Memory cost of quantum contextuality

Matthias Kleinmann,1 Otfried Gihne,1,2 José R. Portillo,3 Jan-Åke Larsson,4 and Adán Cabello5

1Institut für Quantenoptik und Quanteninformation, Österreichische Akademie der Wissenschaften, Technikerstr. 21A, A-6020 Innsbruck, Austria
2Institut für Theoretische Physik, Universität Innsbruck, Technikerstr. 25, A-6020 Innsbruck, Austria
3Departamento de Matemática Aplicada I, Universidad de Sevilla, E-41012 Sevilla, Spain
4Institutionen för Systemteknik och Matematiska Institutionen, Linköpings Universitet, SE-581 83 Linköping, Sweden
5Departamento de Física Aplicada II, Universidad de Sevilla, E-41012 Sevilla, Spain

(Dated: July 21, 2010)

The simulation of quantum effects requires certain classical resources, and quantifying them is an important step in order to understand the difference between quantum and classical physics. We investigate the minimum classical memory needed to simulate the phenomenon of state-independent quantum contextuality in sequential measurements. We derive optimal simulation strategies for several important cases and prove that two bits of classical memory do not suffice to reproduce the results of sequential measurements on a two-qubit system.

PACS numbers: 03.65.Ud, 03.65.Ta, 03.67.-a,

Introduction.—A landmark result at the border between quantum and classical physics is the Kochen-Specker theorem [1,2]. In brief, it states that the result of a measurement may depend on which other compatible observables are measured simultaneously. This property is called contextuality and is in contrast to the classical intuition, which suggests that the answer to a single question should not depend on which other compatible questions are asked at the same time.

Contextuality can be seen as complementary to the well known nonlocality of distributed quantum systems [3]. Both phenomena can be used for information processing tasks, albeit the applications of contextuality are far less explored [4,5]. Although contextuality and nonlocality can be considered as signatures of nonclassicality, they can be simulated by classical models [6,7,8,9]. However, while nonlocal classical models violate a fundamental physical principle (the bounded speed of information), it is not clear whether contextual classical models violate any fundamental principle. Moreover, while the resources needed in order to imitate quantum nonlocality have been extensively investigated [10,11,12,13,14,15], there is no similar knowledge about the resources needed to simulate quantum contextuality.

In this paper we investigate the memory cost of a classical simulation of quantum contextuality. Any model which exhibits contextuality in sequential measurements requires memory and we construct memory-optimal models for two relevant cases. The amount of required memory increases as we consider more and more contextual constraints. This can be used to quantify contextuality in a given quantum setting. We show that certain scenarios breach the amount of two bits needed for the simulation of two qubits. This demonstrates that the memory needed to simulate only a small set of measurements on a quantum system may exceed the information that can be transmitted using this system, which is given by the Holevo bound [13] — a similar effect was observed so far only for the classical simulation of a unitary evolution [19].

Scenario.—We focus on the following set of two-qubit observables, also known as the Peres-Mermin (PM) square [4,20],

\[
\begin{pmatrix}
A & B & C \\
a & b & c \\
\alpha & \beta & \gamma
\end{pmatrix} = \begin{pmatrix}
\sigma_x \otimes 1 & 1 \otimes \sigma_z & \sigma_z \otimes \sigma_x \\
1 \otimes \sigma_x & \sigma_x \otimes 1 & \sigma_y \otimes \sigma_x \\
\sigma_z \otimes \sigma_x & \sigma_x \otimes \sigma_z & \sigma_y \otimes \sigma_y
\end{pmatrix},
\]

(1)

where \(\sigma_x\), \(\sigma_y\), and \(\sigma_z\) denote the Pauli operators. The square is constructed such that the observables within each row and column commute and are hence compatible, and the product of the operators in a row or column yields \(1\), except for the last column which yields \(-1\). Thus the product of the measurement results for each row and column will be +1 except in the third column where it will be -1. In contrast, for a noncontextual model the measurement result for each observable must not depend on whether the observable is measured in the column or row context. Hence the number of rows and columns yielding a product of -1 is always even, as any observable appears twice.

Similar to the Bell inequalities for local models, any noncontextual model satisfies the inequality \(\langle \chi \rangle \leq \langle ABC \rangle + \langle abc \rangle + \langle \alpha \beta \gamma \rangle + \langle Aa \alpha \rangle + \langle Bb \beta \rangle - \langle Cc \gamma \rangle \leq 4\), while for perfect observables quantum mechanics (QM) predicts \(\langle \chi \rangle = 6\) [21]. Here, the term \(\langle ABC \rangle\) denotes the average value of the product of the outcomes of \(A\), \(B\), and \(C\), if these observables are measured simultaneously or in sequence on the same quantum system. The violation is independent of the quantum state, which emphasizes that the phenomenon is a property of the set of observables rather than of a particular quantum state.

Recently, this inequality has been experimentally tested using trapped ions [22], photons [23], and nuclear
magnetic resonance systems [24]. The results show a good agreement with the quantum predictions. In these experiments, the observables are measured in a sequential manner. Since the observed results cannot be explained by a model using only preassigned values, the system necessarily attains different states during some particular sequences, i.e., the system memorizes previous events [31]. This leads to our central question: How much memory is required in order to simulate quantum contextuality?

Before we formulate this question more precisely, let us provide an example of a model that simulates the contextuality in the PM-square. We define the automaton $A_3$ using three internal states $S_1$, $S_2$, and $S_3$ via

$$S_1: \begin{bmatrix} \begin{array}{c} + \end{array} \end{bmatrix}, \begin{bmatrix} \begin{array}{c} + \end{array} \end{bmatrix}, \begin{bmatrix} \begin{array}{c} + \end{array} \end{bmatrix}, \begin{bmatrix} \begin{array}{c} + \end{array} \end{bmatrix}, \begin{bmatrix} \begin{array}{c} + \end{array} \end{bmatrix}, \begin{bmatrix} \begin{array}{c} + \end{array} \end{bmatrix}$$

$S_2$: \begin{bmatrix} \begin{array}{c} + \end{array} \end{bmatrix}, \begin{bmatrix} \begin{array}{c} + \end{array} \end{bmatrix}, \begin{bmatrix} \begin{array}{c} + \end{array} \end{bmatrix}, \begin{bmatrix} \begin{array}{c} + \end{array} \end{bmatrix}, \begin{bmatrix} \begin{array}{c} + \end{array} \end{bmatrix}, \begin{bmatrix} \begin{array}{c} + \end{array} \end{bmatrix}$

