Global Existence of Solutions for Flows of Fluids with Pressure and Shear Dependent Viscosities

J. Málek
Mathematical Institute of Charles University, Sokolovská 83
186 75 Prague 8, Czech Republic

J. Nečas
Department of Mathematical Sciences, Northern Illinois University
DeKalb, IL 60115, U.S.A

K. R. Rajagopal
Department of Mechanical Engineering, Texas A&M University
College Station, TX 77843, U.S.A

(Received and accepted October 2001)

Communicated by Claude Bardos

Abstract—There is clear and incontrovertible evidence that the viscosity of many liquids depends on the pressure. While the density, as the pressure is increased by orders of magnitude, suffers small changes in its value, the viscosity changes dramatically. It can increase exponentially with pressure. In many fluids, there is also considerable evidence for the viscosity to depend on the rate of deformation through the symmetric part of the velocity gradient, and most fluids shear thin, i.e., viscosity decreases with an increase in the rate of shear. In this paper, we study the flow of fluids whose viscosity depends on both the pressure and the symmetric part of the velocity gradient. We find that the shear thinning nature of the fluid can be gainfully exploited to obtain global existence of solution, which would not be possible otherwise. Previous studies of fluids with pressure dependent viscosity require strong restrictions to all data, or assume forms that are clearly contrary to experiments, namely that the viscosity decreases with the pressure. We are able to establish existence of space periodic solutions that are global in time for both the two- and three-dimensional problem, without restricting ourselves to small data. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords—Pressure dependent viscosity, Incompressible fluid, Global-in-time existence, Weak solution.

1. INTRODUCTION

Few models in mathematics have been accorded the intense scrutiny as that for the Navier-Stokes model. The Cauchy stress $T$ in such a fluid is assumed to depend on the density $\rho$, the...
temperature \( \theta \) and linearly on the symmetric part of the velocity gradient \( D \), i.e.,

\[
T = -p(\rho, \theta)I + \lambda(\rho, \theta) \text{tr} \, D + 2\mu(\rho, \theta)D.
\]  

(1.1)

Here, \( p(\rho, \theta) \) is called the thermodynamic pressure, \( \lambda(\rho, \theta) \) and \( \mu(\rho, \theta) \) are the bulk and shear viscosities, and

\[
D = \frac{1}{2} \left[ \nabla v + (\nabla v)^T \right].
\]  

(1.2)

Henceforth, we shall restrict our discussion to isothermal processes, and hence, ignore the dependence of the physical quantities on the temperature. Usually also, the dependence of the viscosities on the densities is ignored and they are assumed to be constant. If the fluid is incompressible, then the stress can be determined only to within a spherical part and it is generally assumed that (1.1) takes the form

\[
T = -pI + 2\mu D,
\]  

(1.3)

where \( pI \) is the indeterminate part of the stress (or to be more precise the constraint stress). Since an incompressible fluid can undergo only isochoric motions, we find that

\[
p = -\frac{1}{3} \text{tr} \, T.
\]  

(1.4)

Tacit in the derivation that an incompressible Navier-Stokes fluid has the form (1.3) is the assumption that the material moduli do not depend on the constraint. However, there is no physical reason that prevents the material moduli from depending on the constraint (see [1,2] for a discussion of related issues), and thus, we could very well suppose that

\[
T = -pI + 2\mu(p)D.
\]  

(1.5)

In the case of the unconstrained model (1.1), if the pressure-density relation is invertible, then we immediately see that material moduli depend upon the thermodynamic pressure. Such is indeed the case for the compressible model.

The notion of incompressibility is an idealization and in using a model such as (1.5) the assumption that is being made is that while changes in the density at even high pressures are insignificant, there could be significant changes in the material moduli. Stokes [3] recognized this fact in his seminal paper and there is a great body of experimental work on the pressure-dependence of the viscosity of liquids at high pressures (see [4] and the numerous references therein, and the more recent experiments of Johnson and Cameron [5], Johnson and Greenwood [6] and Johnson and Tevaarwerk [7]), which clearly show that the variation of the viscosity with the pressure can be exponential.

There are many practical situations where it is imperative to consider the pressure dependence of the viscosity. For instance, in elastohydrodynamic lubrication, one cannot predict the existence of a continuous film unless the pressure dependence of the viscosity is taken into account. It is also obvious that the viscosity of the water on the surface of the Pacific Ocean would be far less than the viscosity near the bottom, and thus, the calculation based on wrong value for the viscosity could lead to a serious error in computing the fuel requirements of a submarine. Suffice it to say that there are many technical situations wherein model (1.5) would be appropriate.

