Domain decomposition method for contact problems with small range contact

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Abstract

A non-overlapping domain decomposition algorithm of Neumann–Neumann type for solving variational inequalities arising from the elliptic boundary value problems in two dimensions with unilateral boundary condition is presented. We suppose that boundary with inequality condition is ‘relatively’ small. First, the linear auxiliary problem, where the inequality condition is replaced by the equality condition, is solved. In the second step, the solution of the auxiliary problem is used in a successive approximations method. In these solvers, a preconditioned conjugate gradient method with Neumann–Neumann preconditioner is used for solving the interface problems, while local problems within each subdomain are solved by direct solvers. A convergence of the iterative method is proved and results of computational test are reported.

Keywords: Domain decomposition; Schur complement; Unilateral contact problems; Parallel computing; Preconditioning

1. Equilibrium of a system of bodies in contact

We consider a system of elastic bodies decomposed into subdomains each of which occupies, in reference configuration, a domain $\Omega_i^M$ in $\mathbb{R}^2$, $i = 1, \ldots, I_M$, $M = 1, \ldots, J$, with boundary $\partial \Omega_i^M$. Suppose that boundary $\bigcup_{i=1}^{J} \partial \Omega_i^M$ consists of four disjoint parts $\Gamma_u$, $\Gamma_\tau$, $\Gamma_c$, and $\Gamma_0$ and that the displacements $u_0 : \Gamma_u \rightarrow \mathbb{R}^2$ and forces $P : \Gamma_\tau \rightarrow \mathbb{R}^2$ are given. The part $\Gamma_c$ denote the part of boundary that may get into unilateral contact with some other subdomain and the part $\Gamma_0$ denote the part of boundary on that the condition of the bilateral contact is prescribed (see Fig. 1).

We shall look for the displacements that satisfy the conditions of equilibrium in the set $K = \{ v \in K \subseteq \mathbb{R}^2 \mid v_{\text{in}} + v_{\text{out}} \leq 0 \text{ on } \Gamma_c \}$ of all kinematically admissible displacements $v$. $V = \{ v \in H^1(\Omega) \mid v = u_0 \text{ on } \Gamma_u, v_{\text{in}} = 0 \text{ on } \Gamma_\tau \}$, $H^1(\Omega)_i = [H^1(\Omega_1^1)]^2 \times \cdots \times [H^1(\Omega_J^1)]^2$ is standard Sobolev space.
The displacement $u \in K$ of the system of bodies in equilibrium then minimizes the energy functional $L(v) = (1/2)a(v, v) - L(v)$:

$$L(u) \leq L(v) \quad \text{for any } v \in K.$$

(1)

Conditions that guarantee existence and uniqueness of the solution may be expressed in terms of coercivity of $L$ and may be found, for example, in [1].

We define $\Gamma^M_j = \partial \Omega^M_j / \partial \Omega^M$, and the interface $\Gamma = \bigcup_{M=1}^{I_M} \bigcup_{i=1}^{I_n} \Gamma^M_i$. Let us introduce

$$\Gamma^M = \{ j \in \{1, \ldots, I_M \} : \bar{\Gamma}_j \cap \partial \bar{\Omega}^M_j = \emptyset \}, \quad M = 1, \ldots, J.$$

(2)

The number of a separate subset $\Gamma'$ is $P_c$, i.e. $\Gamma' = \bigcup_{i=1}^{P_c} \Gamma_i$. Further, we denote

$$\Omega^*_{j} = \bigcup_{\{i,M\} \in \Omega_{j}} \Omega^M_i, \quad j = 1, \ldots, P_c,$$

(3)

$$\vartheta_j = \{ \{i,M\} : \partial \Omega^M_i \cap \Gamma^M_j \neq \emptyset \}, \quad j = 1, \ldots, P_c,$$

(4)

i.e.

$$\Omega^*_{j} = \bigcup_{\{i,M\} \in \vartheta_j} \Omega^M_j, \quad j = 1, \ldots, P_c.$$

We suppose that

$$\Gamma \cap \Gamma' = \emptyset,$$

(5)

then

$$V_{\gamma} = \gamma K |_{\Gamma} = \gamma V |_{\Gamma}.$$

(6)
for trace operator $\gamma': [H^1(\Omega^M)]^2 \to [L^2(\partial\Omega^M)]^2$. We suppose that $\gamma^{-1}: V_{\Gamma} \to V$ is arbitrary linear inverse mapping for which

$$\sum_{M=1}^{J} (\gamma^{-1} \bar{v})_n = 0 \quad \forall \bar{v} \in V_{\Gamma}.$$  \hspace{1cm} (7)

After denoting restrictions $\bar{R}^M_i: V_{\Gamma} \to \Gamma_{M}^i$, $L^M_i: \Omega^M \to \Omega_{M}^i$, $a^M_i(., .): a^M(., .) \to \Omega^M$, $V(\Omega^M)$, we can formulate the Theorem 1.1.

