FREDHOLM AND COMPACT MAPPINGS SHARING A VALUE

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Abstract: Sufficient conditions are given to assert that two differentiable mappings between Banach spaces have common values. The proof is essentially based upon continuation methods.

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I Preliminaries

In this paper, we assume that $X$, $Y$ and $Z$ are Banach spaces over $K = \mathbb{R}$ or $K = \mathbb{C}$. One method to solve the equation

$$F(x) = 0$$

is the continuation method$^{[1-10]}$. This method consists of embedding (1) in a continuum of problems

$$H(t,x) = 0, \quad 0 \leq t \leq 1,$$

where (2) is easily solved when $t = 0$. If it is possible to continue, the solution for $t = 0$ through $t = 1$ (1) is solved.

Sufficient conditions are given in order to prove that a $C^1$-mapping $F$ has at least one zero. Other sufficient conditions to guarantee the existence of zero points have been given by the author in a finite dimensional setting in papers $^{[11-18]}$ and in an infinite dimensional setting in papers $^{[19, 20]}$. Continuation methods are used. The proof supplies the existence of implicitly defined curves reaching zero points$^{[5, 6]}$. Key points are the use of Inverse and Implicit Function theorems and properties of the Fredholm $C^1$-mappings$^{[10]}$.

We briefly recall some concepts about Banach-valued mappings. Assuming that $F : A \rightarrow Y$ is a continuous mapping, where $A \subseteq X$, $F$ is said to be compact whenever $\text{Range} (F(B))$ is relatively compact (i.e. its closure $(F(B))$ is compact in $Y$) for every bounded subset $B \subseteq A$.

Proposition 1: If $a \in A$, $A$ is open, $F$ is compact and the derivative $F'(a)$ exists, then
$F(a) \in \mathbf{L}(X,Y)$ is also compact (see [10], p. 296).

Moreover if $A$ is open, then $F$ is said to be a Fredholm mapping if and only if $F$ is a $C^1$-mapping, and if and only if $F'(x) : X \to Y$ is a linear Fredholm operator for all $x \in A$. An operator $L : X \to Y$ is called a linear Fredholm operator if and only if $L$ is linear and continuous and both the numbers $\dim(\ker(L))$ and $\text{codim}(L(X))$ are finite. The number

$$\text{ind}(L) = \dim(\ker(L)) - \text{codim}(L(X))$$

is called the index of $L$.

Proposition 2 \[ For a linear Fredholm operator $B : X \to Y$, the following is true: the perturbed operator $B + C$ is also a Fredholm operator with $\text{ind}(B + C) = \text{ind}(B)$, if $C \in \mathbf{L}(X,Y)$ and $C$ is compact (see [10], p. 366).

Finally we state the Implicit and Inverse Function Theorems (see [10], pp. 150 - 154,172).

Theorem 1 \[ Assuming that: (i) the mapping $F : U(x_0,y_0) \subset X \times Y \to Z$ is defined in an open neighbourhood $U(x_0,y_0)$ and $F(x_0,y_0) = 0$, where $X$, $Y$ and $Z$ are Banach spaces; (ii) $F_Y$ exists as a partial Fréchet-derivative on $U(x_0,y_0)$, and $F_Y(x_0,y_0) : Y \to Z$ is bijective; (iii) $F$ and $F_Y$ are continuous at $(x_0,y_0)$, then the following are true.

(a) There exist numbers $r_0$ and $r$ such that for every $x \in X$ satisfying $\| x - x_0 \| \leq r_0$, there is exactly one $y(x) \in Y$ for which $\| y(x) - y_0 \| \leq r$, and $F(x,y(x)) = 0$.

(b) The sequence $(y_n(x))_{n \geq 1}$ of successive approximations, defined by $y_0(x) = y_0$, and $y_{n+1}(x) = y_n(x) - F_Y(x_0,y_0)^{-1}F(x,y_n(x))$, converges at the solution $y(x)$, whenever $n \to \infty$, for all points $x$ satisfying $\| x - x_0 \| \leq r_0$.

(c) If $F$ is a $C^m$-mapping, $1 \leq m \leq \infty$, in a neighbourhood of $(x_0,y_0)$ then $y(\cdot)$ is also a $C^m$-mapping in a neighbourhood of $x_0$.

Theorem 2 \[ Let $f : U(x_0) \subset X \to Y$ be a $C^1$-mapping. Then $f$ is a local $C^1$-mapping at $x_0$ if and only if $f'(x_0) : X \to Y$ is bijective.

2\[ Operators Sharing a Value

Clearly, if we rewrite $F$ as $f - g$ then $F$ has a zero if and only if $f$ and $g$ share a value, that is, there is $x \in X$ with $f(x) = g(x)$. We establish our result in terms of $f$, $g$.

Theorem 3 \[ Supposing that $f, g : X \to Y$ are two $C^1$-mappings satisfying the following conditions:

(1) $f$ is a $C^1$-diffeomorphism.

(2) $g$ is compact.

(3) If $f(x) - tg(x) = 0$, $x \in X$, $t \in [0,1]$ then $f'(x) - tg'(x)$ is injective.

Therefore $f$ and $g$ share a value.