There is another aspect to the response of fluids that warrants some consideration. Many fluids shear-thin or shear thicken, that is their viscosities depend on the stretch tensor \( D \), and we shall be interested in such isotropic\(^1\) incompressible fluids, and thus, the viscosity could depend on the second invariant \( \Pi = (1/2)((\text{tr} \, D)^2 - \text{tr} \, D^2) \) and the third invariant \( \Pi = \text{det} \, D \). However, as

\(^1\)If the fluid is isotropic the viscosities depend on the principal invariants of the stretch tensor. However, it is possible for fluids to be anisotropic and in this event the material moduli will depend on the appropriate integrity basis (see [8] for a discussion of this issue).
most of the models that have been used to model such fluids are able to capture their response with a dependence of the viscosity on the second invariant, we shall restrict our attention to such models. We shall thus be interested in modeling the response of the fluids whose viscosity depends on both the mean normal stress and the second invariant of the stretch, and thus, we shall consider the following class of models:

$$\mathbf{T} = -\mathbf{p} I + [\mu(p, |\mathbf{D}|^2)] \mathbf{D}. \quad (1.6)$$

It is worth observing a significant difference between models (1.6) and (1.3), while the former provides an explicit relationship for the Cauchy stress $\mathbf{T}$ in terms of the kinematical quantity $\mathbf{D}$, the latter is an implicit relation between the stress $\mathbf{T}$ and the stretch tensor $\mathbf{D}$ by virtue of (1.4).

We shall specifically suppose that the viscosity $\nu := \mu/p$, $p > 0$ being a constant density, satisfy the following assumptions.

(1) For a given $r \in (1, 2)$ there are positive constants $C_1$ and $C_2$ such that for all symmetric matrices $\mathbf{B}, \mathbf{D} \in \mathbb{R}^{d\times d}$ and all $p \in \mathbb{R}$

$$C_1 (1 + |\mathbf{D}|^2)^{(r-2)/2} |\mathbf{B}|^2 \leq \frac{\partial \nu(p, |\mathbf{D}|^2)}{\partial \mathbf{D}} \cdot (\mathbf{B} \otimes \mathbf{B}) \leq C_2 (1 + |\mathbf{D}|^2)^{(r-2)/2} |\mathbf{B}|^2.$$

(2) For all symmetric matrices $\mathbf{D} \in \mathbb{R}^{d\times d}$ and all $p \in \mathbb{R}$

$$|\partial_{p} \nu(p, |\mathbf{D}|^2)| |\mathbf{D}| \leq \gamma_0 (1 + |\mathbf{D}|^2)^{(r-2)/4} \leq \gamma_0, \quad \text{with } \gamma_0 = \min\left(\frac{1}{2}, \frac{C_1}{4C_2}\right).$$

Examples of viscosities that satisfy the above conditions (see [9] for details) are

$$\nu_i(p, |\mathbf{D}|^2) = (A + \gamma_i(p) + |\mathbf{D}|^2)^{(r-2)/2}, \quad i = 1, 2, 3, \quad (1.7)$$

where $A \in (0, \infty)$, $r \in (1, 2)$ and $\gamma_i(p)$ takes one of the following forms for $\alpha, q > 0$:

$$\gamma_1(p) = (1 + \alpha p^2)^{-q/2}. \quad (1.8)$$

$$\gamma_2(p) = (1 + \exp(\alpha p))^{-q}. \quad (1.9)$$

$$\gamma_3(p) = \begin{cases} \exp(-\alpha qp), & \text{if } p > 0, \\ 1, & \text{if } p \leq 0. \end{cases} \quad (1.10)$$

We note that in the above models, when $r = 2$, the model reduces to the classical Navier-Stokes model, and when $q = 0$ it reduces to the generalized Navier-Stokes model.

Yet another possibility is to consider the model in which the viscosity depends linearly on the pressure, i.e.,

$$\mu(p) = \alpha p, \quad \alpha > 0.$$

In this case, it is possible to establish explicit exact solution for certain flows, and the flows of such a model have been studied in some detail (see [10]). For instance, in the case of unidirectional Poiseuille flow between parallel plates a whole gamut of velocity profiles is possible, the velocity and pressure fields being given by

$$v = u(y)i - \frac{1}{\alpha C_0} \ln \left[\frac{\cosh(C_0 y)}{\cosh C_0}\right], \quad (1.11)$$

$$p(x, y) = L \exp(C_0 x) \cosh(C_0 y), \quad L > 0.$$
Notice that sharp triangular velocity profiles drawn in Figure 1 are possible amongst the many solutions.

