z) \iff \exists uv(\phi_1(c_0, x, u, v) \land \phi_1(u, v, y, z))
\]

\[
F(x) \iff D(x, c_0, c_1)
\]

\[
\phi_3(x, y, z) \iff x = z = c_0 \lor (x = c_1 \land z = y) \lor (F(x) \land x \neq c_0 \land x \neq c_1)
\]

\[
\land \exists uv(\phi_0(c_1, x, u, v) \land \phi_0(u, y, v, z) \land D(u, v, c_0) \land D(y, z, c_0))
\]
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\[ \phi_1^i(\lambda, x) \iff \lambda = c_0 \lor x = c_0 \]

\[ \phi_2^i(\lambda_1, \ldots, \lambda_n, x_1, \ldots, x_n) \iff \exists uv(\phi_1(\lambda_1, x_1, u) \land \phi_1(\lambda_2, x_2, v) \land \phi_2(u, v, w) \land \phi_2^{\mathbb{R}^{-1}}(c_1, \lambda_3, \ldots, \lambda_n, w, x_3, \ldots, x_n)) \]

\[ \text{Free}^n(x_1, \ldots, x_n) \iff \forall \lambda_1 \ldots \lambda_n(F(\lambda_1) \land \ldots \land F(\lambda_n) \land \phi_2^*(\lambda_1, \ldots, \lambda_n, x_1, \ldots, x_n) \rightarrow \lambda_1 = \ldots = \lambda_n = c_0) \]

\[ \text{Fr}(B) \iff \bigwedge_{n < \omega} \forall x_1 \ldots x_n \in B \left( \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j \rightarrow \text{Free}^n(x_1, \ldots, x_n) \right). \]

**Definition** \( Q_{xyz}D(x, y, z) \iff \) there is an uncountable set \( B \) such that \( \text{Fr}(B) \) holds for some choice of \( c_0 \neq c_1 \).

Suppose now that \( V \) is a vector space over a field \( K \). Define

\[ D_V(x, y, z) \iff \exists \lambda \in K(x = \lambda y + (1 - \lambda)z) \]

("\( x, y \) and \( z \) are on the same line").

Then for this interpretation of \( D \) and any choice of \( c_0 \neq c_1 \), \( \text{Fr}(B) \) holds if and only if \( B \) is a free set of vectors. This shows that one can separate dimensions of vector spaces using \( Q \).

**Proposition** The class of countable dimensional vector spaces is definable in \( \mathcal{L}(Q) \).

**Corollary 4** \( \mathcal{L}(Q) \not\subseteq \mathcal{L}_{\omega \omega}(\text{Bin}) \), that is, there is a ternary quantifier which is not definable using binary quantifiers.

**Problems:** Is there an \((n + 1)\)-ary quantifier not definable using \( n \)-ary quantifiers for \( n > 2 \)? Is \( \Delta(\mathcal{L}(Q_1)) \) definable using binary quantifiers?

**NOTE**

1. Recall that \( \Delta(\mathcal{L}(Q_0)) = \mathcal{L}_{\text{HYP}} \).

**REFERENCES**