**Theorem 1.1.** A function $u \in K$ is the solution of the global problem (1) if and only if the function $u$ satisfies

1. \hspace{0.5cm} \sum_{M=1}^{J} \sum_{i,M=1}^{\xi} a^M_i(u^M_i(\bar{u}), \gamma u^M_i(\bar{u})) \geq \sum_{M=1}^{J} a^M_i(\bar{u}, \bar{v}) \quad \forall \bar{v} \in V_{\Gamma}, \bar{u} \in V_{\Gamma}, \quad \text{(10)}

\hspace{0.5cm} \text{such that} \quad u + \phi \in K; \quad \gamma u^M_i(\bar{u})|_{\Gamma_{M}^i} = \bar{R}^M_i \bar{u} \quad \text{for } [i, M] \in \theta^j, \quad \text{(10)}

\hspace{0.5cm} \text{for } j = 1, \ldots, P_c.$$

**Proof.** See [2,5].
2. The local and global Schur complement

We now want to write the interface problem (8) in operator form. For this purpose, we first introduce additional notation. We introduce the local trace spaces

$$V^M = \{ v | v \mid \Gamma = 1 \} = \{ v | v \mid V \},$$  \hspace{1cm} (11)

and the extension $\text{Tr}^{-1}_i^M : V^M \rightarrow V(\Omega^M_i)$ defined by

$$\gamma(\text{Tr}^{-1}_i^M \bar{u}^M_i)|_{\Gamma_i^M} = \bar{u}^M_i,$$ \hspace{1cm} (12)

For subdomains $\Omega^M_j, j = 1, \ldots, P_c$, we completed definition $\text{Tr}^{-1}_i^M$ with boundary condition

$$\sum_{[i,M] \in \vartheta_j} a^M_i(\text{Tr}^{-1}_i^M \bar{u}^M_i, v^M_i) = 0 \hspace{1cm} \forall v^M_i \in V^M_i,$$ \hspace{1cm} (13)

Definition 2.1. The local Schur complement, for $i \in T^M, M = 1, \ldots, J$, is operator $S^M_i : V^M_i \rightarrow (V^M_i)^*$ defined by

$$\langle S^M_i \bar{u}^M_i, \bar{v}^M_i \rangle = a^M_i(\text{Tr}^{-1}_i^M \bar{u}^M_i, \text{Tr}^{-1}_i^M \bar{v}^M_i) \hspace{1cm} \forall \bar{u}^M_i, \bar{v}^M_i \in V^M_i.$$ \hspace{1cm} (14)

In matrix form, we have

$$S^M_i \bar{u}^M_i = (A^M_i - B^M_i \Delta^M_i B^M_i) \bar{u}^M_i,$$ \hspace{1cm} (15)

where we decompose the degrees of freedom $\bar{u}_i$ into internal degrees of freedom $\bar{u}^M_i$ and interface degrees of freedom $\bar{U}_i^M$:

$$\bar{u}_i = [\bar{u}^M_i, \bar{U}_i^M]^T.$$  

With this decomposition, the matrix representation of $a^M_i(.,.)$ on $H^1(\Omega^M_i)$ takes the form

$$A^M_i = \begin{bmatrix} \Delta^M_i & B^M_i \\ B^M_i^T & A^M_i \end{bmatrix}.$$ \hspace{1cm} (16)

Definition 2.2. The combined local Schur complement, for subdomains $\Omega^M_j, j = 1, \ldots, P_c$, is operator

$$S^M_j : (V^M_j, [i, M] \in \theta^j) \rightarrow (V^M_j, [i, M] \in \theta^j)^*.$$  

defined by

\[
\langle S^*_{j}(\bar{u}^M, [i, M] \in \theta^j), (\bar{v}^M, [i, M] \in \theta^j) \rangle
= \sum_{[i, M] \in \vartheta^j} a^M_i (u^M_i(\bar{u}^M_i), \text{Tr}^{-1}iM \bar{v}^M_i) \quad \forall (\bar{v}^M, [i, M] \in \theta^j) \in (V^M, [i, M] \in \theta^j),
\]

(17)

where \( u^M_i(\bar{g}^M_i) \) is the solution of the problem (10) and \( \bar{R}^M \bar{u} = \bar{u}^M, [i, M] \in \theta^j \).