2. GOVERNING EQUATIONS AND MAIN THEOREM

On substituting the constitutive relation (1.6) into the balance of linear momentum, where for simplicity, we set the external body forces to be zero, and taking into account that the fluid can undergo only isochoric motions, that is $\text{div } v = 0$, we obtain the following system of $d+1$ equations

$$
\begin{align*}
\partial_t v + \text{div} (v \otimes v) - \text{div} \left( \nu \left( \frac{1}{2} |D(v)|^2 \right) D(v) \right) &= -\nabla p, \\
\text{div } v &= 0,
\end{align*}
$$

with $\nu$ satisfying Assumptions 1 and 2.

Few studies have been carried out concerning the analysis of equations (2.1) with $\nu(p)$. Renardy [11] and Gazzola [12] have established local-in-time existence of smooth solutions, but for totally unrealistic assumptions on how the viscosity depends on the pressure or under severe restrictions on the data. For instance, Renardy [11] assumes that $\nu(p)/p \to 0$ as $p \to \infty$. On the other hand, the numerous experiments of Andrade, Bridgman, Johnson and others clearly indicate that $\nu(p)$ behaves as though it is an exponential function and all the data clearly and unequivocally show that $\nu(p)/p \to \infty$ as $p \to \infty$.

We look for $(v, p) : (0, T) \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}$ solving (2.1) with the requirements

$$(v, p) \text{ are } L\text{-periodic at each spatial variables } z_i, \quad i = 1, \ldots, d,$$

$$(v(0, \cdot) = v_0, \text{ in } \mathbb{R}^d, \quad \int_\Omega p(t, x) \, dx = g(t), \text{ in } (0, T),$$

where $g : (0, T) \to \mathbb{R}$ is given and $\Omega := (0, L)^d$.

We formulate our main result.

**THEOREM 1.** Assume that Assumption 1 and 2 hold with $r$ satisfying

$$r \in \left( \frac{9}{5}, 2 \right), \text{ if } d = 3 \quad \text{and} \quad r \in \left( \frac{4}{3}, 2 \right), \text{ if } d = 2.$$  

Let $v_0 \in W^{1,2}_{\text{div}}$ and $g \in L^q(0,T)$. Then there is a weak solution $(v, p)$ to (2.1),(2.2) such that

$$v \in C \left( 0, T; \dot{L}^{2}_{\text{weak}} \right) \cap L^r \left( 0, T; W^{1,r}_{\text{div}} \right), \quad p \in L^{5r/6} \left( 0, T; L^{5r/6} \right),$$

and (2.1) holds in the sense of distributions.
Moreover, assuming that \( r \in (5/3, 2) \) if \( d = 3 \) and \( r \in (1, 2) \) if \( d = 2 \) then there exists a strong solution \((v, p)\) to (2.1), (2.2), unique in the class of strong solutions, satisfying

\[
v \in L^\infty \left(0, T^*; \tilde{W}^{1,2}_{\text{div}} \right) \cap L^r \left(0, T^*; \tilde{W}^{2,r}_{\text{div}} \right), \quad p \in L^2 \left(0, T^*; W^{1,2} \right).
\]

Here, \( T^* > 0 \) is arbitrary if \( v_0 \) is sufficiently small, or \( T^* \) is small enough if \( v_0 \) is arbitrary.

In the above theorem \( \tilde{W}^{1,2}_{\text{div}} \) stands for functions that belong to \( W^{1,r} \), are divergeless and have mean value zero; \( W^{1,r} \) and \( L^r \) denoting the usual Sobolev and Lebesque spaces.

**Proof. Brief Sketch.**

**Step 1. Basic Energy Estimates and Their Consequences.**

Energy balance implies the following estimates for convenient approximations \((v^\varepsilon, p^\varepsilon)\) of equations (2.1), (2.2)

\[
\|v^\varepsilon(t)\|_2^2 + \int_0^T \|\nabla v^\varepsilon\|_r^r \, ds \leq K,
\]

from which we can conclude, on using standard interpolation techniques, that

\[
\int_0^T \|v^\varepsilon\|_{3r/3}^{3r/3} \, ds \leq K.
\]

Taking the divergence of (2.1) leads (at least formally) to

\[
p^\varepsilon = \left(-\Delta\right)^{-1} \text{div} \text{div} \left(v^\varepsilon \otimes v^\varepsilon - \nu(p^\varepsilon, |D(v^\varepsilon)|^2) D(v^\varepsilon)\right).
\]

Using Assumptions 1 and 2, it follows from (2.8) and (2.7) that

\[
\int_0^T \|p^\varepsilon\|_{5r/6}^{5r/6} \, ds \leq K.
\]

We also have the following bound on \( \partial_t v^\varepsilon \) (see [9, p. 207, (5.2.25)])

\[
\|\partial_t v^\varepsilon\|_{L^r(0,T;W^{1,2})} \leq K, \quad s > \frac{5}{2}, \quad r' = \frac{r}{r - 1}.
\]

These uniform estimates then imply weak convergence to \((v, p)\) that are clearly sufficient to take the limit of terms with the time derivative and also the pressure, and if \( r > 2d/(d + 2) \) we can also take the appropriate limit for the nonlinear convective term thanks to the Aubin-Lions compactness lemma.