$S_3$: \begin{bmatrix} \begin{array}{c} + \end{array} \end{bmatrix}, \begin{bmatrix} \begin{array}{c} + \end{array} \end{bmatrix}, \begin{bmatrix} \begin{array}{c} + \end{array} \end{bmatrix}, \begin{bmatrix} \begin{array}{c} + \end{array} \end{bmatrix}, \begin{bmatrix} \begin{array}{c} + \end{array} \end{bmatrix}, \begin{bmatrix} \begin{array}{c} + \end{array} \end{bmatrix}$

Those tables define an automaton in the following way: If e.g. the automaton is in state $S_1$ and we measure the observable $\gamma$, consider the first table at the position of $\gamma$ (i.e., the last entry in the third row). The $+$ sign at this position indicates that the automaton returns $+1$, and stays in $S_1$. If we continue and measure $C$, we encounter the entry $+1$, which indicates that the automaton returns $+1$ and, after that, changes its internal state to $S_2$. Being in state $S_2$, the second table defines the behavior for the next measurement and e.g. a measurement of $c$ yields now the output $-1$, while the automaton stays in $S_2$.

Thus, starting in state $S_1$, the output of the automaton for the measurement sequence $\gamma C c$ is the sequence $+1, +1, -1$, so that the product is $-1$ in accordance with the quantum prediction. It is straightforward to verify that this automaton yields $\langle \chi \rangle = 6$. In addition, the observables within each context are compatible in the sense that in sequences of the form $AA, ABA$, or $A_aoA$, the first and last measurement of $A$ yields the same output. In fact, we will later show (cf. Theorem 2) that $A_3$ is memory-optimal. But we shall first be more specific on what conditions the quantum predictions impose and how the required memory is quantified in the general case.

Conditions on automata.—As discussed above, any model that reproduces contextuality eventually predicts that the system attains different states during certain measurement sequences. As an omniscient observer we would know the internal state prior to each measurement and could include it in our measurement record. Thus, knowing the internal state, we can predict the measurement outcome as well as the state of the system that will occur prior to the next measurement. Thus we can write any model that explains the outcomes of sequential measurements as an automaton similar to $A_3$; this class of automata is also known as Mealy machines [25, 26].

The quantum predictions add restrictions on such an automaton and thus increase the number of internal states needed. The first requirement we will use is that the automaton should reproduce the quantum predictions from the rows and columns, i.e., for all sequences $2$ in the set

$$Q_{rc} = \{ABC, abc, \alpha \beta \gamma, Aao, Bb\gamma, Cc\gamma, \text{ and permutat.} \}$$

We require that for any of those sequences the automaton must yield an output that matches the quantum prediction. For example, QM predicts for the sequence $ABC$ that the output is either $+1, +1, +1$ or one of the permutations of $+1, -1, -1$.

More generally, if $Q$ denotes a set of measurement sequences, we say that an automaton $A$ obeys the set $Q$ if the output for any sequence in $Q$ matches the quantum prediction — i.e., if for any sequence in $Q$, the output of $A$ could have occurred with a nonvanishing probability according to the quantum scenario. We say that a sequence yields a contradiction if the output of this sequence cannot occur according to QM. Hereby we consider all quantum predictions from any initial state (it would also suffice to only consider the completely mixed state $q = 1/4$). Furthermore, we assume that prior to the measurement of a sequence, the automaton always is re-initialized. This ensures that the output of the automaton is independent of any action prior to the selected measurement sequence. Note that we only consider the certain quantum predictions which occur with a probability of 1, while we do e.g. not require that for the sequence $A\gamma$ the probability of obtaining $+1, +1$ is equal to the probability of obtaining $+1, -1$. Finally, if an automaton with $k$ states $S_1, \ldots, S_k$ obeys $Q$ and there exists no automaton with less states obeying $Q$, we define the memory cost of $Q$ to be $M(Q) = \log_2(k)$ [32].

The set $Q_{rc}$ merely captures the fact that it is impossible to assign values to the observables in the PM-square, such that the row and column constraints are fulfilled. A basic property of measurements that we have neglected so far is that a direct repetition of a measurement must yield the same result. This natural constraint can be implemented via the set of repeated measurements

$$Q_{repeat} = \{AA, BB, CC, aa, bb, cc, \alpha\alpha, \beta\beta, \gamma\gamma\},$$

where we expect that for any of these pairs that the results in the first and the second measurement coincide.

We now can formulate:

**Lemma 1.** (i) An automaton that reproduces the certain quantum predictions for the column and row measurements possesses at least one bit of memory; $M(Q_{rc}) \geq 1$.

(ii) If in addition the automaton gives the same values for repeated measurements, then one bit does not suffice; $M(Q_{rc} \text{ and } Q_{repeat}) > 1$.

**Proof.** Only statement (ii) remains to be proven. Assume that the automaton has only two internal states
and without loss of generality that it starts in state $S_1$. We consider the case where in the last column there must be a prescribed state change in order to avoid a contradiction, i.e., in $S_1$, the product of the assignments of $C\gamma$ is $+1$, contrary to the quantum prediction. Note that there always exists at least one row or column with such a contradiction, and that the proof for any row or column follows the same lines. If there is only one state change (say after a measurement of $\gamma$), then, while measuring the sequence $C\gamma$, the automaton would remain in $S_1$ until after the last output and therefore yield a contradiction. If there are two (or more) state changes in the last column (say $c$ and $\gamma$), both must go to $S_2$. Then, the constraints from $\mathcal{D}_{\text{repeat}}$ require that $\gamma$ has the same values in $S_1$ and $S_2$ (this is also true for $c$). But then the sequence $C\gamma$ in $\mathcal{D}_{\text{rc}}$ will yield a contradiction. Thus a two-state automaton cannot obey both, $\mathcal{D}_{\text{rc}}$ and $\mathcal{D}_{\text{repeat}}$.