**Lemma 2.1.** The condition (8) for the function \( \bar{u} \) on interface \( \Gamma \) is equivalent to the condition (18):

\[
\sum_{M=1}^{\mathcal{J}} \sum_{i \in I^M} (S^M_i \bar{u}^M_i, \bar{w}^M_i) + \sum_{j=1}^{P_c} (S^*_{j}(\bar{g}^M, [i, M] \in \theta^j), (\bar{w}^M, [i, M] \in \theta^j))
= \sum_{M=1}^{\mathcal{J}} \sum_{i \in I^M} L^M_i (\text{Tr}^{-1}iM \bar{w}^M_i) \quad \forall \bar{w} \in V_{\Gamma}, \text{where } \bar{w}^M = \bar{R}^M \bar{w}, \bar{u}^M = \bar{R}^M \bar{u},
\]

(18)

by using the local Schur complements.

**Proof.** See [5]. \( \square \)

We rewrite the condition (18) in the form

\[
S_u \hat{U} + S_{\text{kon}} \hat{U} = F,
\]

(19)

where

\[
S_u = \sum_{M=1}^{\mathcal{J}} \sum_{i \in I^M} (\bar{R}^M_i)\bar{T} s^M_i \bar{R}^M_i,
\]

\[
F = \sum_{M=1}^{\mathcal{J}} \sum_{i \in I^M} (\bar{R}^M_i)\bar{T} (\text{Tr}^{-1}iM \bar{w}^M_i)
\]

\[
S_{\text{kon}} = \sum_{j=1}^{P_c} \bar{R}^T_j S_{j} \bar{R}^T_j,
\]

and

\[
\bar{R}_j \bar{u} = (\bar{R}^M \bar{u}, [i, M] \in \theta^j)^T, \quad \bar{u} \in V_{\Gamma}, \quad \forall j = 1, \ldots, P_c.
\]

Since the operator \( S_{\text{kon}} \) is non-linear, we solve the Eq. (19) successive approximations method. We choose the solution of the auxiliary linear problem as an initial approximation \( \bar{U}^{(0)} \). In the auxiliary problem, we replace the set \( K \) by

\[
K^0 = \left\{ v \in V \mid \sum_{[i, M] \in \theta^j} (v^M)_{x} = 0 \text{ on } \Gamma_{xj}, j = 1, \ldots, P_c \right\}
\]
and we obtain
\[ u_0 = \arg\min_{v \in K} L(v), \]
\[ \bar{U}_0 = \gamma u_0|_\Gamma. \]

Now, we come back to Eq. (19) and we compute \( \bar{U}_k \) as the solution of the linear problem
\[ S_0 \bar{U}_k = F - S_{\text{KON}} \bar{U}_k - 1, \quad k = 1, 2, \ldots \] (20)

3. The auxiliary problem

We solve the variational equation
\[ u_0 \in K^0, \quad D L(u_0, v) = 0 \quad \forall v \in K^0. \] (21)

There exists a unique solution of the problem (21). For problem (21) we can describe analogy of Theorem 1.1.

**Theorem 3.1.** A function \( u_0 \in K^0 \) is the solution of the auxiliary problem (21) if and only if the function \( u_0 \) satisfies

1. \[ \sum_{M=1}^J \sum_{i=1}^M \left( a_{ij}^M (u_{ij}^M (\bar{u}^0), \phi_{ij}^M) - L_{ij}^M (\gamma^{-1} \bar{u}) \right) = 0 \quad \forall \bar{u} \in V_{\Gamma}, \bar{U} \in V_{\Gamma}, \] (22)

2. Its restriction \( u_{ij}^M (\bar{u}) = u_{ij}^M |_{\Omega M} \) satisfies following conditions:
   (a) \[ a_{ij}^M (u_{ij}^M (\bar{u}), \phi_{ij}^M) = L_{ij}^M (\gamma u_0 |_{\Omega M}), \quad \forall \phi_{ij}^M \in V^0(\Omega M), \quad a_{ij}^M (u_{ij}^M (\bar{u})) \in V(\Omega M), \gamma u_{ij}^M (u_{ij}^M (\bar{u})) = \tilde{R}_{ij} \bar{u}, \] (23)
   for \( i \in T^M, M = 1, \ldots, J, \)
   (b) \[ \sum_{i \in T^M \setminus \emptyset} a_{ij}^M (u_{ij}^M (\bar{u}), \phi_{ij}^M) = \sum_{i \in T^M \setminus \emptyset} L_{ij}^M (\phi_{ij}^M) \quad \forall \phi \equiv (\phi_{ij}^M, [i, M] \in \emptyset), \phi_{ij}^M \in V^0(\Omega M) \] such that \( \phi \in K_0^0 \) (24)

**Proof.** See [5]. \( \square \)

**Definition 3.1.** We define a combined local Schur complements for subdomains \( \Omega^{\ast j} \)
\[ S_{ij}^j : (V_0^M, [i, M] \in \emptyset) \to (V_0^M, [i, M] \in \emptyset)^*, \quad j = 1, \ldots, P. \]
by
\[
\langle S_0^j (u_0^M, [i, M] \in \partial^j), (\tilde{v}_M^i, [i, M] \in \partial^j) \rangle = \sum_{[i, M] \in \partial^j} u_0^M (T_{dM} u_0^M, T_{dM} \tilde{v}_M^i) \forall (\tilde{v}_M^i, [i, M] \in \partial^j) \in (V_M^i, [i, M] \in \partial^j). \tag{25}
\]