Proof of theorem 3

First step \[ By hypothesis $(1)$, there is a single $x_0 \in X$ such that $f(x_0) = 0$. Choosing any open ball $D_1 \subset X$ containing $x_0$, let us define the set

$$V = g(A),$$

where

$$A = \left\{ x \in D_1 : \text{there is } t = t(x) \in [0,1] \text{ such that } f(x) = tg(x) \right\}.$$
It can be observed that $A$ is not empty, since $f(x_0) = 0 = 0g(x_0)$, and so $x_0 \in A$. Moreover, we note that $V \subset g(D_1)$, and $D_1$ is bounded, therefore $g(D_1)$ is relatively compact (by $(\text{i})$) hence $V$ is relatively compact, and the set $[0,1] \times \overline{V}$ is compact in the topological product $\mathbb{R} \times Y$. Let $V$ be the set

$$V = \left\{ ty : t \in [0,1], \ y \in \overline{V} \right\},$$

which is compact in $Y$, since it is the image of $[0,1] \times \overline{V}$ under the continuous mapping

$$(t, y) \in [0,1] \times Y \mapsto ty \in Y.$$

Next, we consider the subset of $X$ given by

$$V'' = f^{-1}(V).$$

Again $V''$ is a compact set, since $f^{-1}$ is continuous (by $(\text{i})$) and $V$ is compact.

The point $x_0$ and the set $V''$ will be used in the third step.

**Second step** We will prove here that $f$ is a Fredholm mapping and that

$$\forall x \in X, \ \forall t \in [0,1], \ f'(x) - tg'(x)$$

is a Fredholm linear operator of index zero. Theorem 2 implies that $\forall x \in X, f'(x) \in L(X,Y)$ is bijective, since $f$ is a local $C^1$-diffeomorphism in $x$. Therefore $f'(x) \in \text{Isom}(X,Y)$, $\forall x \in X$, and hence

$$\text{codim}(\text{Range}(f'(x))) = 0, \ \dim(\ker(f'(x))) = 0, \ \text{Ind}(f'(x)) = 0,$$

that is, $f'(x)$ is a Fredholm linear operator of index zero, for all $x \in X$. Hence $f$ is a Fredholm mapping as previously asserted.

On the other hand, $g$ is a compact mapping and therefore the derivative $g'(x)$ is a linear compact operator (Proposition 1) for each fixed $x$, and therefore $tg'(x)$ is also a linear compact operator for each fixed $t \in [0,1]$.

Since $f'(x)$ is a Fredholm linear operator of index zero and $tg'(x)$ is a compact linear mapping, Proposition 2 implies that

$$(f'(x) - tg'(x))$$

is a Fredholm linear operator of index:

$$\text{ind}(f'(x) - tg'(x)) = \text{ind}(f'(x)) = 0$$

as previously asserted.

**Third step** We will prove here that $f$ and $g$ share a value. We will use Theorem 1 together with the continuation method. Let us construct the following $C^1$-mapping

$$H : \mathbb{R} \times X \to Y$$

given by

$$H(t,x) = f(x) - tg(x).$$

Clearly, when $t \in [0,1]$ this is a homotopy between $f$ and $g$. We have seen in the second step that

$$\text{index}(H(t,x)) = \text{Ind}(f'(x) - tg'(x)) = 0, \ \forall x \in X, \ t \in [0,1].$$

Furthermore, if $(t,x) \in [0,1] \times X$ verifies that $H(t,x) = f(x) - tg(x) = 0$, by hypothesis
(iii) $H_x(t,x)$ is injective, that is

$$\ker(H_x(t,x)) = \{0\}.$$ 

Therefore

$$\dim(\ker(H_x(t,x))) = 0,$$

hence from the definition of index ($H_x(t,x)$),

$$\text{codim}((\text{Range})(H_x(t,x))) = 0,$$

and so

$$\text{Range}(f'(x) - tg'(x)) = \text{Range}(H_x(t,x)) = Y.$$ 

Thus $H_x(t,x)$ is surjective. Succinctly: if $H(t,x) = 0$, $(t,x) \in [0,1] \times X$, $H_x(t,x) \in L(X,Y)$ is bijective.

On the other hand, $H(\cdot,\cdot)$ and $H_x(\cdot,\cdot)$ are continuous since $f$ and $g$ are $C^1$-mappings.

Hence, when

$$H(t_0,x_0) = 0, \quad \Box \Box (t_0,x_0) \in [0,1] \times X,$$

the hypotheses of Theorem 1 are verified. Thus there are $r_0, r > 0$ such that for every $t \in (t_0 - r_0, t_0 + r_0)$ there is exactly one $x(t) \in X$ for which $y(x) - y_0 \leq r$ and $H(t,x(t)) = 0$.

Furthermore, $x(\cdot)$ is a $C^1$-mapping on a neighbourhood of $t_0$. From the proof of Theorem 1 [10, pp. 149 - 155] obtained through the Banach Fixed Point Theorem together with the aforementioned compactness of $[0,1] \times V'$ and the continuity of $H(\cdot,\cdot)$ and $H_x(\cdot,\cdot)$, the constants $r_0$ and $r$ can be taken independently of the particular point considered, that is, $(t_0,x_0) \in [0,1] \times V'$.

$\Box \Box$ Hence the compactness of $[0,1]$ implies we can reach $t = 1$ from level $t = 0$ in a finite number of steps. The fact that $X$ is connected implies there is a solution curve $(t,x(t)) \equiv C \subset [0,1] \times X$.

$H(t,x) = 0$ is verified if $(t_0,x_0) \in C$. The curve $C$ starts at the point $(t_0,x_0)$, as previously considered in Section 1, and finishes at the point $(1,x_0^*) \in [0,1] \times X$. Thus

$$H(1,x_0^*) = 0,$$

which can be written as

$$f(x_0^*) = g(x_0^*)$$

and the proof is completed.

References:


