Mappings sharing a value on finite-dimensional spaces

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Abstract

Sufficient conditions are given to assert that differentiable mappings and differentiable proper mappings between \( \mathbb{R}^n \) and \( \mathbb{R}^m \) share a value. The proof of the result is constructive and is based upon continuation methods. © 2001 Elsevier Science Inc. All rights reserved.

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1. Preliminaries

In this paper priori conditions are given to prove that a function has at least one zero. Other sufficient conditions have been given in previous papers [1–4] to guarantee the existence of zero points. These conditions were essentially compactness conditions. In paper [5], we studied the case of bounded perturbations of proper mappings. Here, we study the existence of zeros of differentiable perturbations of proper mappings between finite dimensional Banach spaces of not necessarily the same dimension.

To prove our theorem we use continuation methods (see [1–14]). They supply the existence of curves reaching zero points [1–4,6,10,15,16]. Essentially, we have to study a suitable homotopy proving its zeros are in a compact set.

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Here we use the Implicit Function theorem [14,17] as a principal tool. Let us briefly recall some concepts that will be used [14,18].

Let $X$ and $Y$ be Banach spaces, and $f : D(f) \subseteq X \to Y$ a mapping. Then $f$ is called compact if and only if $f$ is continuous and maps bounded sets into relatively compact sets. For finite dimensional Banach spaces, continuous and compact mappings are the same whenever $D(f)$ is closed, for instance if $D(f) = X$. Let $f : X \to Y$ be a mapping between Banach spaces $X, Y$. Then $f$ is called proper if and only if the pre-image of every compact set $C$ is also compact. Let $Y$ be a Banach space, and $Y_1, Y_2$ linear subspaces of the linear space $Y$. Suppose that every $y \in Y$ can be written uniquely as $y = y_1 + y_2$, where $y_1 \in Y_1$ and $y_2 \in Y_2$. Then we write

$$Y = Y_1 \oplus Y_2$$

and say $Y$ is a direct sum.

Finally we state the Implicit Function Theorem for future references (see [13, pp. 150–154]).

**Theorem 1.** Suppose that:

(i) The mapping $F : U(x_0, y_0) \subseteq X \times Y \to Z$ is defined in an open neighbourhood $U(x_0, y_0)$, and $F(x_0, y_0) = 0$, where $X, Y$ and $Z$ are Banach spaces over $K = \mathbb{R}$ or $K = \mathbb{C}$.

(ii) $F_y$ exists as a partial Frechét-derivative on $U(x_0, y_0)$ and the partial derivative $F_y(x_0, y_0) : Y \to Z$ is bijective.

(iii) $F$ and $F_y$ are continuous at $(x_0, y_0)$.

Hence the following are true:

(a) Existence and uniqueness. There exist positive numbers $r_0$ and $r$ such that, for every $x \in X$ satisfying $\|x - x_0\| \leq r_0$, there is exactly one $y(x) \in Y$ for which $\|y(x) - y_0\| \leq r$ and $F(x, y(x)) = 0$.

(b) Continuity. If $F$ is continuous in a neighbourhood of $(x_0, y_0)$, then $y(\cdot)$ is continuous in a neighbourhood of $x_0$.

(c) Continuous differentiability. If $F$ is a $C^m$-mapping, $1 \leq m \leq \infty$ in a neighbourhood of $(x_0, y_0)$, then $y(\cdot)$ is also a $C^m$-mapping in a neighbourhood of $x_0$.

2. Mappings sharing a value

Clearly, if we say $F = f - g$, then $F$ has a zero if and only if $f$ and $g$ share a value, that is, $x \in X$ exists with $f(x) = g(x)$. We establish our result in terms of $f, g$.

**Theorem 2.** Let $f, g : \mathbb{R}^n \to \mathbb{R}^m$ be two $C^1$-mappings between the spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, with $n \geq m$. Suppose that:
(i) \( f \) has at least one zero \( x_0 \) and is proper.
(ii) There exists \( M > 0 \), with \( \|x_0\| < M \) such that

\[
\text{if } \|x\| = M \Rightarrow f'(x) \neq tg(x) \quad \forall t \in [0, 1].
\]

(iii) Given \( t \in [0, 1] \) and \( x \in \mathbb{R}^n \), with \( \|x\| < M \), then either \( f(x) \neq g(x) \) or the linear mapping \( f'(x) - tg'(x) \) is surjective.

Hence \( f \) and \( g \) share a value.

**Proof. First step.** From (i), we know that \( f(x_0) = 0 \). Choose any open ball \( B_1 \subset \mathbb{R}^n \) containing \( x_0 \).