**Lemma 3.1.** The condition (22) for the function \( \tilde{u} \) on interface \( \Gamma \) is equivalent to the condition (26):
\[
\sum_{M=1}^J \sum_{i \in \Omega_M} \langle S_M^0 (u_0^M, [i, M] \in \partial^j), (\tilde{v}_M^i, [i, M] \in \partial^j) \rangle = \sum_{M=1}^J \sum_{i \in \Omega_M} \langle S_M^0 (u_0^M, [i, M] \in \partial^j), (\tilde{v}_M^i, [i, M] \in \partial^j) \rangle = \sum_{M=1}^J \sum_{i \in \Omega_M} \langle L_M^i (T_{dM} u_0^M, T_{dM} \tilde{v}_M^i) \rangle \forall \tilde{w} \in V_\Gamma, \text{ where } \tilde{w}^M = \tilde{R}_M^i \tilde{w}, u_0^M = \tilde{R}_M^i u_0^M. \tag{26}
\]

**Proof.** See [5]. \qed

**Definition 3.2.** We define a global Schur complement:
\[
S = \sum_{j=1}^P \tilde{S}_j^0 \tilde{S}_j + \sum_{M=1}^J \sum_{i \in \Omega_M} \langle \tilde{R}_M^i \tilde{S}_j^0 \tilde{R}_M^i \rangle,
\tag{27}
\]
and the condition (26) on the interface \( \Gamma \) has form
\[
S \tilde{U} = F,
\tag{28}
\]
in dual space \( (V_\Gamma)^* \).

Eq. (28) we solve by a preconditioned conjugate gradient method PCG1:
Choose \( \bar{U} \) and \( H^{(0)} \) (by Eq. (51)), \( \bar{P}^{(0)} = 0 \).

Iteration loop on \( n \).

1. Compute the preconditioned direction of descent \( G^{(n)} = M^{-1} H^{(n)} \).
2. Compute \( p^{(n)} = G^{(n)} + ((H^{(n)}, G^{(n)})/(H^{(n)} - \bar{B}^{(n)})) p^{(n-1)} \).
3. On each subdomain \( \Omega^M, i \in \Omega_M, M = 1, \ldots, J \), solve in parallel Dirichlet problem
   \[
   \tilde{A}_{ij} \bar{U}_{ij} = B_{ij} \bar{R}_M^i p^{(n)}.
   \]
4. On subdomain \( \Omega^\ast, j = 1, \ldots, P \), compute \( S_{ij}^0 \tilde{R}_s p^{(n)} \).
5. Compute
   \[
   Z^{(n)} = SP^{(n)} = \sum_{j=1}^P \tilde{R}_s^T S_{ij}^0 \tilde{R}_s p^{(n)} + \sum_{M=1}^J \sum_{i \in \Omega_M} \langle \tilde{R}_M^i \tilde{R}_j \rangle T S_{ij}^0 \bar{R}_M^i p^{(n)}
   \]
   \[
   = \sum_{j=1}^P \tilde{R}_s^T S_{ij}^0 \tilde{R}_s p^{(n)} + \sum_{M=1}^J \sum_{i \in \Omega_M} \langle \tilde{R}_M^i \tilde{R}_j \rangle T (\tilde{A}_{ij} \bar{R}_M^i p^{(n)} - B_{ij} \bar{U}_{ij}).
   \]
\[ \alpha^n = \begin{pmatrix} H^n & G^n \\ \overline{Z}^n & P^n \end{pmatrix}, \]
\[ H^{n+1} = H^n - \alpha^n \overline{Z}^n, \]
\[ \overline{U}^{n+1} = \overline{U}^n + \alpha^n P^n, \]

End loop on \( n \).

This method does not require the explicit construction of the local Schur complement matrix \( S_M \) but does require an efficient preconditioner \( M^{-1} \) (see [3,4]). By construction, the Schur complement operator is defined by the sum (27). Its inverse \( (S_M^{-1}) \) simply consists in associating to the generalized derivative \( g \in (V_M)^* \) the trace \( \gamma \phi_M \) on \( \Gamma_M \) of the solution \( \phi_M \) of the corresponding Neumann problem in variational form

\[ a_M(\phi_M, v) = \langle g, \gamma v \rangle_{\Gamma_M} \quad \forall v \in \hat{V}_j, \phi_M \in V(\Omega_M), \] (29)

Then

\[ (S_M^{-1})^{-1} g = \gamma \phi_M |_{\Gamma_M}. \] (30)