The key point is to show that for any smooth \( L \)-periodic function \( \varphi \) we have

\[
\int_0^T \int_\Omega \nu \left(p^\varepsilon, |D(v^\varepsilon)|^2\right) D_{ij}(v^\varepsilon) D_{ij}(\varphi) \, dx \, dt
\]

\[
- \int_0^T \int_\Omega \nu \left(p, |D(v)|^2\right) D_{ij}(v) D_{ij}(\varphi) \, dx \, dt.
\]

**Step 2. Almost Everywhere Convergence of the Gradients Via the Regularity Method.**

To achieve (2.11) the regularity approach is incorporated. It can be shown that

\[
\frac{d}{dt} \|\nabla v^\varepsilon\|_2^2 + \|\nabla^2 v^\varepsilon\|_r^r \leq C \|\nabla v\|_2^{2\lambda} \|\nabla v\|_r^{r'},
\]

where \( \lambda \) is given by the formulae

\[
\lambda = \frac{2(3 - r)}{3r - 5}, \quad \text{if } d = 3, \quad \lambda = \frac{4 - r}{r}, \quad \text{if } d = 2.
\]
In both cases, as \( \lambda > 1 \) for \( r < 2 \), we observe that this method does not, in general, improve the smoothness of the solution. On the other hand, restricting either to small initial data or to short time interval we observe that (2.12) directly implies those statements of the Theorem 1 concerning strong solutions.

However, it also follows from (2.12) and (2.6) that in general

\[
\int_0^T \frac{||\nabla^2 v||^2_r}{||\nabla v||^{2\lambda}_r} \, ds \leq K. \tag{2.14}
\]

This together with (2.6) then implies that if \( r \) satisfies (2.3), then

\[
\int_0^T \left|\nabla^{(2)} v\right|^{2\beta}_r \, dt \leq K, \quad \text{with } \beta \in \left(0, \frac{1}{3}\right). \tag{2.15}
\]

All these estimates together with an interpolation technique for certain \( \sigma > 0 \) and \( r_0 \in (1, r) \) lead then to the following:

\[
\int_0^T \|v^{r_0}\|^{1+\sigma,r}_1 \, dt \leq K \quad \Rightarrow \quad \int_0^T \|v^{r_0}\|^{\sigma}_1 \, dt \leq K. \tag{2.16}
\]

It is worth noting that despite the fact that \( 2\beta < 1 \), we can find \( r_0 > 1 \) and \( \sigma > 0 \) such that (2.16) holds.

Having (2.16) and (2.9) at hand, we apply the Aubin-Lions compactness lemma again and conclude the strong convergence of \( \nabla v^{r_0} \) to \( \nabla v \) in \( L^\infty(0, T; W^{1,r}) \), which implies almost everywhere convergence of \( \nabla v^{r_0} \) to \( \nabla v \) in \( (0, T) \times \Omega \).

**STEP 3. ALMOST EVERYWHERE CONVERGENCE OF PRESSURES.**

Once we have the strong convergence of \( \nabla v^{r_0} \) in \( L^\infty(0, T; W^{1,r}) \), we use this and equation (2.8) for the pressures to see that Assumptions 1 and 2 lead to the strong convergence of \( p^{r_0} \) at least in \( L^1(0, T; L^1) \). Thus, for conveniently chosen (but not relabelled) subsequences \( (v^{r_0}, p^{r_0}) \) we obtain their almost everywhere convergence to \( (v, p) \). This together with Vitali's theorem yields (2.11).

The details of the proof in three dimensions can be found in [13].

Several interesting open questions remain, of them the most interesting are the status of solutions for the Dirichlet boundary conditions and the other existence of solutions to viscosities that depend only on the pressure (the proof provided here depends critically on the fact that the fluid is shear thinning).

The class of fluids with pressure dependent viscosities have been barely studied and deserve much greater scrutiny. We hope this report will encourage such an effort.

**REFERENCES**