Contextuality and compatibility.—A convincing argument on noncontextuality vs. contextuality in the PM square can only be carried out if the observables in each row are compatible (cf. Ref. [27] for a detailed discussion). More generally, a convincing simulation of the contextuality in the PM square must reproduce all certain predictions, as long as the sequence is restricted to observables from a single context. We hence define

$$\mathcal{D}_{\text{context}} = \{X_1, X_2, \ldots, X_\ell \text{ are mutually compatible}\}, \tag{5}$$

with $X_\ell \in \{A, B, C, a, b, c, \alpha, \beta, \gamma\}$ being observables from the PM square. For instance, $\mathcal{D}_{\text{context}}$ contains the sequence $ACABCA$ for which QM predicts with certainty that the values of $A (C)$ in the first, third and sixth (second and fifth) measurement coincide, and that product of the outcome for $ABC$ yields $+1$. Note that $\mathcal{D}_{\text{context}}$ contains $\mathcal{D}_{\text{rc}}$ and $\mathcal{D}_{\text{repeat}}$ as subsets, hence these predictions are also included.

A particular feature of the PM square is that the observed correlations do not depend on the actual preparation (the initial state) of the quantum system. Consequently, one may consider an extended preparation procedure of the automaton, where the experimenter performs additional measurements between the initialization of the automaton and the actual sequence. The experimenter would e.g. measure the sequence $bABC$ but consider the measurement of the observable $b$ to be actually part of the preparation procedure. We write $[b]ABC$ for a sequence where we are not interested in the result of $b$. If $\mathcal{D}_{\text{PM}}$ denotes the set of all sequences with observables from the PM square, we write

$$\mathcal{D}_{\text{context}}' = \{|T|S| \ S \in \mathcal{D}_{\text{context}}, T \in \mathcal{D}_{\text{PM}}\} \text{ for the set of all sequences in } \mathcal{D}_{\text{context}}, \text{ including arbitrary preparation procedures.}$$

Remarkably, requiring that the automaton obeys $\mathcal{D}_{\text{context}}'$ does not further increase the needed memory; the automaton $A_3$ with an arbitrary choice for the initial state already obeys $\mathcal{D}_{\text{context}}'$, as we demonstrate in Appendix A. Together with Lemma [1] (ii) this proves

**Theorem 2.** The memory cost for the contextuality correlations $\mathcal{D}_{\text{context}}'$ in the PM square is $\log_2(3) \approx 1.58$; $M(\mathcal{D}_{\text{context}}) = M(\mathcal{D}_{\text{context}}') = \log_2(3)$. Consequently, the automaton $A_3$ is memory-optimal.

The set $\mathcal{D}_{\text{context}}'$ contains all sequences of mutually compatible observables, but does not contain sequences like $ABaA$, for which QM also predicts that both occurring values of $A$ are the same. Sequences of this form enforce that all observables compatible with an observable $X$ must not change the measurement result of $X$. This can be covered by the set of all compatibility conditions

$$\mathcal{D}_{\text{compat}} = \{Y[X_1, X_2, \ldots] | X_\ell \text{ are compatible with } Y\}. \tag{6}$$

Again we define $\mathcal{D}_{\text{compat}}'$ to include arbitrary preparation procedures. The automaton $A_3$ does not obey $\mathcal{D}_{\text{compat}}'$, since e.g. starting with state $S_1$, the sequence $B[C]B$ yields the record $+1, [+1, -1], -1$ and hence violates the assumption of compatibility; similar sequences can be found for any initial state. We show in Appendix D that no automaton with three states can obey simultaneously $\mathcal{D}_{\text{compat}}'$ and $\mathcal{D}_{\text{context}}'$, i.e., $M(\mathcal{D}_{\text{compat}}' \text{ and } \mathcal{D}_{\text{context}}') \geq 2$. However, there exist automata with four internal states which obey $\mathcal{D}_{\text{compat}}'$ and $\mathcal{D}_{\text{context}}$. As an example of such an automaton we define $A_4$ via

$$S_1: \begin{bmatrix} + & + & (+, 2) \\ + & + & (+, 3) \\ + & + & + \end{bmatrix}, \quad S_2: \begin{bmatrix} + & + & + \\ - & + & - \\ (−, 4) & (+, 1) & + \end{bmatrix},$$

$$S_3: \begin{bmatrix} + & − & − \\ + & + & + \\ (+, 1) & (−, 4) & + \end{bmatrix}, \quad S_4: \begin{bmatrix} + & − & (−, 3) \\ − & + & (−, 2) \\ − & − & + \end{bmatrix}. \tag{7}$$

Similar to the situation for $A_3$, the initial state for the automaton $A_4$ can be chosen freely; we refer to Appendix B for details. So we have:

**Theorem 3.** The memory cost of an automaton that obeys $\mathcal{D}_{\text{compat}}'$ and $\mathcal{D}_{\text{context}}'$ is two bits; $M(\mathcal{D}_{\text{compat}}' \text{ and } \mathcal{D}_{\text{context}}') = 2$. Hence, the automaton $A_4$ is memory-optimal.

**Extended Peres-Mermin square.**—There are, however, further contextuality effects for two qubits which then require more than two bits for a simulation. Namely, in Ref. [28] an extension of the PM square has been introduced, involving 15 different observables in 15 different contexts. The argument goes as follows: Consider the 15 observables of the type $\sigma_\mu \otimes \sigma_\nu$ where $\mu, \nu \in \{0, x, y, z\}$ and $\sigma_0 = \mathbb{1}$ and the case $\mu = \nu = 0$ is excluded. In this set there are 12 trios of mutually compatible observables such that the product of their results is always $+1$, like $[\sigma_x \otimes \mathbb{1}, \mathbb{1} \otimes \sigma_y, \sigma_x \otimes \sigma_y]$ and
tuality in sequential measurements. We determined the memory needed in order to simulate quantum contextuality and the Kochen–Specker theorem on the other side. While for Bell’s theorem such connections are well explored and have given deep insights in QM [17, 22, 30], for contextuality many questions remain open: If an experiment violates some noncontextuality inequality up to a certain degree, but not maximally, what memory is required to simulate this behavior? Can nondeterministic machines help to simulate contextuality? What amount of memory and randomness is required to simulate all quantum effects in the PM square, especially in the distributed setting [11]? Finally, for the Bell inequality of Clauser, Horne, Shimony, and Holt, it has been extensively investigated why QM does not allow the maximal possible violation [29, 30]. This inequality can also be used to exclude noncontextual models [21, 27]. Can concepts from information theory also help to understand the nonmaximal violation in this situation?

The authors thank E. Amselem, P. Badziąg, J. Barrett, I. Bengtsson, M. Bourennane, Č. Brukner, P. Horodecki, A. R. Plastino, M. Rådmark, and V. Scholz for discussions. This work has been supported by the FWF (START prize and SFB FOQUS) and the EU (QICS, NAMEQUAM). J.R.P. acknowledges support from Project No. MTM2008-05866. A.C. acknowledges support from Project No. FIS2008-05596.

APPENDIX A: $A_3$obeys $\mathcal{L}_{\text{context}}$

In this Appendix we demonstrate that the automaton $A_3$ indeed obeys $\mathcal{L}_{\text{context}}$. For that it is enough to show, that for any choice of the initial state, the automaton will obey $\mathcal{L}_{\text{context}}$. So we assume that $S_1$ is the initial state; the reasoning for $S_2$ and $S_3$ is similar. If we now measure a sequence with observables from the first row only, we may jump between the states $S_1$ and $S_2$, but the output for all observables in the first row are the same for either state. A similar argument holds for all rows and the first and second column. For a sequence with measurements from the third column, assume that the first observable in the sequence, that is not $\gamma$, is the observable $c$. Then the state changes to $S_3$, in which the last column does not yield a contradiction. Since only the output $C$ was changed, but $c$ was not measured so far, we cannot get any contradiction. A similar argument can be used for the case where the first observable in the sequences, that is not $\gamma$, is the observable $C$. ■
APPENDIX B: $A_4$obeys $\mathcal{L}_{\text{context}}$ and $\mathcal{L}_{\text{compat}}$

In this Appendix we demonstrate that the automaton $A_4$ indeed obeys $\mathcal{L}'_{\text{context}}$ and $\mathcal{L}'_{\text{compat}}$. The proof for $\mathcal{L}'_{\text{context}}$ is completely analogous to the one in Appendix A.

For $\mathcal{L}'_{\text{compat}}$, we consider a fixed observable, e.g. $B$. Then $S_1$ and $S_2$ yield +1 while $S_3$ and $S_4$ give −1. However, using arbitrary measurements compatible to $B$, (i.e., $A$, $B$, $C$, $b$, and $\beta$) we can never reach $S_3$ or $S_4$ if we start from $S_1$ or $S_2$ and vice versa. Hence no contradiction occurs for any sequence of the type $[T|B|X_1X_2\ldots]I$. A similar argument holds for all observables, if we note in addition that e.g. after a measurement of $C$ the automaton can only be in $S_2$ or $S_3$.

APPENDIX C: DEFINITIONS AND BASIC RULES USED IN THE OPTIMALITY PROOFS

As we already did in the main text, we denote the observables from the PM square by

$$\begin{bmatrix} A & B & C \\ a & b & c \\ \alpha & \beta & \gamma \end{bmatrix},$$

(8)

the rows by $R_i$, and the columns by $C_j$. The value tables of for each memory state $i$ is denoted by $T_i$ and the update table by $U_i$. We write an entry of zero in $U_i$, if the state does not change for an observable. Furthermore, we write measurement sequences as $A^i_1B_2C_3a_4$, meaning that when the sequence $ABCa$ was measured, the results were $+,-,-,+$, and the memory was initially in state $S_1$ and changed like $S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow S_3$.

It is also useful to note some rules about the structure of the tables and update rules. This is convenient for our later discussion.

1. **Sign flips**: Let us assume that we have an automaton obeying a set $\mathcal{L}$ and pick a $2 \times 2$ square of observables (e.g., the set $\{A, B, a, b\}$ or $\{A, B, \alpha, \beta\}$ or $\{A, C, \alpha, \gamma\}$). Then, if we flip in each $T_i$ the signs corresponding to these observables, we will obtain another valid automaton.

   This holds true, because the mentioned sign flips do not change any of the certain quantum predictions. This rule will allow us later to fix one or two entries in a given $T_i$.

2. **Number of contradictions**: Any table $T_i$ contains either one, three, or five contradictions to the PM conditions.

   This follows directly from that fact that any fixed assignment fulfills $\prod_k R_k C_k = +1$, while the PM conditions require $\prod_k R_k C_k = -1$.

3. **Condition for fixing the memory**: Let us assume that we have an automaton obeying $\mathcal{L}'_{\text{context}}$ and let there be a table $T_i$ which assigns to an observable (say $A$) a value different from all other tables. Then, the update table $U_i$ must contain only zeroes in the corresponding row and column (here, $R_1$ and $C_1$).

   The observables in the row and column correspond to compatible observables, which are not allowed to change the value of the first observable. However, any change of the memory state would change the value, as $T_i$ is the only table with the initial assignment.

4. **Contradictions and transformations**: Let us assume that we have an automaton obeying $\mathcal{L}'_{\text{context}}$ and let there be in $T_i$ some contradiction in the column $C_j$ (or the row $R_j$). Then, in the update table $U_i$ there cannot be two zeroes in the the column $C_j$ (or the row $R_j$).

   If there were two zeroes, it could happen that one measures two entries of $C_j$ without changing the memory state. But then measuring the third one will reveal the contradiction in $T_i$. (Note that the automaton first provides the result and then updates its state.)

5. **Contradictions and other tables**: Let us assume that we have an automaton obeying $\mathcal{L}'_{\text{context}}$ and let there be in $T_i$ a contradiction in the column $C_j$ (or the row $R_j$). Then, there must be two different tables $T_k$ and $T_l$ where in both $C_j$ has no contradictions anymore, but the assignments of $T_k$ and $T_l$ differ in two observables of $C_j$. Furthermore, in the column $C_j$ of the update table $U_i$ there must be two entries leading to two different states.

   First, note that there must be at least one other table $T_k$ where the contradiction does not exist anymore. This follows from the fact that we may measure $C_j$ starting from the memory state $i$. After having made these measurements, we arrive at some state $k$, and from the contextuality correlations $\mathcal{L}'_{\text{context}}$ it follows that $C_j$ in $T_k$ has no contradiction.

   The table $T_k$ differs from $T_i$ in at least one observable $X$ in $C_j$. On the other hand, starting from $T_i$ one might measure $X$ as a first observable. Then, making further measurements on $C_j$ one must arrive at a table $T_l$ without a contradiction. Since $T_k$ and $T_l$ have both no contradiction, they must differ in at least two places, one of them being $X$. The rest follows from Rule 4.

   Note that this rule is similar to Lemma 1(ii) in the main text. While in Lemma 1 the correlations $\mathcal{L}'_{\text{context}}$ were not used, the main idea, namely that
Appendix D: A₄ Is Memory-Optimal

Here, we prove the optimality of the 4-state automaton A₄, in the sense of obeying D₁₄ and D₁₅ with a minimum number of states. We use the definitions and rules as introduced in Appendix C.