Similarly, we define inverse operator

\[ (S_{0j}^{-1})^{-1} \colon (V_M, [i, M] \in \theta_j)^* \rightarrow (V_M, [i, M] \in \theta_j), \quad j = 1, \ldots, P. \]

We solve the Neumann problem for

\[ \sum_{\{i, M\} \in \theta_j} a_M^*(\phi_M, v_M) = \sum_{\{i, M\} \in \theta_j} \langle g^M, \gamma v_M \rangle_{\Gamma_M} \quad \forall v, \phi \in \hat{V}_j \] (31)

where \( \hat{V}_j = \{(v_M, [i, M] \in \theta_j) | v_M \in V(\Omega_M), \sum_{\{i, M\} \in \theta_j} \langle v_M \rangle = 0 \text{ on } \Gamma_j \}). \)

Then

\[ (S_{0j}^{-1})^{-1}(g^M, [i, M] \in \theta_j) = (\gamma \phi_M |_{\Gamma_M}, [i, M] \in \theta_j). \] (32)

**Definition 3.3.** We define an injection

\[ D_M^i : V_M^i \rightarrow V_T, \quad i \in T_M, \quad M = 1, \ldots, J, \]
\[ D_{0j} : (D_M^i, [i, M] \in \theta_j) \rightarrow V_T, \quad D_{0j} = (D_M^i, [i, M] \in \theta_j), \quad j = 1, \ldots, P, \] (33)

such that on each interface degree of freedom is

\[ D_M^i \bar{v}(P) = \bar{v}(P) \frac{\partial \bar{v}^M}{\partial y}, \quad i = 1, 2, \ldots, I_M, \quad M = 1, \ldots, J, \] (34)

if the \( i \)th degree of freedom of \( V_T \) corresponds to the \( k \)th degree of freedom of \( V_M^M \) and

\[ D_M^i \bar{v}(P) = 0, \quad \text{if not.} \] (35)
Here, $\varrho_M^i$ is a local measure of the stiffness of subdomain $\Omega^i$ (for example, an average Young modulus on $\Omega^i$) and

$$
\varrho_T = \sum_{i \in \mathcal{G}^*} \varrho_M^i,
$$

is the sum of $\varrho_M^i$ on all subdomains $\Omega^i$ containing $P_i$.

We define

$$
M^{-1} = \sum_{M=1}^{J} \sum_{i \in TM} D_M^i (S_M^i)^{-1} (D_M^i)^T + \sum_{j=1}^{P_c} D_j (S_j^*)^{-1} D_j,
$$

(36)

In operator form, the action of $M^{-1}$ on $L \in (V_\Gamma)^*$ is thus given by

$$
M^{-1} L = \sum_{M=1}^{J} \sum_{i \in TM} D_M^i \bar{U}_M^i + \sum_{j=1}^{P_c} D_j \bar{U}_j,
$$

and

$$
\langle S_M^i \bar{U}_M^i, \bar{V}_M^i \rangle = \langle L, D_M^i \bar{U}_M^i \rangle \quad \forall \bar{V}_M^i \in V_M^i, \quad \bar{U}_M^i \in \tilde{V}_M^i,
$$

for $i \in TM$, $M = 1, \ldots, J$.

(37)

We introduce a closed orthogonal complement space $Q(\Omega_M^i)$ of $Z_M^i$ in $V(\Omega_M^i)$ and a closed orthogonal complement space $Q(\Omega^*)$ of $Z^*$ in $\tilde{V}^*$. Let then $\phi_0^M \in Q(\Omega_M^i)$ be the particular solution of the variational problem (29) defined by

$$
\langle L, D_M^i \gamma^M \rangle = 0 \quad \forall \gamma^M \in Z_M^i, \quad i \in TM, \quad M = 1, \ldots, J.
$$

(40)

$$
\langle L, D_j \gamma^* \rangle = \sum_{i \in TM} \langle L, D_M^i \gamma^M \rangle = 0 \quad \forall \gamma^* \in Z^*, \quad j = 1, \ldots, P_c.
$$

(41)

We introduce the invariant property

$$
\langle L, D_M^i \gamma^M \rangle = 0 \quad \forall \gamma^M \in Z_M^i, \quad i \in TM, \quad M = 1, \ldots, J.
$$

(42)
and \( \phi_0^j = (\phi_0^M, [i, M] \in \partial J^j) \in Q(\Omega^M) \) be the particular solution of the variational problem (31) defined by
\[
\sum_{[i, M] \in \partial J^j} a^M_i (\phi_0^M, v^M_i) = \sum_{[i, M] \in \partial J^j} (L, D^M_i (v^M_i))_{\Gamma M_i} \quad \forall v^j \in V^j.
\] (43)

Eqs. (42) and (43) are well posed variational problems set on \( Q(\Omega^M) \), \( Q(\Omega^M^*) \).