Let \( V \) be the set

\[
V = g(A),
\]

where

\[
A = \{ x \in B_1 : \text{there is } t = t(x) \in [0, 1] \text{ such that } f(x) = tg(x) \}.
\]

Observe that \( A \) is not empty, because \( f(x_0) = 0 = 0 \cdot g(x_0) \), and so \( x_0 \in A \). We also deduce that \( V \subset g(B_1) \), \( B_1 \) is bounded, and \( g \) is a compact mapping, since \( g \) is continuous and \( D(g) = \mathbb{R}^n \). Therefore \( g(B_1) \) is relatively compact and hence \( V \) is relatively compact. We also conclude that the subset \( [0, 1] \times \overline{V} \) is compact in the topological product space \( \mathbb{R} \times \mathbb{R}^n \). Therefore the set

\[
V' = \{ ty : t \in [0, 1], \ y \in \overline{V} \}
\]

is also compact in \( \mathbb{R}^n \), because it is the range of the continuous mapping

\[
\phi : [0, 1] \times \overline{V} \subset \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad \phi(t, y) = ty.
\]

Hence, \( f \) being proper, the set \( V'' = f^{-1}(V') \) is also compact. Obviously the origin of \( \mathbb{R}^n \) is in \( V' \), so we obtain \( x_0 \in V'' \).

In the following steps compact sets will be used, such as \( V'' \) contained in the ball \( B(0, M) \).

Now, we construct the homotopy

\[
H : [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad H(t, x) = f(x) - tg(x),
\]

which will also be used in the following steps.

**Second step.** We will see here that if \( H(t_0, a) = 0 \) for some pair \((t_0, a) \in [0, 1] \times V''\), then there exists a neighbourhood of \( t_0, N(t_0) \), such that for each \( t \in N(t_0) \) there is

\[
x \in V'' \quad \text{with } H(t, x) = 0.
\]

Suppose \( H(t_0, a) = 0 \) and \( \|a\| < M \). Then (iii) implies that

\[
f'(a) - tg'(a) \quad (= D_2 H(t_0, a))
\]

is surjective.
Let us consider \( \ker(D_2 H(t_0, a)) = X_1 \). We obtain
\[
\mathbb{R}^n = X_1 \oplus X_2,
\]
with \( X_1 \) and \( X_2 \) Banach spaces, because they are norm linear finite dimensional spaces. Furthermore, \( D_2 H(t_0, a) \equiv A \) restricted to \( X_2 \) is bijective onto \( \mathbb{R}^m \), since if \( Ax_2 = 0 \), we obtain \( x_2 \in X_1 \), and therefore \( x_2 = 0 \).

Let \( G \) be the mapping
\[
G : ([0, 1] \times X_1) \times X_2 \to \mathbb{R}^m, \quad G((t, x_1), x_2) = (t, a + x_1 + x_2).
\]
Clearly \( G \) is a \( C^1 \)-mapping from the product of Banach spaces \([0, 1] \times X_1 \) and \( X_2 \) on the Banach space \( \mathbb{R}^m \). It holds that:

(i) \( G((t_0, 0), 0) = H(t_0, a) = 0 \),

(ii) \( G \in C^1(N(t_0, 0), 0) \),

(iii) \( D_2 G((t_0, 0), 0) = D_2 H(t_0, a) \), and so \( D_2 G((t_0, 0), 0) \) is an isomorphism.

So the hypotheses of Theorem 1 are fulfilled. Thus, there are \( r_0, r > 0 \), such that for every
\[
(t, x_1) \in [0, 1] \times X_1 \quad \text{with} \quad \| (t, x_1) - (t_0, 0) \| = \sup \{ |t - t_0|, \|x_1\| \} \leq r_0
\]
there exists a unique point \( v(t, x_1) \in X_2 \) with \( \|v(t, x_1)\| \leq r \), and
\[
G(t, x_1, v(t, x_1)) = H(t, a + x_1 + v(t, x_1)) = 0.
\]
Furthermore, \( a + x_1 + v(t, x_1) \in V'' \) because of step 1.

Third step. Now we will prove that \( f \) and \( g \) share a value.

From the proof of the Implicit Function Theorem [14, pp. 150–154] obtained from the Banach Fixed Point Theorem with the compactness of \([0, 1] \times V''\) and the continuity of \( H(t, x) \) and \( D_2 H(t, x) \), the constants \( r_0 \) and \( r \) can be taken independently of the particular point \((t_0, a) \in [0, 1] \times V''\). Then from the compactness of \([0, 1] \) we can reach level \( t = 1 \) from level \( t = 0 \) in a finite number of steps by an iteration process of Theorem 1. Note that the continuation method lets us reach level \( t = 1 \) since, by (ii) there is a solution \( x \) of the equation \( H(t, x) = 0 \) in \( B(0, M) \) for each \( t \in [0, 1] \).

\[ \Box \]

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References