Let us assume that we would have a three-state automaton obeying D₁₅. Rule 1 has a contradiction, and we can assume, without loss of generality, that it is C₃. Then, according to Rule 1 we can, without loss of generality, that it is C₃. Together with Rule 5 this leads to the conclusion that the three states Tᵢ are, without loss of generality, of the form:

\[
T_1: \begin{bmatrix} + \\ + \\ 2 \\ 3 \end{bmatrix}, \quad T_2: \begin{bmatrix} + \\ + \\ 0 \\ 0 \end{bmatrix}, \quad T_3: \begin{bmatrix} + \\ - \\ 0 \\ 0 \end{bmatrix}. \tag{9a}
\]

Let us now discuss the extended PM square from Ref. 28. Again, we refer to Appendix C for basic definitions and rules. As already mentioned, one considers for that the array of observables

\[
\begin{bmatrix}
\chi_{01} & \chi_{02} & \chi_{03} \\
\chi_{10} & \chi_{11} & \chi_{12} & \chi_{13} \\
\chi_{20} & \chi_{21} & \chi_{22} & \chi_{23} \\
\chi_{30} & \chi_{31} & \chi_{32} & \chi_{33}
\end{bmatrix} = \begin{bmatrix}
\sigma_x \otimes \mathbb{I} & \sigma_y \otimes \mathbb{I} & \sigma_z \otimes \mathbb{I}
\end{bmatrix}.
\] \tag{11}

These observables can be grouped into trios, in which the observables commute and their product equals ±1. Nine trios are of the form {χₐ₀, χₐ₁, χₐ₂}, three trios where the product equals +1 are {χ₁₁, χ₂₃, χ₃₂}, {χ₁₂, χ₂₁, χ₃₃}, and {χ₁₃, χ₂₂, χ₃₁}. Three trios where the product equals −1 are {χ₁₁, χ₂₂, χ₃₃}, {χ₁₂, χ₂₁, χ₃₁}, and {χ₁₃, χ₂₂, χ₃₂}. From this, one can derive the inequality

\[
\sum_{k,l} (\langle \chiₖ₀ | χₖ₁ | χₖ₂⟩ + |χ₁₁|χ₂₃χ₃₂⟩ + |χ₁₂|χ₂₁χ₃₃⟩ + |χ₁₃|χ₂₂χ₃₁⟩ \leq 9 \tag{12}
\]

for noncontextual models, while QM predicts a value of 15, independently of the state.
the 10 PM squares, if we consider them separately. Since in a PM square the number of contradictions cannot be two (Rule 2), this means that one of the PM squares has to have three contradictions.

Let us now assume that we have a valid automaton for this extended PM square with four memory states. Of course, this would immediately give a valid four-state automaton of any of the 10 PM squares. For one of these PM squares, at least one table has to have three contradictions. So it suffices to prove the following Lemma:

**Lemma 5.** There is no four-state automaton obeying $\mathcal{L}_{\text{comput}}$ and $\mathcal{L}_{\text{context}}$, where one table $T_i$ has three contradictions.

In course of proving this Lemma we will also prove the following:

**Proposition 6.** The four-state automaton $A_4$ is unique, up to some permutation or sign changes.

To prove the Lemma, we proceed in the following way: Without loss of generality, we can assume that the first three tables $T_i$ look like the $T_i$ in Eq. (15). Then, we can add a fourth table $T_4$. For the last column of this table, there are $2^{3} = 8$ possible values. We will investigate all eight possibilities and show that either we arrive directly at a contradiction, or that only an automaton similar to $A_4$ is possible, in which any table has only one contradiction. This proves the Lemma.

We will first deal with the four cases, where $T_4$ has also a contradiction in $C_3$. This will lead to Observation [7] which will be useful in the following four cases.

**Case 1:** For $T_4$ one has $[C, c, \gamma] = [+ + +]$:

In this case, a simple application of the previous rules implies that several entries are fixed:

$$T : \begin{bmatrix} + \\ + \\ + \\ + \end{bmatrix}, \begin{bmatrix} + \\ + \\ - \\ + \end{bmatrix}, \begin{bmatrix} + \\ - \\ + \\ + \end{bmatrix}, \begin{bmatrix} + \\ + \\ + \\ + \end{bmatrix}.$$  \hfill (13)

$$U : \begin{bmatrix} 2 \\ 0 0 0 \\ 0 0 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 0 0 \\ 0 0 0 \end{bmatrix}.$$  \hfill (14)

Here and in the following, we write the $T_i$ and $U_i$ just as a row for notational simplicity, starting from $T_1$ to $T_4$. The entries in $U_1$ and $U_2$ are fixed from the following reasoning: Let us assume that one measures $c$ in $T_1$, then, since the values $C(T_i)$ are the same in all $T_i$, one has to change immediately to a table with no contradiction in $C_3$, and where the value of $c$ is still the same. The only possibility is $T_2$. Furthermore, $R_2$ in $U_2$ and $R_3$ in $U_3$ must be zero due to the same argument which led to Eq. (10b).

It follows (Rule 4) that $T_2$ and $T_3$ have both exactly one contradiction, which must be in $R_1$. So, in $R_1$ of $U_2$ there must be the entries “1” and “4” [an entry “3” would not solve the problem, because in $R_1(T_3)$ has also a contradiction]. As we can still permute the first and second column, we can without loss of generality assume that the first row in $U_2$ is $[1 \ 4 \ 0]$. Due to Rule 1, we can also assume, without loss of generality, that $A(T_2) = +$. Similarly, in $R_1(U_3)$ there must be the entries “1” and “4”, resulting in two different cases:

If $R_1(U_3) = [1 \ 4 \ 0]$, we must have the following tables,

$$T : \begin{bmatrix} + \\ + \\ + \\ + \end{bmatrix}, \begin{bmatrix} + \\ + \\ - \\ + \end{bmatrix}, \begin{bmatrix} + \\ - \\ + \\ + \end{bmatrix}, \begin{bmatrix} + \\ + \\ + \\ + \end{bmatrix}.$$  \hfill (15)

$$U : \begin{bmatrix} 2 \\ 0 0 0 \\ 0 0 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 0 0 \\ 0 0 0 \end{bmatrix}.$$  \hfill (16)

where the added values in $R_1$ of the $T_i$ follow from $R_1(U_2)$ and $R_1(U_3)$.

Now, if we start from $T_2$ and measure the sequence $a_2 A_2 a_1$, we see that we must have $a(T_1) = a(T_2)$. Similarly, from $T_3$ we can measure $a_2 A_2 a_1$, implying that $a(T_1) = a(T_2) = a(T_3)$. Similarly, we find that $b(T_2) = b(T_3) = b(T_4)$. But this gives a contradiction: In $R_2(T_2)$ and $R_2(T_3)$ there is no contradiction and $c(T_2) \neq c(T_3)$. Therefore, it cannot be that $a(T_2) = a(T_3)$ and at the same time $b(T_2) = b(T_3)$.

As the second case, we have to consider the possibility that $R_1(U_3) = [4 \ 1 \ 0]$. Then, also the values of $R_1(T_3)$ must be interchanged, $R_1(T_3) = [+ + +]$. Then, starting from $T_2$, the sequence $a_2 A_2 a_1 \gamma_3$ shows directly that $a(T_2) = a(T_3)$. Similarly, starting from $T_3$, the sequence $a_2 A_2 a_1 \gamma_3$ shows that $a(T_2) = a(T_3)$. But since $A(T_2) \neq A(T_3)$, this is a contradiction.

**Case 2:** For $T_4$ one has $[C, c, \gamma] = [+ - -]$:

As in Case 1, one can directly see that several entries are fixed:

$$T : \begin{bmatrix} + \\ + \\ + \\ + \end{bmatrix}, \begin{bmatrix} + \\ - \\ + \\ + \end{bmatrix}, \begin{bmatrix} + \\ - \\ + \\ + \end{bmatrix}, \begin{bmatrix} + \\ + \\ + \\ + \end{bmatrix}.$$  \hfill (17)

$$U : \begin{bmatrix} 2 \\ 0 0 0 \\ 0 0 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 0 0 \\ 0 0 0 \end{bmatrix}.$$  \hfill (18)

The zeroes in $U_2$ and $U_3$ come from the following argumentation: Starting from $T_1$, the measurement sequence $c_1^+ X_2 \gamma_1 c_3^-$ with $X$ compatible to $c$ shows that in $R_2(U_2)$ and $C_2(U_2)$ there can be no “3” or “4”. But there can be also no “1”, because then the sequence $c_1^+ X_2 \gamma_1 c_3^-$ would lead to a contradiction. Therefore, $R_2(U_2)$ and $C_2(U_2)$ have to be zero. Starting from $T_4$ and measuring $\gamma$ one can
similarly prove that the entries for \( R_3(U_2) \) have to be zero and analogous arguments prove also the zeroes in \( U_3 \).

It is now clear (Rule 4) that the contradictions in \( T_2 \) and \( T_3 \) have to be in \( R_1 \) and the missing entries in \( U_2 \) and \( U_3 \) can only be “4” and “1”. As we still can permute the first and second column, there are only two possibilities:

**Case 2A:** First, we consider the case that \( R_1(U_2) = R_1(U_3) = [1 4 0] \). As in Case 1, we can directly see that \( a(T_2) = a(T_1) = a(T_3) \) and \( b(T_2) = b(T_1) = b(T_3) \). Hence, \( R_2(T_2) \) and \( R_2(T_3) \) differ exactly in the value of \( c \), but in both cases there is no contradiction in \( R_2 \). This is not possible.

**Case 2B:** Second, we consider the case that the first rows of \( U_2 \) and \( U_3 \) differ, and we take \( R_1(U_2) = [1 4 0] \) and \( R_1(U_3) = [4 1 0] \). Then, we apply Rule 1 to fix for \( A(T_3) = a(T_3) = + \). Then, the tables have to be:

\[
T : \begin{bmatrix}
- & - & + \\
- & + & + \\
+ & + & - \\
+ & - & + \\
\end{bmatrix}
\]

(19)

\[
U : \begin{bmatrix}
2 \\
3 \\
2 \\
3 \\
\end{bmatrix}
\]

(20)

Here, \( C_2(T_1) \) and \( C_1(T_4) \) come from measurement sequences like \( a_3^+ \gamma_1 a_3^+ \), starting from \( T_3 \).

Again, we have two possibilities for the value of \( b \) in \( T_2 \). If we set \( b(T_2) = - \), then all values in all \( T_i \) are fixed and each table has exactly one contradiction. This is, up to some relabelling, the four-state automaton \( A_4 \) from the main text (indeed, this is the way how this solution was found). If we set \( b(T_2) = + \), then also all \( T_i \) can be filled, and we must have:

\[
T : \begin{bmatrix}
- & - & + \\
- & + & + \\
+ & + & - \\
+ & - & + \\
\end{bmatrix}
\]

(21)

\[
U : \begin{bmatrix}
2 \\
3 \\
2 \\
3 \\
\end{bmatrix}
\]

(22)

Here, the tables \( T_1 \) and \( T_3 \) have three contradictions (two new ones in \( R_2 \) and \( R_3 \)) and the new entries in \( U_1 \) and \( U_3 \) must be introduced according to Rule 5 [note that \( a(T_1) \) and \( b(T_1) \) are for all tables the same]. Then, however, starting from \( T_1 \), the sequence \( a_3^+ \gamma_1 a_3^+ \gamma_1 \) shows that this is not valid solution.

**Case 3:** For \( T_4 \) one has \( [C, c, \gamma] = [- + -] \):

In this case, a simple reasoning according to the usual rules fixes the entries:

\[
T : \begin{bmatrix}
+ \\
+ \\
- \\
+ \\
\end{bmatrix}
\]

(23)

\[
U : \begin{bmatrix}
3 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

(24)

Here we have an obvious contradiction in \( T_4/U_4 \): \( C_3(T_4) \) contains a contradiction, but (due to Rule 3) one is not allowed to change the memory state when measuring it. Therefore, the memory can never be in the state 4. But then, one would have effectively a three-state solution, which is not possible, as we know already.

**Case 4:** For \( T_4 \) one has \( [C, c, \gamma] = [- - +] \):

This is the same as Case 3, where \( R_2 \) and \( R_4 \) have been interchanged.

Now we have dealt with all the cases, where \( T_4 \) contains a contradiction in \( C_3 \), just as \( T_1 \). We have seen that in this cases there can only be a solution if each table contains exactly one contradiction, and this solution is unique, up to some permutations or sign flips. Moreover, we could have made the same discussion with rows instead of columns. Therefore from the first four cases we can state an observation which will be useful in the remaining four cases:

**Observation 7.** If in any four-state solution two tables \( T_i \) and \( T_j \) have both a contradiction in the same column \( C_k \) (or row \( R_k \)), then there has to be exactly one contradiction in each value table of the automaton.

So, if there is a four-state solution where one table has three contradictions, then it cannot be that two tables have both a contradiction in the same column or row.

Then we can proceed with the remaining cases.

**Case 5:** For \( T_4 \) one has \( [C, c, \gamma] = [+ - +] \):

This is the critical case, as it is difficult to distinguish the tables \( T_2 \) and \( T_4 \) here. First, the following entries are directly fixed:

\[
T : \begin{bmatrix}
+ \\
+ \\
- \\
+ \\
\end{bmatrix}
\]

(25)

\[
U : \begin{bmatrix}
2 \\
0.4 \\
0.4 \\
0.4 \\
\end{bmatrix}
\]

(26)

Here, \( c(U_1) = 2 \) has been chosen without loss of generality. It is clear that \( c(U_1) = 2 \) or \( c(U_1) = 4 \), as \( T_2 \)
and $T_4$ are equivalent at the beginning, we can choose $T_2$ here. The entries of the type $i,j$ in $U_2$ and $U_4$ mean that the numbers can be $i$ or $j$, but nothing else. The values of $c(U_2)$ and $c(U_4)$ cannot be $1$, because then the sequence $c_1^C C_2^R c_3$ directly reveals a contradiction. Furthermore, the zeroes in $R_3(U_3)$ and $R_2(U_4)$ follow similarly as Eq. (10E) or from Rule 3. In addition, $C(U_2) \neq 1$, because otherwise the sequence $c_1^C C_2^R c_3$ reveals a contradiction to the PM conditions. Also, $C(U_2) \neq 3$, because of $c_1^C C_2^R c_4$. Similarly, 1 and 3 are excluded as values for $a(U_2)$ and $b(U_2)$, due to the sequences $c_1^C a_2^R c_3^+$ and $c_1^C a_2^R c_3$.

Furthermore, we can use our Observation 4. If in a four-state solution one column has a contradiction in two of the $T_i$, then there can be only one contradiction in any $T_i$. Here we can use it as follows: It is clear that $T_3$ has its contradiction in $R_1$. Since we aim to rule out a four-state solution where one table has three contradictions, we can assume that there is no contradiction in $R_1$ in all the other $T_i$ (especially in $T_2$ and $T_3$). Otherwise, we would already know that no solution exists with three contradictions in a table. We can distinguish two cases:

**Case 5A:** Let us assume that $\gamma(U_2) = 0$. Then, the tables must read:

$$T : \begin{bmatrix} + & + & + & + \\ + & - & + & + \end{bmatrix},$$

$$U : \begin{bmatrix} 2 & 0.4 & 0.4 & 0.4 \\ 3 & 0.4 & 0.4 & 0 \end{bmatrix}.$$

The new entries in $U_2$ follow from $\gamma(U_2) = 0$ in combination with $\gamma(T_1) = \gamma(T_3) = \gamma(T_2) = 0$.

Due to Rule 5, the table $T_2$ must have a contradiction in $C_1$, $C_2$, or $R_1$. From Observation 7 we can assume that it is not in $R_1$. Due to possible permutations of $C_1$ and $C_2$ we further assume without loss of generality that the contradiction is in $C_1$. Then we have:

$$U : \begin{bmatrix} 2 & 1 & 0.4 & 0.4 \\ 3 & 0.4 & 0.4 & 0 \end{bmatrix}.$$

We cannot have $A(U_2) = 3$, since there is a contradiction in $R_1(T_3)$ and $C(T_1) = +$ for all tables. In addition, due to Rule 5, it is not possible that $A(U_2) = 4$. Finally, we choose $a(U_2) = 4$, the other option would be $\alpha(U_2) = 4$, which will be discussed below. Then, we can conclude that in $R_1(U_1)$ and $C_1(U_1)$ we cannot have the entries “2” and “4”, and in $R_2(U_3)$ and $C_1(U_3)$ we cannot have the entries “2” and “1”. To see this, note that we must have $A(T_2) = A(T_1) \neq A(T_4)$ and, if $B(U_1) = 2$, we can consider the measurement sequence $A_2 B_1 A_2 A_4$ or, if $B(U_1) = 4$, the sequence $A_2 B_1 A_4$ etc. Hence, we have:

$$U : \begin{bmatrix} 0.3 & 2 & 4 & 0.4 & 0.4 \\ 0.3 & 3 & 0.4 & 0.4 & 0 \end{bmatrix},$$

Here, we used in $R_1(U_1)$ that $R_1(T_3)$ has a contradiction and $C(T_4) = +$ for all tables, so it is not possible to go there.

Now, by Rule 1, we may fix $A(T_2) = a(T_2) = +$. Then we arrive at

$$T : \begin{bmatrix} + & + & + & + \\ + & + & + & + \end{bmatrix}.$$

$$U : \begin{bmatrix} 0.3 & 2 & 4 & 0.4 & 0.4 & 0 \\ 0.3 & 0.4 & 0.4 & 0 \end{bmatrix}.$$
In this case, the tables read:

**Case 7:**

For $\omega = 5$, we must have:

$$T : \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & + \\ - & - & - \\ + & - & - \\ + & + & + \end{bmatrix}, \quad U : \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 4 \\ 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (34)$$

However, if $b(U_1) = 3$, then the sequence $B_1^+ B_1 A_3 B_1^-$ leads to a contradiction, while, if $\beta(U_1) = 3$, then the sequence $B_1^+ \beta_1 A_3 B_1^-$ leads to a problem.

Finally, if we would have taken $\alpha(U_2) = 4$ or $\alpha(T_1) \neq \alpha(T_4)$ the proof would proceed along the same lines, but this time the contradiction in $T_4$ would be in the second row.

**Case 5B:** Let us assume that $\gamma(U_2) = 4$. Then, many entries on $U_4$ are fixed and we have:

$$T : \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & + \\ + & + & + \\ + & + & + \\ + & + & + \end{bmatrix}, \quad U : \begin{bmatrix} 2 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (36)$$

Here we cannot have $a(U_4) = 1$, due the sequences $c_2 \gamma_2 a_4 c_1$ [if $c(U_2) = 0$] or $c_2 a_4 c_1$ [if $c(U_2) = 4$], and also not $a(U_3) = 3$, due to similar sequences. The same arguments apply to $b(U_4)$. The entries in $R_3(U_4)$ and $C_3(U_4)$ come from possible sequences like $\gamma_4 a_4\gamma_4$ if $\gamma(U_4) = 0$ or $\gamma_4 a_4\gamma_4$ if $\gamma(U_4) = 2$.