**Definition 3.4.** We define our new Neumann–Neumann preconditioner \( M^{-1}(z^0) \) by
\[
M^{-1}(z^0)L = \sum_{M=1}^{J} \sum_{i=1}^{I_M} D^M_i (\phi_0^M + z^0 M_i) |_{\Gamma^M_j},
\] (44)
with the solution \( z^0 \) of the minimization problem
\[
z^0 = \arg \min_{z \in \Pi Z} \langle S (M^{-1}(z) - S^{-1} L), (M^{-1}(z) - S^{-1} L) \rangle_{\Pi Z},
\] (45)
\( \Pi Z = \bigotimes_{i=1}^{J} \bigotimes_{j=1}^{Pc(Z^*)} \).

By construction, and since \( L \) satisfies (40) and (41), we have
\[
J(z) = \left\langle S \sum_{M=1}^{J} \sum_{i=1}^{I_M} D^M_i (\phi_0^M + z^0 M_i), \sum_{M=1}^{J} \sum_{j=1}^{Pc} D^M_j (\phi_0^M) \right\rangle_{\Pi Z} + \text{constant}.
\] (46)
Its minimum is attained for the function \( z^0 \) which cancels its gradient, that is for the solution of the variational coarse equality
\[
\left\langle S \sum_{M=1}^{J} \sum_{i=1}^{I_M} D^M_i (\phi_0^M + z^0 M_i), \sum_{M=1}^{J} \sum_{j=1}^{Pc} D^M_j (\phi_0^M) \right\rangle_{\Pi Z} = - \left\langle S \sum_{M=1}^{J} \sum_{j=1}^{Pc} D^M_j (\phi_0^M), \sum_{M=1}^{J} \sum_{j=1}^{Pc} D^M_j (\phi_0^M) \right\rangle_{\Pi Z} \forall z \in \Pi Z.
\] (47)

The upgraded Neumann–Neumann preconditioner (44) is therefore obtained by first solving the local Neumann problems (42) and (43) and then the variational coarse problem (47) set on the coarse product space \( \Pi Z \).

We introduce the coarse trace space
\[
V_H = \bigotimes_{M=1}^{J} \bigotimes_{j=1}^{Pc} D^M_j Z^M_j + \bigotimes_{j=1}^{Pc} D_j Z^j,
\] (48)
a set \( V_H^* \subset (V_F)^* \) given by
\[
L \in V_H^* \Leftrightarrow \langle L, z \rangle = 0 \quad \forall z \in V_H
\]
and the \( S \)-orthogonal projection \( P_S \) from \( V_F \) onto \( V_H \) given by
\[
\langle S c, \hat{U} - P_S \hat{U} \rangle = 0 \quad \forall c \in V_H, \forall \hat{U} \in V_F, P_S \hat{U} \in V_H.
\] (49)
Under this notation, the coarse problem (47) can be written as
\[
\sum_{M=1}^{J} \sum_{i=1}^{I_M} D^M_i \phi^0_i = - P_T \sum_{M=1}^{J} \sum_{i=1}^{I_M} D^M_i \phi^1_i,
\]
and thus the new Neumann–Neumann preconditioner (44) takes the final form
\[
M^{-1}(\phi^0 - z_0)L = (I - P_T) \sum_{M=1}^{J} \sum_{i=1}^{I_M} D^M_i \phi^0_i.
\]
(50)

Lemma 3.2. Suppose that \( H^{[0]} \in V_T^\perp \), \( P^{[0]} = 0 \) in algorithm PCG1 using preconditioner (44). Then \( H^{[n]} \in V_T^\perp, n = 1, 2, \ldots \)

Proof. See [5]. \( \square \)

According to Lemma 3.2, we must only suppose that \( H^{[0]} \in V_T^\perp \), which is achieved by setting the initial solution \( \bar{U}^{[0]} \in V_T^\perp \) to the solution of the coarse problem
\[
\langle H^{[0]}, z \rangle = \langle F - S_k \bar{U}^{[0]}, z \rangle = 0 \quad \forall z \in V_H.
\]
(51)

This coarse problem is identical to (47) within a change of right-hand side and thus the conditions (40) and (41) do not restrict the generality of the proposed preconditioner.

4. The original problem

Now, we solve by the successive approximations method, Eq. (20). We must effectively compute the solution \( \bar{U}^k \) of the linear problem
\[
S_0 \bar{U}^k = b^k,
\]
(52)

with
\[
S_0 = \sum_{M=1}^{J} \sum_{i \in T_M} (\bar{R}_i^M)^T S^M_i \bar{R}_i^M, \quad b^k = F - S_{KON} \bar{U}^{k-1},
\]
\[
F = \sum_{M=1}^{J} \sum_{i \in T_M} (\bar{R}_i^M)^T (T_{iM})^T L^M_i, \quad S_{KON} = \sum_{j=1}^{P} \bar{R}_j^N S_{ij} \bar{R}_j^N.
\]

The Eq. (52) we solve by a preconditioned conjugate gradient method PCG2 (we construct the sequence of the iterations \( \tilde{\omega}^{[n]} \rightarrow \hat{U}^k \) for \( n \rightarrow \infty \))

Choose \( \tilde{\omega}^{[0]} \) (by the Eq. (64)), \( n^{[0]} = b^k - S_0 \tilde{\omega}^{[0]} \), \( \pi^{[0]} = 0 \).