But then the proof can proceed exactly as in the Case 5A, with $T_2$ and $T_4$ interchanged: The only significant difference comes from $c(U_1) = 2 \neq 4$, but this was never used in the proof.

**Case 6:** For $T_4$ one has $[C, c, \gamma] = [+]$:

This is the same as the Case 5 with a permutation of $R_2$ and $R_3$.

**Case 7:** For $T_4$ one has $[C, c, \gamma] = [-+]$:

In this case, the tables read:

$$T : \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & + \\ - & - & - \\ + & - & - \\ + & + & + \end{bmatrix}, \quad U : \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (39)$$

Here, the entries in $U_1$ have been chosen without loss of generality: From Rule 4 and 5 it follows that one can restrict the attention to the cases where $C_2(U_1) = [\omega, 2, 3], C_3(U_1) = [\omega, 2, 4], \text{ or } C_3(U_1) = [\omega, 4, 3]$. We only consider the first possibility, in the other cases the proof is analogous and is left to the gentle reader as an exercise.

The zeroes in $U_2, U_3$, and $U_4$ come from Rule 3. The entries $0_2$ in $U_4$ come from possible measurement sequences like $c_2 a_4 c_1$ or $c_2 a_4 c_3$ which prove that there cannot be the entries “3” or “1”. The other entries can be derived accordingly.

From Rule 5, it follows that in $T_4$ the contradiction cannot be in the rows, so it has to be in the first or second column. Let us assume, without loss of generality, that it is in $C_1(T_4)$. Further, we can assume without loss of generality, that the values $A$ and $a$ in $T_4$ are both “+”.

Then, the tables can be more specified as

$$T : \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & + \\ - & - & - \\ + & - & - \\ + & + & + \end{bmatrix}, \quad U : \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (40)$$

To see this, one first fills $T_4$, then, together with the entries of $C_1(U_4)$, many values of $T_2$ and $T_3$ are fixed. The entries $0_1$ are justified similar to the reasoning above.

In $T_2$ as well as in $T_3$ the contradiction has to be either in $R_1$ or $R_2$. However, there cannot be a contradiction in $R_1$. To see this, assume that there were a contradiction in $R_1(T_2)$. Then, starting from $T_2$ we may measure the sequence $C_2 A_2 B$ or $C_2 B_2 A$. According to Rule 5, we must end in two different $T_1$. But the memory state can never change to $T_4$ [because $C(T_4) = -$]. So we must have $B(U_2) = 3$, but this will not escape the contradiction, since the values for $A$ and $C$ coincide in $T_2$ and $T_3$. So there is only $T_1$ left, and we arrive at a contradiction.

Consequently, the contradictions have to be both in $C_2(T_2)$ and $C_2(T_3)$. In principle, our Observation implies already that we cannot find a solution with three contradictions in one table. But one can also directly prove that there is no solution at all. We have:

$$T : \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & + \\ - & - & - \\ + & - & - \\ + & + & + \end{bmatrix}, \quad U : \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (42)$$

$$U : \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (43)$$

Here, we must have $B(T_1) = B(T_2) = B(T_3) = +$ due to measurement sequences like $B_2^+ B_1^+ B_1^+$ or $B_2^+ B_1^+$ and
\[ \beta(T_1) = \beta(T_2) \text{ due to } \beta_2^+ B_2^\pm \beta_1^{-} \text{ and } b(T_1) = b(T_3) \text{ due to } b_1^+ B_1^\pm b_1^+ \text{ reveals a contradiction to the PM conditions.} \]

Case 8: For \( T_4 \) one has \([C, c, \gamma] = [\cdots] \):

In this case, we directly have:

\[
T : \begin{bmatrix} + & + & + & - \\ + & + & - & - \\ + & - & - & - \end{bmatrix}, \quad U : \begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix} \quad \text{(44)}
\]

Starting from \( T_2 \) we may measure the sequence \( C_2 A_2 B \) or \( C_2 B_2 A \). According to Rule 5, we must end in two different \( T_i \). But the memory state can neither change to \( T_1 \) [because \( C(T_1) = - \)] nor to \( T_3 \) [as \( R_1(T_3) \) contains a contradiction]. So there is only \( T_1 \) left, and we arrive at a contradiction.

In summary, by considering all eight different cases we have shown that no four-state solution exists in which one table has three contradictions. This proves the claim. \( \blacksquare \)

**APPENDIX F: A 10-STATE AUTOMATON OBEYING ALL SEQUENCES**

In this Appendix we show an example of a 10-state automaton that obeys the set of all sequences \( \mathcal{L}_{PM} \). For that, we define 10 eigenstates of two compatible observables. For instance, let \( |A^{-} B^+\rangle \) be a quantum state with \( A|A^{-} B^+\rangle = -|A^{-} B^+\rangle \) and \( B|A^{-} B^+\rangle = +|A^{-} B^+\rangle \). In this fashion we define the 10 states \( |A^+ B^+\rangle, |A^{-} B^+\rangle, |C^+ c^+\rangle, |C^{-} c^+\rangle, |\gamma^+ \beta^+\rangle, |\gamma^{-} \beta^+\rangle, |\alpha^+ a^+\rangle, |\alpha^{-} a^+\rangle, |\alpha^+ b^+\rangle, \) and \( |B^+ b^+\rangle \). Any measurement of an observable from the PM square projects with finite probability any state of the set onto another state of the set. If e.g. the automaton is in state \( |A^{-} B^+\rangle \) and we measure \( c \), QM predicts a chance of 50% to get the outcome \(+1\) yielding the state \( |C^{-} c^+\rangle \), and a 50% chance to obtain \(-1\) and the state \( |C^{-} c^-\rangle \). The former state is in the set of 10 states and hence our automaton would return \(+1\) and change to the state \( |C^{-} c^+\rangle \). We furthermore define that, if both states predicted by QM are in the set of 10 states, then we prefer the state corresponding to the output of \(+1\).

Together with an arbitrary choice of the initial state, this completes the definition of the automaton. By construction, this automaton is deterministic and obeys \( \mathcal{L}_{PM} \). \( \blacksquare \)