Iteration loop on \( n \):

Compute the preconditioned direction of descent \( \kappa^{[n]} = M^{-1} \pi^{[n]} \)

Compute \( \pi^{[n]} = \kappa^{[n]} + (\pi^{[n]}, \kappa^{[n]})/\pi^{[n-1]} \kappa^{[n-1]} \pi^{[n-1]} \)
On each subdomain $\Omega^M_i, i \in T^M, M = 1, \ldots, J$, solve in parallel Dirichlet problem

$$\hat{A}_M \hat{G}^M_i = B_M \hat{R}^M_i \pi^{[n]}.$$ 

Compute

$$\xi^{[n]} = S_0 \pi^{[n]} = \sum_{M=1}^J \sum_{i \in T^M} (\bar{R}^M_i)^T (\hat{A}_M \hat{R}^M_i \pi^{[n]} - B_M \vec{u}^{[n]}_M),$$

$$\alpha^{[n]} = \frac{\langle \eta^{[n]}, \kappa^{[n]} \rangle}{\langle \xi^{[n]}, \pi^{[n]} \rangle},$$

$$\eta^{[n+1]} = \eta^{[n]} - \alpha^{[n]} \kappa^{[n]},$$

$$\vec{u}^{[n+1]} = \vec{u}^{[n]} + \alpha^{[n]} \bar{u}^{[n]}.$$ 

End loop on $n$:

Now, we define a preconditioner $M^{-1}_0$:

$$M^{-1}_0 = \sum_{M=1}^J \sum_{i \in T^M} D^M_i (\xi^M_i)^{-1} (D^M_i)^T$$

with a new injection $D^M_i$.

**Definition 4.1.** We define an injection

$$D^M_i : V^M_i \to V_{\Gamma}, \quad i \in T^M, \quad M = 1, \ldots, J,$$

such that on each interface degree of freedom is

$$D^M_i \tilde{v}(P_k) = \tilde{v}(P_k) \left. \right|_{i \in T_i^M \subset \partial \Omega^j} \quad \text{for any} \ j \in \{1, \ldots, P_c\},$$

$$D^M_i \tilde{v}(P_k) = \tilde{v}(P_k) \left. \right|_{i \notin T_i^M \subset \partial \Omega^j} \quad \forall j = 1, \ldots, P_c,$$

if the $i$th degree of freedom of $V_{\Gamma}$ corresponds to the $k$th degree of freedom of $V^M_i$ and

$$D^M_i \tilde{v}(P_k) = 0 \quad \text{if not}.\quad (57)$$

Let $\phi^{0M}_i \in Q(\Omega^M_i)$ be the particular solution of the variational problem (29) defined by (42). Similarly to the auxiliary problem we now define a new preconditioner.

**Definition 4.2.** We define our new Neumann–Neumann preconditioner $M^{-1}_0$ by

$$M^{-1}_0 (\xi^M_L) = \sum_{M=1}^J \sum_{i \in T^M} D^M_i (\phi^{0M}_i + \xi^{[n]}_i)|_{P_c}.$$
with the solution $z_0$ of the minimization problem
\[
z_0 = \arg\min_{z \in P_0 Z} \langle S_0 \left( M^{-1}_0(z) - S_0^{-1} M_0 \right) L, M^{-1}_0(z) - S_0^{-1} M_0 \rangle L \tag{59}\]
and $P_0 Z \equiv \bigcup_{i \in T^M} M_0(z) \in \Pi_0 \langle \bar{Z}^M \rangle$.

Its minimum is attained for the function $z_0$ which cancels its gradient, i.e. for the solution of the variational coarse equality
\[
\left\{ S_0 \sum_{M=1}^{J} \sum_{j \in T^M} D^M_j \gamma z_j^M, \sum_{M=1}^{J} \sum_{i \in T^M} D^M_i \gamma z_i^M \right\} = -\left\{ S_0 \sum_{M=1}^{J} \sum_{i \in T^M} D^M_i \gamma \phi_i^M, \sum_{M=1}^{J} \sum_{i \in T^M} D^M_i \gamma \phi_i^M \right\} \quad \forall z \in \Pi_0 \langle \bar{Z} \rangle. \tag{60}\]

We introduce the coarse trace space
\[
V_0 H = \sum_{M=1}^{J} \sum_{i \in T^M} D^M_i \gamma Z_i^M, \tag{61}\]
and the $S_0$-orthogonal projection $P_0 Z$ from $V_{\Gamma}$ onto $V_0 H$ given by
\[
L \in V_{\text{def}} \Leftrightarrow \langle L, z \rangle = 0 \quad \forall z \in V_{\text{def}}. \tag{62}\]

Under this notation, the coarse problem (60) can be written as
\[
\sum_{M=1}^{J} \sum_{i \in T^M} D^M_i \gamma z_i^M = -P_0 \sum_{M=1}^{J} \sum_{i \in T^M} D^M_i \gamma \phi_i^M, \tag{63}\]
and thus the new Neumann–Neumann preconditioner (58) takes the final form
\[
M^{-1}_0(z_0) L = (I - P_0) \sum_{M=1}^{J} \sum_{i \in T^M} D^M_i \gamma \phi_i^M. \tag{64}\]

**Lemma 4.1.** Suppose that $\eta^{(0)} \in V_0^\perp$, $\alpha^{(0)} = 0$ in algorithm PCG2 using by preconditioner (58), then $\eta^{(n)} \in V_0^\perp$, $n = 1, 2, \ldots$.

According to Lemma 4.1 we must only suppose that $\eta^{(0)} \in V_0^\perp$, which is achieved by setting the initial solution $\bar{b}^{(0)} \in V_0^\perp$ to the solution of the coarse problem
\[
\langle \eta^{(0)}, z \rangle = \langle \bar{b} - S_0 \phi^{(0)}, z \rangle = 0 \quad \forall z \in V_{\text{def}}. \tag{64}\]
A convergence theorem requires to introduce some definitions. Let \( \Theta \) be an orthogonal complement of \( V_oH \) in \( V_\Gamma \). We introduce seminorms

\[
| \tilde{R}_j \tilde{v} |_{a_j} = \sqrt{ \sum_{[i,M] \in \varTheta_j} a_i^M (T_{M,i}^H \tilde{R}_M, T_{M,i}^H \tilde{v})}, \quad j = 1, \ldots, P_c.
\]

**Lemma 4.2.** The expression

\[
\| \tilde{u} \|_Q = \langle S_0 \tilde{u}, \tilde{u} \rangle
\]

is a norm on \( \Theta \) where

\[
Q = \times_{M=1}^{P_c} \times_{M=1}^{J} Q(M) \]

**Definition 4.3.** Let \( T : \Theta \rightarrow \Theta \) be a mapping defined by

\[
\langle S_0 (T \tilde{u}), \tilde{v} \rangle = \langle F - S_{KON} (\tilde{y}), \tilde{v} \rangle \quad \forall \tilde{v} \in \Theta.
\]

(65)

**Theorem 4.1.** Assume that there exists a constant \( \lambda \leq (1/\sqrt{2P_c}) \) such that the following condition hold:

\[
| \tilde{R}_j \tilde{u} |_{a_j} \leq \lambda \| \tilde{u} \|_Q \quad \forall \tilde{u} \in \Theta, \quad \forall j \in \{1, \ldots, P_c\}.
\]

(66)

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**Fig. 2.** Model problem—(a) geometry and (b) deformations.
Then the mapping \( T \) is the contraction on \( \Theta \). If \( \bar{U}_0 \in \Theta \) then the sequence of the iterations \( \tilde{U}_k \), computed by (52), are convergent and the limit is a fixed point \( \bar{U} \) of the mapping \( T \). The following error estimate holds

\[
\| \tilde{U}_k - \bar{U} \|_Q \leq \frac{(2\lambda^2 P_c)}{1 - 2\lambda^2 P_c} \| \bar{U}_0 - \bar{U}_0 - T \bar{U}_0 \|_Q.
\]

**Proof.** See [5].

5. Numerical experiments

In this section, we illustrate the practical behavior of our algorithm on the solution of a model problem. The introduced algorithm has been implemented in the program system MATLAB Version 5.2.1 and in MPI Version 1.2.0 by using FORTRAN 77 compiler. A geometry of the problem is in Fig. 2(a).

**Material parameters:** Three regions with Young’s modulus \( E = 10^{10} \) (Pa) and Poisson’s ratio \( \nu = 0.3 \).


**Discretization statistics:** Eight subdomains, 1748 nodes, 3040 elements, 3304 unknowns, 28 unilateral contact conditions, 68 interface elements, dimension of the coarse equality for the auxiliary problem is 13, dimension of the coarse equality for the original problem is 7.

**Convergence statistics:** Nine iterations of the PCG1 algorithm for the auxiliary problem, 17 iterations of the successive approximations method, total 38 iterations of the PCG2 algorithm for the original problem. Fig. 2(b) represents deformations in model problem.

**References**


